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A Generic Method for Azimuthal Map Projection

Narayan Panigrahi* and Cyan Subhra Mishra#

*Centre for Artificial Intelligence and Robotics, Bangalore - 560 093, India
#National Institute of Technology, Rourkela- 769 008, India
*E-mail: n_panigrahi@yahoo.com

ABSTRACT

Map projections are mathematical methods for projecting spherical coordinates in the form of (φ, λ) to the map coordinates in the form of (*X,Y*) in Cartesian reference frame. Numerous methods for map projection have been derived and are being used for preparation of cartographic products. These map projections take into account specific position of the viewer on the datum surface for derivation of the map projections. A generic method for azimuthal map projection where the position of the viewer can be taken at an arbitrary point on the datum surface is derived. Using this generic method all the specific azimuthal map projections can be derived.

Keywords: Map projection, azimuthal projection, cartographic transformation, GIS

1. INTRODUCTION

Map projections are a class of cartographic transformation which is important for preparation of maps in different forms. Mathematically map projection is a pair of transformation equations which intakes the locations of objects on 3-D earth surface described through a geodetic datum and transforms them to locations into a 2-D map container. In other words map projection is a pair of 3-D to 2-D transformation[1-3].

Another definition of map projection can be transformation of spherical co-ordinate system (in the form of latitude longitude of a datum surface) into Cartesian co-ordinate system on map container. The map container can be a paper map, a digital terrain elevation model, a digital surface model or a vector or raster digital map. Mathematically it can be described as pair of Eqns[2,3]:

$$X = f_1(\phi, \lambda)$$
$$Y = f_2(\phi, \lambda)$$

The reverse process of locating objects on the datum surface given its corresponding location in the map is known as the reverse map projection or inverse map projection. Reverse map projections are described by the following Eqns:

$$\phi = f_1^{-1}(X, Y)$$
$$\lambda = f_2^{-1}(X, Y)$$

Also it can be observed that map projections is a mixture of many-to-one, one-to-one and one-to-many transformation process, where many locations of the spherical datum surface are projected on the map surface[2,4-6].

Also it can be inferred that map projection inherently depends upon the geodetic datum which models the natural surface of the map. There are many theoretical possibilities of map projection. Generally map projections are derived depending upon the datum, viewer's position and the position of the map container with respect to the datum surface. Many map projections are possible, but out of infinite theoretical possibilities, the important map projections are discussed and few of them are applied. Projections which are important from the point of view of its application can be classified into following categories:

(i) Azimuthal map projection
(ii) Conical map projection
(iii) Cylindrical map projection
(iv) Mercator map projection

Generally map projections are derived for known position of the eye on the main meridian of the datum surface (e.g. the axis of rotation) and target point of contact of the map surface.

Described and derived the mathematical equation of azimuthal map projection which takes into account the viewer's position at any arbitrary place on the datum surface. The projection for the azimuthal category with arbitrary viewer position is further validated for specific cases of gnomic, stereographic and orthographic projections. The geometric manifestation of the proposed generic azimuthal map projection (GAP) is also described.

2. GEOMETRIC MANIFESTATION OF THE GAP

Let us consider a generalised case of the azimuthal projection where the observer can be anywhere on the surface of the earth as depicted in Fig. 1.

Figure 1, let *V* be the generic position of the viewer on the datum surface, which is at *hR* distance(where *R* is the radius of the earth) from the map surface $(\overline{SS'})$.The map surface is tangent to the datum surface at point *F* (Pole of the earth).

FF' is the prime meridian where-as EE' is the equator. The other specifications of the figure are:

- *O* is the centre of the earth i.e. scale reduced of the earth

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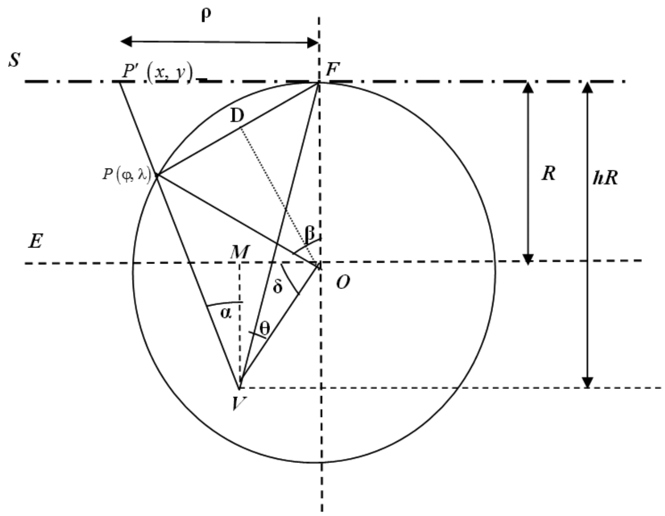

Figure 1. Geometric construction of GAP.

with a spherical datum.

- R is the mean radius of the earth.
- V is the view point or eye point of the observer.
- F is the point representing the pole of the earth which coincides with the point on the map container.
- $P\ (\varphi,\ \lambda)$ is a generic point on the surface of earth.
- α is view angle i.e. the angle subtended by the eye looking at the point P.
- β is the angle subtended by chord $\overline{PF}$ at the centre of the earth.
- $P'\ (x,\ y)$ is the azimuthal projection of point P on the map surface.
- M is the foot of perpendicular dropped on the equator from point V.
- F' is the foot of the perpendicular dropped from V to the prime meridian(polar axis).
- N is the foot of perpendicular dropped from V on map surface $\overline{SS'}$ and $V-M-N$.

3. MATHEMATICAL FORMULATION OF GAP

Figure 1 shows, the geometrical construction of the generic azimuthal projection is described, where

$$OF' = hR - R = R(h-1)$$

in $\Delta OVF'$ $\sin\delta = \dfrac{OF'}{OV}$

$$\Rightarrow OV = R(h-1)csc\delta$$

$$\Rightarrow VF' = R(h-1)cot\delta$$

From Pythagoras theorem in $\Delta FVF'$

$$VF^2 = VF'^2 + FF'^2 = R^2(h-1)^2 cot^2\delta + h^2R^2$$

$$\Rightarrow VF = R\sqrt{(h-1)^2 cot^2\delta + h^2} = RL \tag{1}$$

(where $L = \sqrt{(h-1)^2 cot^2\delta + h^2}$)

Using sine rule in ΔVFO

$$\frac{VF}{\cos\delta} = \frac{R}{\sin\angle FVO}$$

$$\Rightarrow \angle FVO = sin^{-1}\frac{\cos\delta}{VF}$$

$$\Rightarrow \angle VFO = sin^{-1}\frac{\cos\delta}{L} = \theta \tag{2}$$

$$\Rightarrow \angle MVF = \frac{\pi}{2} - \delta - \theta \tag{3}$$

$$\Rightarrow \angle PVF = \alpha + \left(\frac{\pi}{2} - \delta - \theta\right) = \alpha + \eta \tag{4}$$

(where $\eta = \dfrac{\pi}{2} - \delta - \theta$)

In ΔVPO $\angle VPO = \pi - (\pi - \beta - \alpha) = \beta - \alpha$

$$\angle VPF = \angle OPF + \angle VPO = \frac{\pi}{2} + \frac{\beta}{2} - \alpha \tag{5}$$

In ΔOPF, D is the midpoint of $\overline{PF}$.

So from standard geometry $\overline{OD}$ is the perpendicular bisector of $\overline{PF}$ and $\overline{OD}$ also bisects $\angle POF$.

$$\Rightarrow PD = DF$$

and $\angle POD = \angle DOF = \dfrac{\beta}{2}$

$$\Rightarrow PD = DF = Rsin\frac{\beta}{2}$$

$$\Rightarrow PF = 2Rsin\frac{\beta}{2} \tag{6}$$

Applying sine rule in ΔPVF

$$\frac{VF}{\sin\angle VPF} = \frac{PF}{\sin\angle PVF}$$

From Eqns (4), (5), (6)

$$\frac{L}{\cos\left(\frac{\beta}{2} - \alpha\right)} = \frac{2\sin\frac{\beta}{2}}{\sin(\alpha + \eta)}$$

$$\Rightarrow L\sin(\alpha + \eta) = 2\sin\frac{\beta}{2}\left(\cos\frac{\beta}{2}\cos\alpha + \sin\frac{\beta}{2}\sin\alpha\right) \tag{7}$$

Solving Eqn (7) one can obtain

$$\tan\alpha = \left[\frac{\sin\beta - L\sin\eta}{L\cos\eta - 2\,sin^2\frac{\beta}{2}}\right] \tag{8}$$

Now radial distance $= \rho = P'F = P'N + NF = P'N + VF'$

$$\Rightarrow \rho = hR\tan\alpha + R(h-1)\cot\delta$$

From Eqn (8)

$$\rho = hR\left[\frac{\sin\beta - L\sin\eta}{L\cos\eta - 2\,sin^2\frac{\beta}{2}}\right] + R(h-1)\cot\delta \tag{9}$$

Therefore the transformation from spherical coordinate to the Cartesian coordinate is given by:

$$X = \rho \sin(\lambda - \lambda_0)$$

$$\Rightarrow X = hR\left[\left(\frac{\sin\beta - L\sin\eta}{L\cos\eta - 2sin^2\frac{\beta}{2}}\right) + R(h-1)\cot\delta\right]\sin(\lambda-\lambda_0) \tag{10}$$

$$Y = \rho\cos(\lambda - \lambda_0)$$

$$\Rightarrow Y = hR\left[\left(\frac{\sin\beta - L\sin\eta}{L\cos\eta - 2sin^2\frac{\beta}{2}}\right) + R(h-1)\cot\delta\right]\cos(\lambda-\lambda_0) \tag{11}$$

4. DERIVATION OF SPECIFIC PROJECTIONS

There are various special cases of map projections, where the observer position is made fixed for the convenience of calculation. The major map projections are:

(a) Gnomic projection
(b) Stereographic projection
(c) Orthographic projection

4.1 Gnomic projection

In case of gnomic projection the observer is positioned at the centre of the earth. Hence in this case $h = 1$ and $\delta = \frac{\pi}{2}$.

So $L = \sqrt{(h-1)^2 cot^2\delta + h^2} = 1$

Substituting $h = 1$, $\delta = \frac{\pi}{2}$ and $L = 1$ in the Eqns (10) and (11) one can obtain:

$$X_{Gnomic} = R\left[\left(\frac{\sin\beta - \sin 0}{\cos 0 - 2sin^2\frac{\beta}{2}}\right) + R(1-1)\cot\frac{\pi}{2}\right]\sin(\lambda-\lambda_0)$$

$$= \left[\frac{\text{R}\sin\beta}{1 - 2sin^2\frac{\beta}{2}}\right]\sin(\lambda-\lambda_0)$$

$$\Rightarrow X_{Gnomic} = R\tan\beta\,\sin(\lambda-\lambda_0) \tag{12}$$

Similarly

$$Y_{Gnomic} = R\tan\beta\,\cos(\lambda-\lambda_0) \tag{13}$$

4.2 Stereographic Projection

In stereographic projection the observer is placed at the opposite pole of the map surface. Hence in this case h=2, $\delta = \frac{\pi}{2}$

So $L = \sqrt{(h-1)^2 cot^2\delta + h^2} = 2$

Substituting $h = 2$, $\delta = \frac{\pi}{2}$ and $L = 1$ in Eqns (10) and (11) one can obtain:

$$X_{Stereographic} = 2R\left[\left(\frac{\sin\beta - 2\sin 0}{2\cos 0 - 2sin^2\frac{\beta}{2}}\right) + R(2-1)\cot\frac{\pi}{2}\right]\sin(\lambda-\lambda_0)$$

$$= \left[\frac{2\text{R}\sin\beta}{2 - 2sin^2\frac{\beta}{2}}\right]\sin(\lambda-\lambda_0) = \left[\frac{\text{R}\sin\beta}{1 - sin^2\frac{\beta}{2}}\right]\sin(\lambda-\lambda_0)$$

$$= \left[\frac{\text{R}\sin\frac{\beta}{2}cos\frac{\beta}{2}}{cos^2\frac{\beta}{2}}\right]\sin(\lambda-\lambda_0) = \left[\frac{\text{R}\sin\frac{\beta}{2}cos\frac{\beta}{2}}{cos^2\frac{\beta}{2}}\right]\sin(\lambda-\lambda_0)$$

$$X_{Stereographic} = 2R\tan\frac{\beta}{2}\sin(\lambda-\lambda_0) \tag{14}$$

Similarly

$$Y_{Stereographic} = 2R\tan\frac{\beta}{2}\cos(\lambda-\lambda_0) \tag{15}$$

4.3 Orthographic Projection

In orthographic projection the observer lies on the polar line of the earth, but at infinite distance. Hence in this case $h = \infty$ and $\delta = \frac{\pi}{2}$. So we have to apply the limiting case.

Applying limiting conditions in Eqn (2):

$$\theta = \lim_{h\to\infty}\left(sin^{-1}\frac{\cos\delta}{\sqrt{(h-1)^2 cot^2\delta + h^2}}\right) = 0$$

Since $\delta = \frac{\pi}{2}$

$$\Rightarrow \eta = \frac{\pi}{2} - \delta - \theta = 0 \tag{16}$$

$$\rho = hR\left[\frac{\sin\beta - L\sin\eta}{L\cos\eta - 2\,sin^2\frac{\beta}{2}}\right] + R(h-1)\cot\delta$$

$$= hR\left[\frac{\sin\beta}{L - 2\,sin^2\frac{\beta}{2}}\right] = R\left[\frac{\sin\beta}{\frac{L}{h} - \frac{2\,sin^2\frac{\beta}{2}}{h}}\right]$$

Now

$$\lim_{h\to\infty}\left(\frac{\sqrt{(h-1)^2 cot^2\delta + h^2}}{h}\right) = \frac{h\sqrt{cot^2\delta + 1}}{h}$$

$$= csc^2\delta = 1 \qquad (\because \delta = \frac{\pi}{2})$$

$$\Rightarrow \rho = R\sin\beta$$

$$X_{Orthographic} = R\sin\beta\sin(\lambda-\lambda_0) \tag{17}$$

$$Y_{Orthographic} = R\sin\beta\cos(\lambda-\lambda_0) \tag{18}$$

5. INVERSE TRANSFORMATION OF GAP

In the previous section the forward transformation of GAP was described. Similarly the inverse transformation (i.e. the transformation from Cartesian coordinate system to spherical datum) is also of great importance. So we have to find the value of ϕ and λ as a function[1-3] of X and Y.

Here the inverse transformation for the special types of generic azimuthal map projection (GAP) has been derived.

5.1 Inverse Transformation for Gnomic Projection

From Eqns (12) and (13)

$$X_{Gnomic} = R\tan\beta\,\sin(\lambda-\lambda_0) = X_G$$

$$Y_{Gnomic} = R\tan\beta\,\cos(\lambda-\lambda_0) = Y_G$$

$$\Rightarrow X_G^{\ 2} + Y_G^{\ 2} = R^2 tan^2\beta$$

$$\Rightarrow \frac{X_G^{\ 2} + Y_G^{\ 2}}{R^2} = tan^2\beta$$

It is clear from Fig. 1 that

$$\phi = \frac{\pi}{2} - \beta \tag{19}$$

$$\Rightarrow \frac{X_G^{\ 2} + Y_G^{\ 2}}{R^2} = tan^2\left(\frac{\pi}{2} - \phi\right) = cot^2\phi$$

$$\Rightarrow \phi = cot^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{R}\right) \tag{20}$$

5.3 Inverse Transformation for Stereographic Projection

From Eqns (14) and (15)

$$X_{Stereographic} = 2R\tan\frac{\beta}{2}\sin(\lambda-\lambda_0) = X_S$$

$$Y_{Stereographic} = 2R\tan\frac{\beta}{2}\cos(\lambda-\lambda_0) = Y_S$$

$$\Rightarrow X_S^{\ 2} + Y_S^{\ 2} = 4R^2 tan^2\frac{\beta}{2}$$

$$\Rightarrow â = 2\,tan^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{2R}\right)$$

$$\Rightarrow \frac{ð}{2} - \phi = 2tan^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{2R}\right)$$

$$\Rightarrow \phi = \frac{ð}{2} - 2tan^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{2R}\right) \tag{21}$$

5.3 Inverse Transformation for Orthographic Projection

$$X_{Orthographic} = R\sin\beta\sin(\lambda-\lambda_0) = X_O$$

$$Y_{Orthographic} = R\sin\beta\cos(\lambda-\lambda_0) = Y_O$$

$$\Rightarrow X_O^{\ 2} + Y_O^{\ 2} = R^2 sin^2\beta$$

From Eqn (19):

$$X_O^{\ 2} + Y_O^{\ 2} = R^2 cos^2\phi$$

$$\Rightarrow \phi = cos^{-1}\frac{\sqrt{X_O^{\ 2} + Y_O^{\ 2}}}{\mathrm{R}} \tag{22}$$

5.4 Inverse Transformation for λ

From Eqns (10) and (11)

$$\Rightarrow \lambda - \lambda_0 = \tan^{-1}\frac{X}{Y}$$

$$\Rightarrow \lambda = \lambda_0 + \tan^{-1}\frac{X}{Y} \tag{23}$$

From the above derivations:

$$\phi_{Gnomic} = cot^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{R}\right) \tag{24}$$

$$\lambda_{Gnomic} = \lambda_0 + \tan^{-1}\frac{X}{Y} \tag{25}$$

$$\phi_{Stereographic} = \frac{ð}{2} - 2tan^{-1}\left(\frac{\sqrt{X_G^{\ 2} + Y_G^{\ 2}}}{2R}\right) \tag{26}$$

$$\lambda_{Stereographic} = \lambda_0 + \tan^{-1}\frac{X}{Y} \tag{27}$$

$$\phi_{Orthographic} = cos^{-1}\frac{\sqrt{X_O^{\ 2} + Y_O^{\ 2}}}{\mathrm{R}} \tag{28}$$

$$\lambda_{Orthographic} = \lambda_0 + \tan^{-1}\frac{X}{Y} \tag{29}$$

6. CONCLUSIONS

A generic map projection method such as GAP will prove to be quite useful to generate relatively more accurate digital map corresponding to spatial data in comparison to traditional map projection methods. GAP has the following advantages in comparison to the traditional map projection methods (a) that this projection can generate digital maps for users at any part of the globe for accurate location, navigation and measurement. (b) The map projection is independent of the position of the viewer unlike traditional map projection methods which demand that the viewer's position be fixed at the centre, perspective of the globe. So we can get the projection of any point of the datum surface with respect to any point of reference.

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CONTRIBUTORS

Dr Narayan Panigrahi received his BSc (Physics Hons) from Khallikote College, Berhampur and was the Best Graduate of the year 1987, MSc from J.K. Institute of Applied Physics, University of Allahabad and MTech (CSDP) from Indian Institute of Technology, Kharagpur. He obtained his PhD from the Indian Institute of Technology, Mumbai. Presently working as a Scientist in Center for Artificial Intelligence and Robotics (CAIR), DRDO, Bengaluru, India. He has published over thirty scientific and research papers and three intellectual properties. He has authored three books in the field of geographical information science and system. His research interest include: design and development of GIS algorithms, transformations and tools for analysis of spatio-temporal data

Mr Cyan Subhra Mishra is pursuing his MTech in Department of Electronics and Communication Engineering, specialising in VLSI design and Embedded Systems. He worked as an intern at IIT, Bombay and has experience of research in Hyper spectral Image processing. His research area included design of intelligent embedded systems, robotics, machine learning and VLSI design.