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Calculation Method for Three Dimensional Turbulent Boundary Layers

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Abstract. A momentum integral method is developed to predict the growth of three dimensional turbulent boundary layers. Four equations, have been derived in streamline coordinates and solved: (a) momentum integral equation in streamwise direction (b) momentum integral equation in crosswise direction (c) moment of momentum equation in streamwise direction (d) a skin friction equation. The streamwise velocity profile is assumed to be a combination of the logarithmic law of the wall and a linear law of the wake while Mager's expression is used to represent cross flow velocity profile.

1. Basic Equations

Timman[1] has derived equations of motion and continuity in streamline coordinates wherein the longitudinal direction (x_1 direction) follows a streamline outside the boundary layer and is termed streamwise direction. Crosswise direction (x_3 direction) is described by the lines orthogonal to these streamlines and parallel to the surface. Transverse direction (x_2 direction) is defined by the outward normal to the surface, (Fig. 1). In three dimensional boundary layers developing on surfaces of arbitrary shapes velocity profile is skewed. At every x_2 location the velocity vector $\vec{U}$ changes its direction and can be resolved into streamwise (u_1) and crosswise (u_3) components. The skew angle α made by $\vec{U}$ with x_1 direction gradually approaches zero magnitude as the edge of the layer is reached, it attains its maximum value α_0 at the wall. The equations for an incompressible steady mean flow are

Equation of motion in streamwise direction

$$\frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2 \partial u_1}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_1}{\partial x_3} + k_3 u_1 u_3 - k_1 u_3^2 = \frac{-1}{\rho h_1}\frac{\partial p}{\partial x_1} + \frac{1}{\rho}\frac{\partial \tau_{12}}{\partial x_2} \quad (1)$$

Equation of motion in crosswise direction

$$\frac{u_1}{h_1}\frac{\partial u_3}{\partial x_1} + u_2\frac{\partial u_3}{\partial x_2} + \frac{u_3}{h_3}\frac{\partial u_3}{\partial x_3} k_1 u_1 u_3 - k_3 u_1^2 = -\frac{1}{\rho h_3}\frac{\partial p}{\partial x_3} + \frac{1}{\rho}\frac{\partial \tau_{32}}{\partial x_2} \quad (2)$$

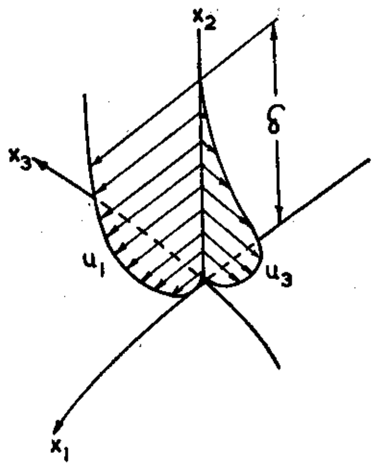

Figure 1. Orthogonal curvilinear coordinates for three dimensional boundary layers.

Continuity equation

$$\frac{1}{h_1 h_3}\left[\frac{\partial(h_3 u_1)}{\partial x_1}+\frac{\partial(h_1 u_3)}{\partial x_3}+\frac{\partial(h_1 h_3 u_2)}{\partial x_2}\right]=0 \tag{3}$$

In a turbulent boundary layer, the instantaneous velocity components are denoted as $u_1 + u_1'$, $u_2 + u_2'$ and $u_3 + u_3'$ along x_1, x_2 and x_3 directions respectively where u_1, u_2 and u_3 are mean values and u_1', u_2' and u_3' are the corresponding fluctuations. p denotes the mean static pressure and ρ the mean density of the fluid. h_1, h_2 and h_3 are the three Lame coefficients along x_1, x_2 and x_3 respectively. τ_{12} and τ_{32} are shear stress components in x_1 and x_3 directions respectively.

$$\tau_{12} = \mu\frac{\partial u_1}{\partial x_2} - \overline{\rho u_1' u_2'} \text{ and } \tau_{32} = \mu\frac{\partial u_3}{\partial x_2} - \overline{\rho u_3' u_2'} \tag{4}$$

where μ is absolute viscosity. Terms $-\overline{\rho u_1' u_2'}$ and $-\overline{\rho u_3' u_2'}$ are Reynolds stresses. The geodesic curvatures k_1 and k_3 are

$$k_1 = \frac{1}{h_1 h_3}\frac{\partial h_3}{\partial x_1} \text{ and } k_3 = \frac{1}{h_1 h_3}\frac{\partial h_1}{\partial x_3} \tag{5}$$

Assuming the flow in the free stream to be irrotational the following relationship is obtained

$$\frac{1}{U_\infty}\frac{\partial U_\infty}{h_3 \partial x_3} = -k_3 \tag{6}$$

U_∞ is the local free stream velocity.

2. Derivation of Equations

Integrating Eqn. (1) with respect to x_2 within the limits of 0 to ∞, after eliminating u_2 through Eqn. (3), the streamwise momentum integral equation is obtained

$$\frac{\partial \theta_{11}}{\partial s} + \frac{1}{U_\infty}\frac{\partial U_\infty}{\partial s}[2\theta_{11} + \delta_1^*] + \frac{\partial \theta_{33}}{\partial n} + \frac{1}{U_\infty}\frac{\partial U_\infty}{\partial n}[\delta_3^* - 2\theta_{31}]$$

$$+ k_1(\theta_{11} + \theta_{33}) + k_3(\delta_3^* - 2\theta_{31}) = \frac{\tau_{01}}{\rho U_\infty^2} = \frac{Cf_1}{2} = \omega^2 \tag{7}$$

where

$$\delta_1^* = \int_0^\infty \left(1 - \frac{u_1}{U_\infty}\right) dx_2 \tag{8}$$

$$\theta_{11} = \int_0^\infty \frac{u_1}{U_\infty}\left(1 - \frac{u_1}{U_\infty}\right) dx_2 \tag{9}$$

$$\theta_{13} = \int_0^\infty \frac{u_3}{U_\infty}\left(1 - \frac{u_1}{U_\infty}\right) dx_2 \tag{10}$$

$$\theta_{33} = \int_0^\infty \left(\frac{u_3}{U_\infty}\right)^2 dx_2 \tag{11}$$

$$\theta_{31} = \int_0^\infty \frac{u_3 u_1}{U_\infty^2}\, dx_2 \tag{12}$$

$$\delta_3^* = \int_0^\infty \frac{u_3}{U_\infty}\, dx_2 \tag{13}$$

τ_{01} and Cf_1 are wall shear stress and skin friction coefficient in x_1 direction respectively while $\omega \left(= \sqrt{\frac{Cf_1}{2}}\right)$ is a skin friction parameter. Terms $\partial s (= h_1 \partial x_1)$ and $\partial n (= h_3 \partial x_3)$ are infinitesimal arc lengths along x_1 and x_3 directions respectively.

Similarly integrating Eqn. (2) with respect to x_2 between the limits 0 to ∞ and using Eqn. (3) to eliminate u_2, crosswise momentum integral equation is obtained

$$\frac{\partial \theta_{31}}{\partial s} + \frac{\partial \theta_{33}}{\partial n} + \frac{2}{U_\infty}\frac{\partial U_\infty}{\partial s}\theta_{31} + 2k_1\theta_{31} + \frac{2}{U_\infty}\frac{\partial U_\infty}{\partial n}\theta_{33}$$

$$+ k_3(\theta_{33} + \delta_1^* + \theta_{11}) = -\frac{\tau_{03}}{\rho U_\infty^2} = -\frac{Cf_3}{2} = -\epsilon_0 \omega^2 \tag{14}$$

where τ_{03} and Cf_3 are respectively the wall shear stress and skin friction coefficient in x_3 direction. It is assumed that for small values of α_0

$$\epsilon_0 = \tan \alpha_0 \approx \frac{\tau_{03}}{\tau_{01}} \tag{15}$$

Multiplying Eqn. (1) by x_2 and substituting for u_2 through Eqn. (3) and integrating the resulting equation with respect to x_2 between the limits 0 to ∞, moment of momentum equation in streamwise direction is obtained

$$\frac{\partial}{\partial s}\left[U_\infty^2 \int_0^\infty x_2 \frac{u_1}{U_\infty}\left(1-\frac{u_1}{U_\infty}\right)dx_2\right] + U_\infty \int_0^\infty \left[\left(1-\frac{u_1}{U_\infty}\right)\frac{\partial}{\partial s}\left\{U_\infty x_2\right.\right.$$

$$\left.\left. + \int_0^{x_2} u_1 dx_2\right\}\right]dx_2 + U_\infty k_1\left[\int_0^\infty x_2 u_1 dx_2 + \int_0^\infty \left(\int_0^{x_2} u_1 dx_2\right)dx_2\right]$$

$$- k_1 \int_0^\infty u_1^2 x_2 dx_2 - k_1 \int_0^\infty \left(u_1 \int_0^{x_2} u_1 dx_2\right)dx_2 + k_1 \int_0^\infty x_2 u_3^2\, dx_2$$

$$+ U_\infty k_3\left[\int_0^\infty x_2 u_3 dx_2 + \int_0^\infty \left(\int_0^{x_2} u_3 dx_2\right)dx_2\right] - 2\int_0^\infty x_2 k_3 u_1 u_3 dx_2$$

$$- \int_0^\infty \left(u_1 \int_0^{x_2} k_3 u_3 dx_2\right)dx_2 + \frac{\partial}{\partial n}\left[U_\infty^2 \int_0^\infty x_2 \frac{u_3}{U_\infty}\left(1-\frac{u_1}{U_\infty}\right)dx_2\right]$$

$$+ U_\infty \int_0^\infty \left[\left(1-\frac{u_1}{U_\infty}\right)\frac{\partial}{\partial n}\left\{\int_0^{x_2} u_3 dx_2\right\}\right]dx_2 - \int_0^\infty u_3 \frac{\partial}{\partial n}\left(U_\infty x_2\right)dx_2$$

$$= \frac{1}{\rho}\int_0^\infty \tau_{12} dx_2 \tag{16}$$

Simplification of Integral Equations

Equations (7), (14) and (16) are simplified by guessing the order of magnitude of the various integral quantities. It is assumed that variation of integral quantities in streamwise direction is large as compared with the variation in crosswise direction; also $u_3 \ll u_1$.

Velocity Profiles in Three Dimensional Turbulent Boundary Layers

In the present method it is assumed that the variation of u_1 in x_2 direction is given by Rotta's[2] velocity profile :

$$\frac{u_1}{u_\tau} = \frac{1}{k}\left(ln\,\frac{u_\tau x_2}{\nu}\right) + \frac{A}{k}(2\eta) + C \text{ for } 0 \leqslant x_2 \leqslant \delta \tag{17}$$

and

$$\frac{u_1}{U_\infty} = 1 - \frac{\omega}{k}\left[2A(1-\eta) - ln\eta\right] \tag{18}$$

where $\delta (= x_2/u_1 = 0.995\ U_\infty)$ is absolute boundary layer thickness, $u_\tau\ (= \tau_{01}/\rho)^{1/2}$ is shear velocity, k is von kármán constant (0.41), ν is kinematic viscosity, η is non-dimensionalised distance $(= x_2/\delta)$, C is a constant and ln is Napierian logarithm. A is a free parameter.

Mager's[3] profile has been taken to represent the velocity in crosswise direction

$$\frac{u_3}{U_\infty} = \epsilon_0 (1 - \eta)^2 \frac{u_1}{U_\infty} \tag{19}$$

Final Form of Integral Equations

Substitution of Rotta's and Mager's profiles into the simplified integral equations yield and streamwise momentum integral equation

$$\omega(G_1 - \omega G_2) \frac{\partial \delta}{\partial s} + \delta(G_1 - 2G_2\omega) \frac{\partial \omega}{\partial s} + \omega\delta \left(\frac{1}{k} - \omega G_4\right) \frac{\partial A}{\partial s}$$

$$+ \frac{2}{U_\infty} \frac{\partial U_\infty}{\partial s} \delta\omega(1.5G_1 - \omega G_2) + k_1\delta\omega(G_1 - \omega G_2) - \omega^2 = 0 \tag{20}$$

where

$$G_1 = \frac{1}{k}(A + 1) \tag{21}$$

$$G_2 = \frac{1}{k^2}(1.33A^2 + 3A + 1) \tag{22}$$

$$G_4 = \frac{1}{k^2}(2.66A + 3) \tag{23}$$

Crosswise Momentum Integral Equation

$$\delta\epsilon_0(2G_5\omega - G_3) \frac{\partial \omega}{\partial s} + (dn)\ \epsilon_0 \frac{\partial \delta}{ds}\ \delta\omega\epsilon_0 \left(\omega G_6 - \frac{1}{k}\right) \frac{\partial A}{\partial s}$$

$$+ \left[\delta\epsilon_0(dn) \frac{2}{U_\infty} \frac{\partial U_\infty}{ds} + 2k_1\delta\epsilon_0(dn) + k_3\delta\omega(2G_1 - \omega G_2) + \epsilon_0\omega^2\right]$$

$$+ \delta(dn) \frac{\partial \epsilon_0}{\partial s} = 0 \tag{24}$$

where

$$G_3 = \frac{1}{k}(A + 1.22) \tag{25}$$

$$G_5 = \frac{1}{k^2}(0.793A^2 + 2.084A + 1.574) \tag{26}$$

$$(dn) = (0.33 - \omega G_3 + \omega^2 G_5) \tag{27}$$

Streamwise Moment of Momentum Equation

$$\delta^2(G_7 - \omega G_8)\frac{\partial\omega}{\partial s} + \delta^2\omega(G_9 - \omega G_{10})\frac{\partial A}{\partial s} + \delta\omega(4G_7 - \omega G_8)\frac{\partial\delta}{\partial s}$$

$$+ \delta^2\omega\frac{2}{U_\infty}\frac{\partial U_\infty}{\partial s}(2G_7 - 0.5\omega G_8) + k_1\delta^2\omega(2G_7 - \omega G_{11}) = \frac{\delta}{U_\infty^2}\int_0^1 \frac{\tau_{12}}{\rho}\,d\eta \quad (28)$$

where

$$G_7 = \frac{1}{k}(0.33A + 0.25) \quad (29)$$

$$G_8 = \frac{1}{k^2}(1.16A^2 + 2.112A + 1) \quad (30)$$

$$G_9 = \frac{0.33}{k} \quad (31)$$

$$G_{10} = \frac{1}{k^2}(1.16A + 0.945) \quad (32)$$

$$G_{11} = \frac{1}{k^2}(A^2 + 1.556A + 0.75) \quad (33)$$

Clauser's[4] eddy viscosity model along with Rotta's profile have been used to determine the 'Shear Stress Integral' appearing on the right hand side of the Eqn. (28)

$$\frac{\delta}{U_\infty^2}\int_0^1 \frac{\tau_{12}}{\rho}\,d\eta = \omega^2\delta\,[\eta_0(1 + 2A - ln\eta) - A\eta_0^2] \quad (34)$$

$$\eta_0 = \frac{\alpha}{k^2}(1 + A) \quad (35)$$

η_0 is the non-dimensionalised distance at which the Clauser's inner layer meets the outer layer and α (= 0.018) is a constant.

Skin Friction Equation

Equation for skin friction has been derived directly from the streamwise velocity profile, Eqn. (17)

$$\frac{1}{\delta}\frac{\partial\delta}{ds} + 2\frac{\partial A}{\partial s} + \frac{\partial\omega}{\partial s}\left[\frac{1}{\omega} + \frac{0.41}{\omega^2}\right] = -\frac{1}{U_\infty}\frac{\partial U_\infty}{\partial s} \quad (36)$$

Numerical Solution

The equations (20), (24), (28) and (36) form a set of four equations involving δ, A, ω and ϵ_0 which have been solved by a Fourth order Runge Kutta method. In order to assess the viability of the calculation method developed, the experimental data of Wakhaloo[5,6] has been used. Fig. 2 shows the pattern of

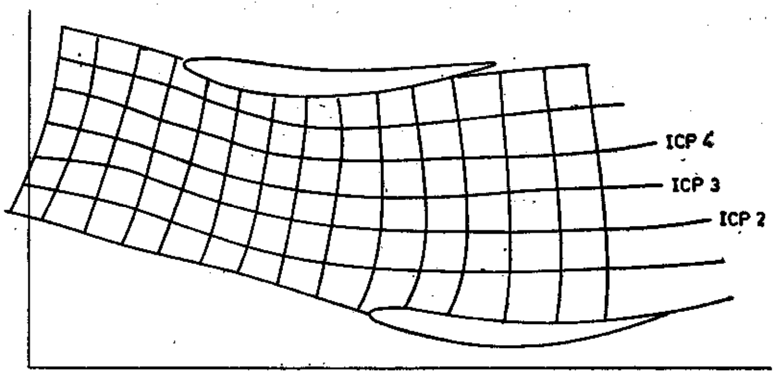

Figure 2. External streamline pattern in isolated cascade passage.

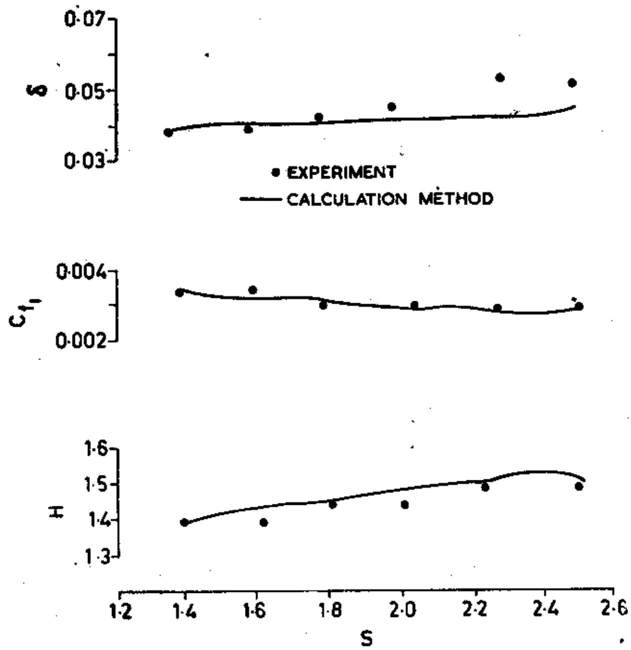

Figure 3. Comparison of predicted values of δ, Cf_1 and H with experimental data.

external streamlines obtained by Wakhaloo in his experiment on three dimensional turbulent boundary layers developing in an isolated compressor cascade passage. Results pertaining to one of the streamlines ICP2 are shown in Figs. 3, 4 and 5.

3. Results

Fig. 3 shows the variation of δ, Cf_1 and H $(= \delta_1^* / \theta_{11})$ along the streamline ICP2. The experimental values match closely with the predicted values. Fig. 4 shows the variation of δ_1^*, θ_{11} and θ_{31} while Fig. 5 shows the variation of ϵ_0 and A along the

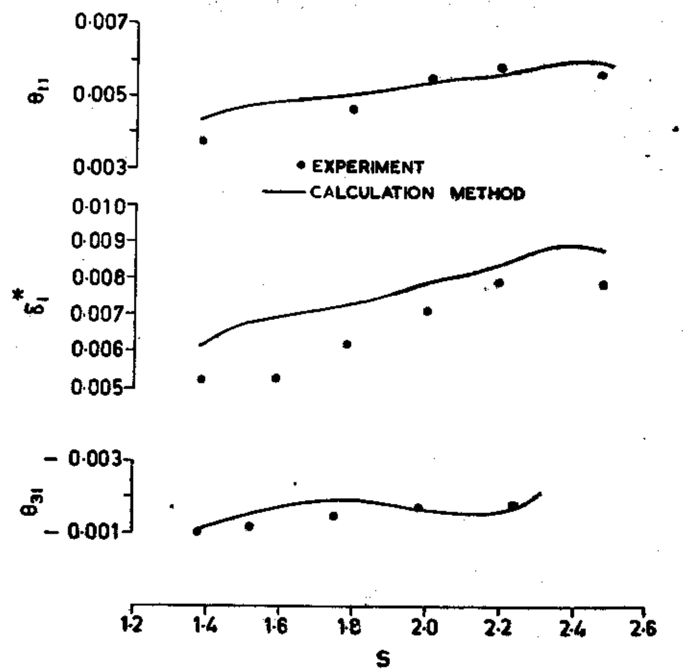

Figure 4. Comparison of predicted values of θ_{11}, δ_1^* and θ_{31} with experimental data.

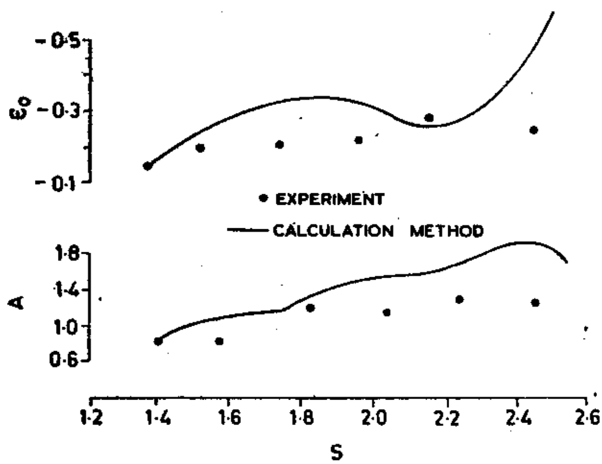

Figure 5. Comparison of predicted values of ϵ_0 and A with experimental data.

same streamline. The deviations could be due to combined effect of reasons cited below.

The change of curvature of external streamline for ICP2 occurred in the range $s = 2.0$ to $s = 2.4$. Within this region, the calculated values exhibit a greater deviation from those obtained experimentally. In this zone s shaped cross flow velocity profiles were obtained. Mager's model of cross flow velocity profile is incapable of representing such profiles, and hence the error.

The computer program needs the curvatures of external streamlines and equipotentials as input data. Wakhaloo obtained the geometry of external streamlines through potential flow solution assuming that the fluid moved tangentially over the blade surfaces. Actually separation of flow from the blade surfaces was observed which resulted in the alteration of the geometry of the external streamlines, hence the input data to the computer must have been uncertain to an extent.

The magnitude of A at the starting point of computation was calculated using interpolated values of H and ω. This too could introduce some error.

The magnitude of α in Eqn. (35) was taken to be 0.018, any other suitable value of α can be chosen to attain a better agreement between predicted and experimental values. Such an approach has already been suggested, Rotta[2].

Terms which have been neglected in the integral equations during simplification are also partly responsible towards the deviations in predicted and experimental data.

4. Conclusion

The calculation method developed in this paper predicts the growth of three dimensional turbulent boundary layers quite satisfactorily.

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