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Flow of a Second Order Fluid Between Two Infinite Porous Rotating Disks

H. G. SHARMA

Department of Mathematics, University of Roorkee, Roorkee

ANAND PRAKASH

Armament Research & Development Establishment, Pune-411021

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Abstract. The flow of an incompressible second-order fluid between two infinite porous rotating disks has been studied with the assumption that the rate of injuction of the fluid at one disc is equal to the rate of suction at the other. The velocity components have been expressed in terms of three dimensionless functions, which in turn are obtained in ascending powers of the Reynold number (taken to be small), defined in terms of the angular velocities of the disks.

1. Introduction

The steady flow of a viscous incompressible fluid between two infinite rotating disks has been investigated by Von-Kârmán[1]. Later Batchelor[2] extended Von-Kármán's solution to the case of two disks He has also given a qualitative discussion of the nature of the flow field for various values of the ratio of the angular velocities of the two disks. The same problem has again been studied by Stewartson[3] and criticized some of Batchelor's[2] results. Further Stuart[4] has investigated the effects of suction on a single infinite rotating disk. Lance and Rogers[5] have discussed the problem on the assumption that the similarity solution is valid by using the method of numerical integration for various values of Reynolds number and for different values of the ratio of the angular velocities of the two disks. The flow of viscous incompressible liquid between two infinite porous rotating disks has recently been studied by Gaur[6].

The paper deals with the steady flow of an incompressible Second-order fluid between two infinite porous disks, when the Reynolds number R, defined in terms of the angular velocities of the disks, is small. It is assumed that the rate of suction of the fluid at the lower disks is equal to the rate of injection of the fluid at the upper disk. The series solution for the velocity components has been obtained by expanding the series in ascending powers of Reynolds number for all values of the porosity parameter.

The effects of elastico-viscous, cross-viscous and porosity on the velocity profiles are illustrated graphically for the cases when (a) the disks rotate in the same sense, (b) opposite sense and (c) one disk rotates and the other is at rest.

The constitutive equation for a Second-order fluid as suggested by Coleman and Noll[7] can be written as

$$T_{ij} = -p\delta_{ij} + 2\mu_1 d_{ij} + 2\mu_2 e_{ij} + 4\mu_3 d_i^{\alpha} d_{\alpha j}$$

where

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \text{ and } e_{ij} = \frac{1}{2}(a_{i,j} + a_{j,i} + 2\, v^m_{,i}\, v_{m,j}), \tag{1}$$

T_{ij} is the stress tensor, P is the pressure, δ_{ij} is the K-roneckar delta, a_i and v_i are the acceleration and velocity vectors respectively and μ_1, μ_2, μ_3 are the material constants, known as Newtonian-viscosity, elastico-viscosity and cross-viscosity respectively.

The equations of motion and continuity for steady motion in the absence of body forces are respectively given by

$$\rho\, v^j\, v_{i,j} = T^j_{i,j} \tag{2}$$

and

$$v^i, i = 0 \tag{3}$$

Where ρ is the density of fluid and comma denotes covariant differentiation.

2. Formulation of the Problem

We consider the motion of an incompressible Second-order fluid whose rheological behaviour characterised by the constitutive Eqn. (1), confined between two coaxial rotating porous disks. The disk coinciding with the plane $z = 0$ is rotating with an angular velocity Ω while the other disk coinciding with $z = h$, with an angular velocity $s\,\Omega$ about an axis $r = 0$ perpendicular to their own planes.

We assume the components of velocity u, v w along the axes of reference in a cylindrical polar system of co-ordinates (r, θ, z). Thus the boundary conditions can be taken as :

$$z \quad 0,\ u = 0,\ u = r\,\Omega,\ w = \quad w_0,$$

$$z = h,\ u = 0,\ u = r\,s\,\Omega,\ w = -w_0. \tag{4}$$

Following Batchelor[2], the non-dimensional from of the velocity components and that of pressure[8] are assumed as follows :

$$= r\Omega F(\xi), \quad = r\Omega G(\xi), \; w = \sqrt{\nu_1 \Omega}\, H(\xi), \tag{5}$$

and

$$p = \mu_1 \Omega \left[-p_1(\xi) + \left(\frac{r^2}{d^2}\right) \left\{ (2T \quad K)(F'^2 + G'^2) + \lambda \right\} \right] \tag{6}$$

where $\xi = \dfrac{z}{h}$, $0 < \xi < 1$, $T = \dfrac{\Omega \nu_2}{\nu_1}$, $K = \dfrac{\Omega \nu_3}{\nu_1}$, $\nu_n = \dfrac{\mu_n}{\rho}$ $(n = 1, 2, 3)$,

λ being a constant and depends on Reynolds number R defined as

$$R = \frac{\Omega h^2}{\nu_1} \tag{7}$$

and prime denotes differentiation with respect to ξ.

Making use of Eqns (1) to (6), the equations of motion and equation of continuity provide

$$\frac{F''}{R} = F^2 + \frac{HF'}{\sqrt{R}} \quad G^2 + \frac{K}{\sqrt{R}} \Big[(1+2\alpha) F'^2 + (3+4\alpha) G'^2 + 2FF''$$

$$- \frac{\alpha H F'''}{\sqrt{R}} \Big] + \frac{\lambda}{R} \tag{8}$$

$$\frac{G''}{R} = 2FG + \frac{HG'}{\sqrt{R}} - \frac{K}{R} \left[2(1+\alpha) F' G' - 2F G'' + \frac{\alpha H G''}{R} \right] \tag{9}$$

$$p_1' = \frac{H''}{\sqrt{R}} - HH' + K \left[4(7+11\alpha) FF' \quad \frac{2\alpha H F''}{\sqrt{R}} \right] \tag{10}$$

and

$$2F + \frac{H'}{\sqrt{R}} = 0 \tag{11}$$

where $\alpha = T/K$.

The boundary conditions (4) now become

$$\begin{aligned} \xi = 0 &: F(\xi) = 0,\; G(\xi) = 1,\; H(\xi) = A, \\ \xi = 1 &: F(\xi) = 0,\; G(\xi) = s,\; H(\xi) = A \end{aligned} \tag{12}$$

$\sqrt{\ }$

3. Solution of the Problem

To obtain the solution of the Eqns (8) to (11) for small values of Reynolds number, the unknown function in these equations can be expanded in ascending powers of R.

Let us assume

$$\begin{aligned} F(\xi) &= R\{R\,F_1(\xi) + R^2\,F_2(\xi) + R^3\,F_3(\xi) + \dots\}, \\ G(\xi) &= \sqrt{R}\{G_0(\xi) + RG_1(\xi) + R^2\,G_2(\xi) + \dots\}, \\ H(\xi) &= A + R^2\{RH_1(\xi) + R^2\,H_2(\xi) + R^3\,H_3(\xi) + \dots\}, \\ \lambda &= R\{R\,\lambda_1 + R^2\,\lambda_2 + R^3\,\lambda_3 + \dots\}. \end{aligned} \tag{13}$$

Substituting Eqn. (13) in the Eqns. (8) to (11) and equating the coefficients of the various powers of R. The following sets of differential equations are obtained

Set I :

$$\left.\begin{aligned} G_0' &= 0, \quad G_1' = 0, \\ F_1''' &= -\,G_0^2 + 2K(3+u\alpha)\,G_0'\,G_2' + \lambda_l, \\ H_1' &= -\,2F_1 \end{aligned}\right\} \tag{14}$$

Set II :

$$\left.\begin{aligned} F_2''' &= -\,2\,G_0\,G_1 + K\left[(3 + u\,\alpha)\,(G_1'^2 + 2\,G_0'\,G_2')\right] + \lambda_2, \\ G_2'' &= -\,K\left[2\,(1 + \alpha)\,F_1'\,G_0' - 2\,F_1'\,G_0''\right], \\ H_2' &= -\,2\,F_2. \end{aligned}\right\} \tag{15}$$

Set III :

$$\left.\begin{aligned} F_3''' &= -\,G_1^2 - 2G_0\,G_1 + K\Big[(1+2\alpha)\,F_1'^2 + (3+4\alpha)\,(2\,G_1'\,G_2' \\ &\qquad + 2\,G_0'\,G_3') + 2\,F_1\,F_1'\Big] + \lambda_3, \\ G_3'' &= 2\,F_1\,G_0 - K\Big[(1+\alpha)\,(2\,F_1'\,G_1' + 2\,F_2'\,G_0') \\ &\qquad - 2\,(F_2'\,G_0'' + F_1'\,G_1'')\Big] \\ H_3' &= -\,2\,F_3 \end{aligned}\right\} \tag{16}$$

and so on

The boundary conditions (12) reduce to

$$\begin{aligned} \xi = 0 &: F_n(0) = 0,\ G_0(0) = 1,\ G_n(0) = 0,\ H_n(0) = 0, \\ \xi = 1 &: F_n(0), = 0,\ G_0(1) = s,\ G_n(1) = 0,\ H_n(1) = 0 \end{aligned} \tag{17}$$

where $n = 1, 2, 3$

Using Eqn. (17), the solution of the above mentioned sets of differential equations for the expressions F, G, H and λ have been obtained. For $A = 0$, we get the results for the flow of a Second-order fluid between two rotating disks. For $K = 0$ and $K = A = 0$, the expression (14) is in agreement with those obtained by Gaur[6] and Lance and Rogers[5] respectively.

It has been observed after choosing the series (13) of this type that there is no effect of porosity on radial and transverse components while the effect of porosity has been occured only in the axial component. The behaviour of radial, transverses and axial components has been computed, for the various values of $s = 0.5, 0.0 - 0.5, -1.0$; $K = 0.01, 0.1$ and depicted graphically only for $R = 0.5$.

4. Radial Velocity

Fig. (1) depicts the radial velocity profiles for $K = 0.01, 0.1$ and $s = 0, 0.5$ i.e. when the lower disk rotates and the upper disk is at rest or when the angular velocity of the upper disk is half of that of the lower disk and both rotate in the same sence. It is observed that the fluid moves radially away from the axis of rotation near the lower (faster) disk, while near the upper disk it moves towards the axis. It is also seen that the radial velocity increases with an increase in K for fixed values of s, while for fixed values of K it decreases near the lower disk and increases near the upper disk with an increase in s. It can also be noted that there are always three planes between the two disks where the radial velocity vanishes.

In this way similar is the behaviour for $s = 0.5$. However the behaviour is remarkably different for $s = -1.0$ i.e. when the disks rotate in the opposite sense with equal angular velocities. It is observed from Fig (2) that the flow is radially outward near both the disks. The radial velocity increases with K in this case also but there

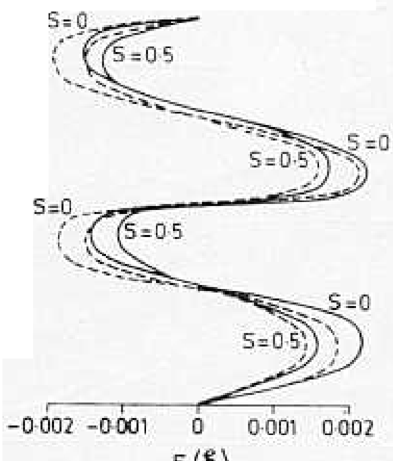

Figure 1. Variation of radial velocity with K and S for poly-iso-butylene cetane-type solution.

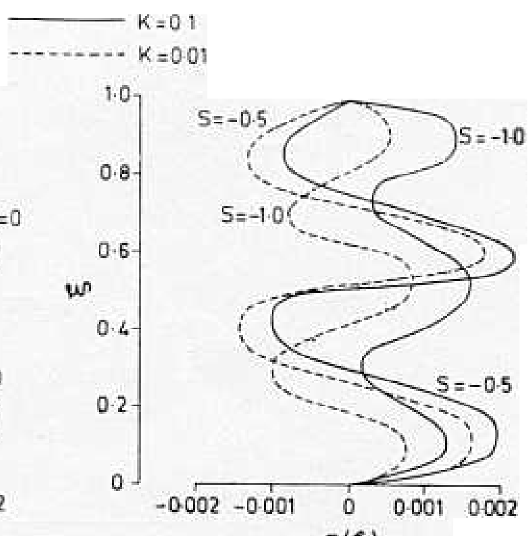

Figure 2. Variation of radial velocity with K and S for poly-iso-butylene cetane-type solution.

are four planes between the disks where the velocity vanishes for $K = 0.01$ while for $K = 0.1$, there is no plane where the radial velocity vanishes.

5. Transverse Velocity

Transverse velocity profiles have been shown in Figs. (3) and (4) for different values of K and s. In the case $s \geqslant 0$, the transverse velocity increases with an increase in K throughout the region for fixed values of s and also when s increases from 0 to 0.5

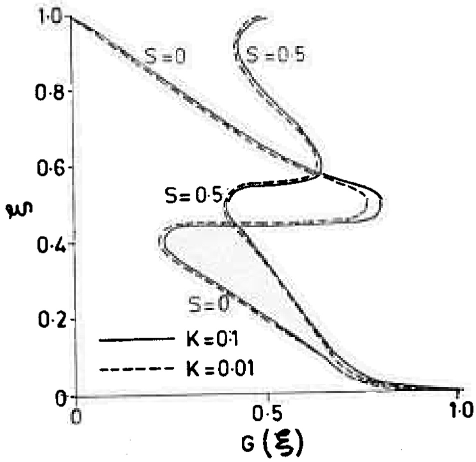

Figure 3. Variation of transverse velocity with K and S for poly-iso-butylene cetane-type solution.

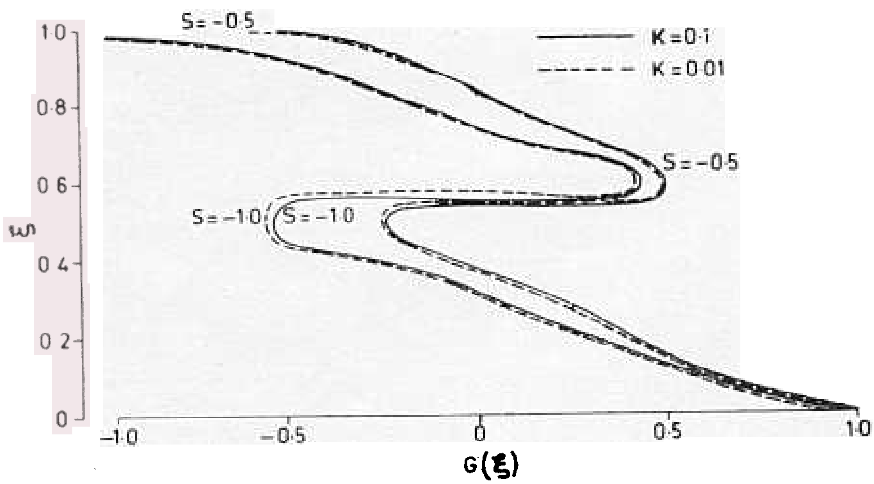

Figure 4. Variation of transverse velocity with K and S for poly-iso-butylene

for fixed values of *K*. On the other hand it increases with an increase in *K* and decreases, when *s* changes from −0.5 to −1.0.

6. Axial Velocity

Fig (5) and (6) show the variation of the axial velocity. It is seen that the axial velocity increases near the lower disk and decreases near the upper disk. It also

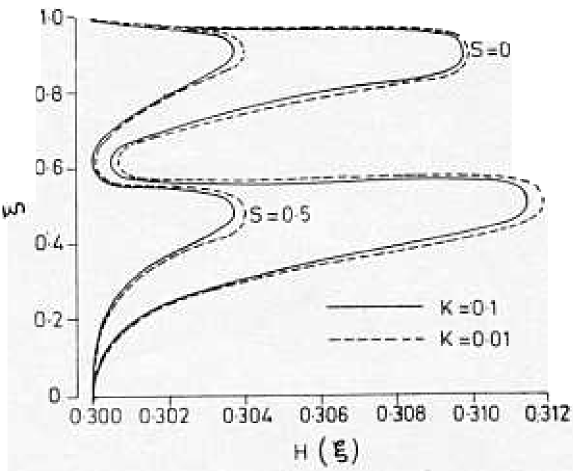

Figure 5. Variation of axial velocity with *K* and *S* for fixed A (=0.3) for poly-iso-butylene cetane-type solution.

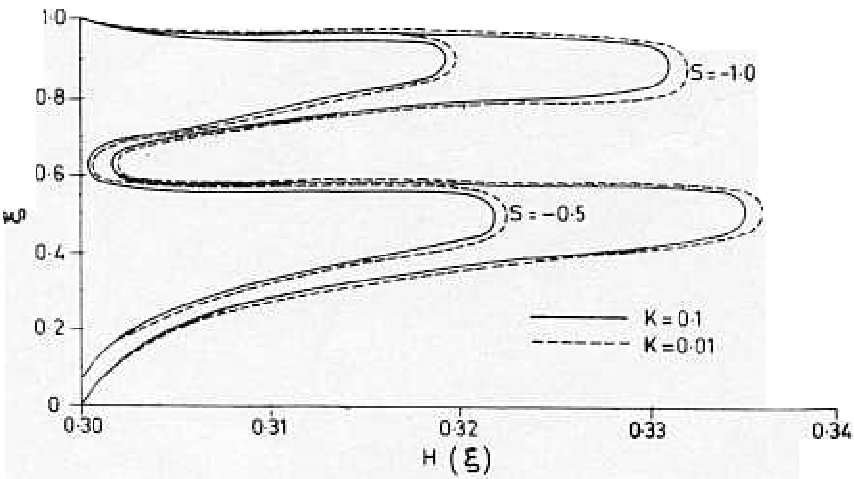

Figure 6. Variation of axial velocity with *K* and *S* for fixed A(=0.3) for poly-iso-butylene cetane-type solution.

decreases with an increase in K or with a positive or a negative increase in s throughout the region. The pattern for positive and negative values of s is the same except that the axial velocity has a little more value for $s < 0$. It is also evident that with an increase in the porosity parameter A, the axial velocity decreases. Thus the effect of porosity is to stabilize the flow.

The case of particular interest is, when $s = 1.0$, i.e. when both the disks rotate in the same sense with the same angular velocity. In this case the solutions of the Eqns. (8) to (11), throughout the region between the two disks, are

$$F(\xi) = 0, \; G(\xi) = 1, \; H(\xi) = A.$$

Thus in this case the flow between the two disks is helical.

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