1. INTRODUCTION

Entanglement plays a major role in quantum information theory}. Entangled quantum states find many applications in the fields like quantum cryptography, quantum computation, teleportation. Entanglement properties are deployed in many areas as a resource to get effective results. There are a number of facets of entanglement such as concurrence, distillable entanglement and entanglement cost.

It is important to know the strength of entanglement in applications of quantum communication under noisy environment. We use 0 and 1 as classical information for transmission under noisy quantum channels, for e.g., amplitude damping and Pauli channels, for the scenario where Alice and Bob act as transmitter (sender) and receiver following the postulates of quantum physics to encrypt and decrypt the information being sent. The simulation results show the amount of average error probability to judge the effect of entanglement under noisy channels for the quality of quantum communication being done. The aim is to achieve error free communication between Alice and Bob.

Cryptography based on the laws of quantum mechanics takes advantage of, say a photon as an information carrier that provides unconditional security against eavesdroppers for practical long distance communication with the help of optical fibers. This can only be success full when very efficient quantum repeaters are deployed to maintain the strength of the quantum signals at the end of the receiver side.

Quantum key distribution (QKD) is one of the important techniques in the area of secure communication network. It is based on key exchange phenomenon that is opposite to classical cryptography where key distribution is used for security. For symmetric key cryptosystems, the same secret key is required for both the users to perform encryption and decryption. This drawback is solved in public key cryptography but it is unsecure because of various attacks. Diffie-Hellman key exchange is a classical key exchange protocol but more complex to perform in polynomial time for some selected problems. All these methods are not unconditionally secure and data can be altered and duplicated by an eavesdropper, say Eve, in between the communication link at any point even without notice of the communicating parties. QKD is based on the No-Cloning theorem that is quantum mechanically and unconditionally secure and any changes in original data alerts the transmitter and receiver, hence providing high security from eavesdroppers.

Quantum key distribution simulation and error reconciliation is implemented by OptiSystem and other related softwares installed on a PC to obtain a high key rate for transmission of the quantum information. Field programmable gate array (FPGA) is one of the efficient hardwares used to perform practical QKD protocols. Features of FPGA include its simplicity for bit-wise operation, fast and parallel computing and large integrated RAM.

2. PRELIMINARIES

We start our discussion from the fundamentals of quantum computing that is required to understand the further theory used in quantum communication under noisy conditions. The transmission of any quantum state under noisy conditions can be represented with the help of a trace preserving completely positive map. This mapping occurs due to the interaction between environment and the system of interest. Mathematically, in quantum prospect this can be written as

\[ \rho \rightarrow \Phi(\rho) = \text{tr}_{\text{env}} \left( U (\rho \otimes \rho_{\text{env}}) U^\dagger \right) \]

where \( U \) as well as \( \rho \) are written for the unitary interaction between system and environment and density operators on a Hilbert space \( H_d \) of dimension-\( d \) respectively. Partial trace
over environment’s Hilbert space is denoted by $tr_m$. The Kraus representation is used to represent all noisy models. Alternatively, we can write Eqn. (1) as follows:

$$\phi(p) = \sum_j A_j \rho A_j^\dagger$$  \hspace{1cm} (2)

where the $A_j$’s must satisfy the completeness condition

$$\sum_j A_j A_j^\dagger = I$$  \hspace{1cm} (3)

The operators which fulfill the criteria from Eqn. (1) to (3) is essentially considered under noisy channel condition. Similarly, for two transmissions through the channel the Kraus operator representation is as follows

$$R \to \phi(R) = \sum_{j,i} (A_i \otimes A_j) R (A_i \otimes A_j)$$ \hspace{1cm} (4)

here $R$ represents density operator for $H_d \otimes H_d$ Hilbert space that is on the $d^2$ dimension.

Encoding at Alice’s side is performed in one of the two states $R_0$ and on the specified Hilbert space $H_d \otimes H_d$. This encoding is performed on the transmitted bit sent to Bob. After this, Alice sends these states via channel. Bob uses the action of the channel given in Eqn. (4) to get output in one of the two possible operators $R_0$ and $R_1$ at the receiving end. $tr(R_5 E_B)$ is the probability value in Bob’s measurement, which depends on the bit value sent by Alice and the method of maximum likelihood that depends on the values of bits $b$ and $s$ to make $tr(R_5 E_B)$ maximum. The average error probability is written as

$$P_e = \frac{1}{2} \sum_s \min\{|tr(R_0 E_B), tr(R_1 E_B)|\}$$  \hspace{1cm} (5)

Bob performs von Neumann measurement to get $P_e$ minimum, hence

$$P_e = \frac{1}{2} - \frac{1}{4} tr[R_1 - R_0]$$  \hspace{1cm} (6)

Concurrence is used to measure the entanglement amount, the entanglement strength and considered as an independent entanglement measure. Concurrence for a pure state $|\psi\rangle$ with bipartite two-level systems is defined as

$$C(|\psi\rangle) = \langle \psi | \sigma_2 \otimes \sigma_2 | \psi \rangle$$  \hspace{1cm} (7)

where $|\psi\rangle$ is the complex conjugate of pure state $|\psi\rangle$, $\sigma_2$ is the Pauli y-matrix defined as

$$\sigma_2 = \begin{pmatrix} 0 & -i \\
-i & 0 \end{pmatrix}$$

Concurrence for a given mixed state is written as

$$C(\rho) = \inf \{ p_i, |\psi_i\rangle, \sum_i p_i |\psi_i\rangle \langle \psi_i| \}$$ \hspace{1cm} (8)

where $p_i > 0$, $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, and

$$C(\rho) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|, |\lambda_4|\}$$ \hspace{1cm} (9)

where $\lambda_i$ represents decreasing eigenvalues of the Hermitian matrix $R = \sqrt{\rho} \rho^{1/2}$ and $\rho = (\sigma_2 \otimes \sigma_2)^* (\sigma_2 \otimes \sigma_2)$. Hence, different values of the $\lambda_i$ are the square roots corresponds to the eigenvalues of non-Hermitian matrix $\rho \rho^*$. Over environment’s Hilbert space is denoted by $tr_m$.

3. THE EFFECT OF VARIOUS KINDS OF NOISES ON THE QUANTUM STATES

Here we consider only some specific noise models like amplitude damping, phase-damping and two types of collective noises for the interaction with the quantum states transmitting between Alice and Bob. We follow the method used for the information transfer under noisy environment. Let the quantum density state is $\rho = |\psi\rangle \langle \psi|$, here $|\psi\rangle$ is any n qubit initial pure quantum state before transmission. Under noisy conditions the transmitted pure state evolves as follows:

$$\rho_k = \sum_{i,j} E_i^k \otimes E_j^k \otimes \cdots \otimes \cdots$$  \hspace{1cm} (10)

where $E_i^k$ denotes Kraus operator for specific type of noisy channel under consideration applied on desired qubit of the travelling quantum state. The evolution of density matrix under collective noise models can be written as

$$\rho_k = U_i^{\dagger} \rho U_i$$  \hspace{1cm} (11)

where $U_i$ represents type of noisy channel used, and $U_i$ is a 2 x 2 matrix (it works on a single qubit) for collective dephasing and collective rotation noises. After obtaining the transformed density matrix $\rho_k$, the normalization can be achieved by calculating the trace to be 1.

$$P_k = \frac{1}{tr(\rho_k)}$$

where $P_k$ is the normalized transformed matrix. It is expected that the receiver gets the exact state $|\psi\rangle$ sent by the sender in the absence of any noise and eavesdropping, hence we can compare this state with $\rho_k$ (quantum state obtained in presence of noise). This comparison parameter is known as fidelity and can be expressed as

$$F = \langle \psi | \rho_k | \psi \rangle$$  \hspace{1cm} (12)

The above mentioned equation of fidelity differs from conventional expression of fidelity. For two different quantum states say $\sigma$ and $\rho$ the fidelity is written as

$$F(\sigma, \rho) = tr[\sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}]$$

The subscript $F_c$ denotes the conventional fidelity. In ideal cases the fidelity is one, but it decreases under noisy environment as shown by the simulated results in the following figures.

4. THE NOISE MODELS UNDER STUDY AND THE SIMULATION RESULTS

For the practical implementation of QKD protocol, there are dedicated hardware available like XILINX based SPARTAN v3 FPGA clocked at 24 MHz. It is similar to an embedded hardware which includes all the necessary devices and components mounted on a single chip, where clock is used for USB interfaced with PC, 1 Mbps counter value as the quantum key bits are decided by the divider circuit.
power section and phase-locked loop (PLL) is monitored by
controlled commands sent by FPGA. The gated avalanche
photodiode (APD) needs a pulse generation of approximately
20 ns for quantum channel, this pulse generation is generated
by a digital clock manager (DCM). DCM is also responsible
for the generation of 48 MHz clock to regulate the functioning
of the FPGA board. Static random-access memory (SRAM) on
the FPGA hardware is an important building block used to store
and process the data\textsuperscript{32,33}. Any eavesdropping attempt between
the communicating parties perturbs the quantum information,
hence as per no-cloning theorem Eve’s presence can be
detected. The DPS QKD protocol as shown in Fig. 1 is used for
long distance communication between the repeater nodes and
practically less complex compared to other existing quantum
communication systems. The DPS QKD system is functionally
compatible with optical devices and networks because of it’s
integrity with these devices hence it is an important component
for the whole area of network security\textsuperscript{34}.

Figure 1 shown, the DPS QKD system uses optical-
fiber as a quantum channel and both the sender and receiver
communicates via weak coherent pulses with encoding of
logical bits in terms of relative phase of these pulses. Encoding
and decoding of the logical bits are performed with two signals
both at transmitter and receiver side. The working principle
of DPS QKD system is similar to B92 protocol, the encoding
and decoding of sent pulses depends on the relative phases,
if these are in phase that means encoded and decoded as 0, if
relative phase is $\pi$, the logical bit is 1. Moreover some kind of
security threats has to be considered because of weak coherent
pulses (WCP) leave some of loop holes for eavesdroppers
hence photon number splitting attack is the main concern of
DPS QKD protocol\textsuperscript{34}.

\begin{equation}
\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H,\rho] + \sum_j \alpha (L_j,\alpha \sigma^\dagger L_j,\alpha - \frac{1}{2}L_j,\alpha L_j,\alpha \sigma^\dagger - \frac{1}{2}L_j,\alpha \sigma^\dagger L_j,\alpha L_j,\alpha)
\end{equation}

Here $H$ is the Hamiltonian of the system and $H = 0$ is considered
just for the simplicity. The Lindblad operators show the coupling
between system and the environment under consideration,
hence in $L_j,\alpha = \sqrt{\eta_j,\alpha} \sigma^\dagger \alpha$ operator, $\sigma^\dagger \alpha$ represents the Pauli
operators of the $j$-th qubit, where $\alpha = x, y, z$ and $L_j,\alpha$ denotes
it’s action on the $j$-th qubit to represent the coupling between
system-environment under consideration.

Here the Figs. 2 and 3 shows the python simulation results
for master equation mentioned above and the Bloch sphere
representation for qubit distribution respectively.

\begin{equation}
E_0^A = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta_J} \end{bmatrix};
E_1^A = \begin{bmatrix} 0 & \sqrt{\eta_J} \\ 0 & 0 \end{bmatrix}
\end{equation}

where $\eta_J (0 \leq \eta_J \leq 1)$ describes the probability of error due to
Amplitude-damping noisy environment when a travel qubit pass through it. $\eta_J$ is also referred to as decoherence rate.
Similarly, phase-damping noise model is characterised by the following Kraus operators

\begin{equation}
E_0^P = \sqrt{1-\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};
E_1^P = \sqrt{\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};
E_2^P = \sqrt{\eta_P} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{equation}

In this section, considering noisy conditions on quantum
key distribution (QKD) protocol, the travel qubits interact with
the amplitude-damping noise or the phase-damping noise. The
Kraus operators for Amplitude-damping noise model is written
as\textsuperscript{13}
where $\eta_P (0 \leq \eta_P \leq 1)$ is the decoherence rate for the phase-damping noise. The simulated results for fidelity versus $\eta$ under amplitude-damping and phase-damping noise are shown in Figs. 4 and 5, respectively. These figures show that high values of fidelity describes less effect of noise at different values of $\eta$. The Bell pair used in BBM protocol is $|\phi^+\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, whereas basis used in B92 protocol are $|0\rangle$ and $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$. In the following simulated results, the Blue and Red lines correspond to B92 and BBM protocols, respectively.

4.2 Effect of the Collective Noise on the QKD Protocol

For the practical implementation, it is necessary to consider the channel noise that is generated due to the interaction of photons with the environment. This process is known as collective noise, with the assumption that the time variations of the noise is greater than the photons travelling time within a particular time window. Collective noise model is described by the following two types of noises. $U_{dp}$ is known as the unitary operator for collective-dephasing and $U_r$ is the unitary operator for collective-rotation noise. Each of these noises can be explained by the following unitary operations. The unitary operations for collective-dephasing noise is as follows

$$U_{dp} |0\rangle = |0\rangle, U_{dp} |1\rangle = e^{i\phi} |1\rangle$$ (17)

where $|0\rangle$ and $|1\rangle$ are the horizontal and vertical polarizations of photons respectively and $\phi$ is the noise parameter varying with respect to time. Similarly, the unitary operations for collective-rotation noise are as follows

$$U_r |0\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle, U_r |1\rangle = -\sin \theta |0\rangle + \cos \theta |1\rangle$$ (18)

here $\theta$ is also the noise parameter varying with respect to time. These noises are characterised by the following Kraus operators

$$U_{dp} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}, U_r = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$ (19)

The fidelity- $\eta$ plots for collective noise is shown in Figs. 6 and 7. Higher values of fidelity indicates less effect of noise at particular values of $\theta$ and $\phi$ for collective rotation and collective dephasing noise respectively.

4.3 Effect of the Pauli Noise on the QKD Protocol

Pauli noise model is characterised by the following Kraus operators

$$U_{dp} |0\rangle = |0\rangle, U_{dp} |1\rangle = e^{i\phi} |1\rangle$$ (17)

$$U_r |0\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle, U_r |1\rangle = -\sin \theta |0\rangle + \cos \theta |1\rangle$$ (18)

These noises are characterised by the following Kraus operators

$$U_{dp} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{bmatrix}, U_r = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$ (19)

The fidelity- $\eta$ plots for collective noise is shown in Figs. 6 and 7. Higher values of fidelity indicates less effect of noise at particular values of $\theta$ and $\phi$ for collective rotation and collective dephasing noise respectively.
operators
\[
W_{01} = \sqrt{P_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; W_{11} = \sqrt{P_2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
W_{10} = \sqrt{P_3} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; W_{00} = \sqrt{P_4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
(20)

where \( P_1 + P_2 + P_3 + P_4 = 1 \), that is sum of probabilities is equal to unity. The fidelity expressions for the BBM protocol under various types of noises are as follows

\[
F_{\text{AD}} = \frac{1}{4} (2 + 2\sqrt{1 - \eta - \eta})
\]
(22)

\[
F_{\text{PD}} = \left(1 - \frac{\eta}{2}\right)
\]
(23)

\[
F_{\text{CA}} = \cos^2\left(\frac{\theta}{2}\right)
\]
(24)

\[
F_{\text{CS}} = \cos^2(\theta)
\]
(25)

\[
F_{\text{CS}} = P_4
\]
(26)

On the other hand, the B92 protocol has the following fidelity expressions under noisy environment

\[
F_{\omega} = \frac{1}{4} \left(3 + \sqrt{1 - \eta}\right)
\]
(27)

\[
F_{\phi} = \frac{1}{2} \left(2 - \frac{\eta}{2}\right)
\]
(28)

\[
F_{\text{CD}} = \frac{3}{4} + \cos\left(\frac{\phi}{4}\right)
\]
(29)

\[
F_{\text{CR}} = \cos^2\left(\frac{\theta}{2}\right)
\]
(30)

\[
F_{\text{P}} = \frac{P_1 + P_2 + 2P_3}{2}
\]
(31)

here \( F_{\omega}, F_{\phi}, F_{\text{CA}}, F_{\text{CS}} \) and \( F_{\text{CS}} \) are the fidelities corresponds to the amplitude-damping, phase-damping, collective dephasing, collective rotation and Pauli noise, respectively. \( \eta \), \( \theta \) and \( \phi \) are the noise parameters mentioned in Kraus operator equations. \( P_1, P_2, P_3 \) and \( P_4 \) are the associated probabilities with Pauli noise model with the condition that

\[
P_1 + P_2 + P_3 + P_4 = 1
\]

4.4 QKD Protocol under Squeezed Generalized Amplitude Damping Channel

The squeezed generalised amplitude damping (SGAD) channel is derived by the master Eqn. (10). This follows the concept of completely positive map given in Eqn. (3). For more information about the effects of squeezing and corresponding Kraus operators\(^6\).

5. CONCLUSIONS

Authors observed that different types of noises in the master equation form are responsible for varying nature of different quantum states under quantum communication between Alice and Bob. The simulated fidelity graph results show that, the fidelity values vary with the above mentioned noise parameters under particular type of noise. The various types of noises mentioned in quantum algorithms\(^{25,26}\) are of different kind compared with the noises mentioned here. Entanglement purification\(^{23}\) and quantum error correction\(^{24}\) are the effective methods by which errors generated from decoherence can be corrected. The following observations are made from the above results

(i) The Bell-state of the form \( |\phi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \) used in BBM protocol is highly corrupted under Amplitude-damping noise, so it is not practically reliable under AD noise as shown in Fig. 4.

(ii) From Fig. 5 it is clear that under PD noise, single qubit based B92 protocol provides high value of fidelity compared to entangled state based BBM protocol, hence while transmission under ND noise, B92 is less affected by this kind of noise compared to BBM protocol.

(iii) From Figs. 4 to 7, it can be concluded that single qubit based B92 protocol(with \(|0\rangle\) and \(|+\rangle\) basis) is immune to all type of noises compared to the BBM protocol.

(iv) In BBM protocol \( |\phi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \) is used for quantum communication hence, any disturbance in sent sequence of \( |\phi^-\rangle \) is detected as Eavesdropping.

(v) Under CR noise, the fidelity graph of the B92 protocol shows variations with \( \theta \) which is not considered as an efficient approach for practical point of view, because this degrades the originality of the information being sent at long distances, hence in practical implementation the information will further decay due to device imperfections and losses in optical fibers used as a medium for data transmission, so noise consideration with the fidelity is an important point of research to be considered to avoid the loss of originality of the information sent from Alice to Bob.

Differential phase shift (DPS) QKD protocol is capable of communicating the quantum information for long distance using optical fiber as a quantum channel which is practically important between the quantum repeater nodes because of its high key rates and ease of practical implementation. But, the various attacks must be considered for DPS QKD protocol for security issues. Similar to B92 protocol, in DPS QKD protocol also Alice randomly prepares the quantum states and sends two non-orthogonal states to Bob, but opposite to B92 protocol, in DPS QKD protocol no need of a bright reference pulse but requires a weak coherent pulse (WCP) with less than 1 amount of average photon number, hence practically easy and simple to perform compared to that of B92 protocol.

The quantum noise generated due to the fiber imperfections could provide large amount of noise, so it is the main concern to deal with such noise while considering optical fibers for transmission of quantum information. Proper care should be taken when modulation speed of 40 Gb/s and above because
it may severely degrades the quantum signal strength. Hence, we need a practical quantum communication system which is flexible, easily implementable and efficient with low communication complexity.

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