THE NON-UNIFORM ROTATION OF A NON-NEWTONIAN LIQUID FILLED IN BETWEEN TWO COAXIAL CYLINDERS OF INFINITE LENGTH

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ABSTRACT

The non-uniform rotation of a non-Newtonian, incompressible liquid contained between two co-axial cylinders of infinite length is considered. Initially, the outer cylinder is rotating with constant angular velocity. When the inner cylinder is at rest, the liquid is supposed to be in steady motion but, when it is given an impulsive twist, it also begins to rotate with a constant angular velocity and the motion of the liquid becomes unsteady. The various states of motion of the liquid are discussed.

Introduction

D. D. Mallick[1] has discussed the non-uniform rotation of a homogeneous, incompressible, viscous liquid extending to infinity in the presence of an infinitely long circular cylinder rotating with constant angular velocity about its axis. The inner cylinder is initially fixed and the outer rotating with constant angular velocity. When the steady state is attained the inner cylinder is given an impulsive twist, the angular velocity of the outer remaining unaltered. In our case it is found that of the flow invariants K_1, K_2, K_3 (Rivlin[2] $K_1=0$, $K_3=0$ so that the coefficients Θ and ψ of shear and cross viscosity respectively are functions of K_2. The coefficient of cross-viscosity ψ occurs only in the expression for the pressure and consequently presents no difficulty. The coefficient of shear viscosity Θ appears in the equation for determining the velocity and necessitates an assumption as to its value for the determination of the velocity. We have assumed Θ to be equal to μr^n where μ is a constant both when the motion is steady and when it is not, as is the case after the impulsive twist has been given to the inner cylinder. The assumption in the unsteady case when K_2 being a function of r alone is a function of both r and the time t is justified in view of the fact that the invariance of K_2 is with respect to a transformation of the axes of reference, so that even when K_2 is a function of Θ and t, the basic stress strain velocity relations for a non-Newtonian liquid remain valid. Equations of motion are set up by taking $\Theta = \mu r^n$. These equations are then integrated to give us the solution of the problem for a particular case when $n=2$. It can be easily seen that Mallick's problem forms a particular case of the present problem. The frictional couples on the two cylinders per unit of their lengths have been calculated.

Formulation of the problem

An incompressible non-Newtonian liquid is contained between two coaxial circular cylinders of radii a and b ($a < b$). The axis of z lies along the common axis of the cylinders. The cylinders are of infinite length. Initially the outer cylinder is rotating with a constant angular velocity Ω_2 and the inner is at rest so that the liquid can be supposed to be in a steady state of motion. Now an impulsive twist is given to the inner cylinder such that it also begins to rotate with a constant angular velocity Ω_1, and the motion of the liquid becomes unsteady.

The incompressible non-Newtonian liquid considered here is that for which the stress tensor t_{ik} is related to the rate of strain tensor $e_{ik}\left[= \frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} + \frac{\partial u_i}{\partial x_k}\right) \text{ in Cartesians}\right]$ by

$$\left.\begin{aligned} t_{ik} &= 2\,\Theta\, e_{ik} + 2\,\psi\, e_{ij}\, e_{jk} - p\delta_{ik} \\ e_{jj} &= 0 \end{aligned}\right\} \quad \ldots \quad (2\cdot1)$$

with the usual summation convention.

The equation of motion is

$$\rho\left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k}\right) = \rho X_i + \frac{\partial t_{ik}}{\partial x_k} \quad \ldots \quad (2\cdot2)$$

where Xi is the extraneous force and ρ the density of the liquid.

If v_r, v_θ, v_z are the velocity components in the directions of r, θ, and z, then for two dimensional motion v_z is zero. Also because of symmetry we may assume v_r and v_θ to be independent of θ.

Writing $u_1=u$, $u_2=v$, $u_3=w$ and $x_1=x$, $x_2=y$, $x_3=z$ and applying the transformations

$$\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

and

$$u = v_r\cos\theta - v_\theta\sin\theta$$

$$v = v\sin\theta + v_\theta\cos\theta$$

$$w = v_z$$

in the equation of continuity, we have

$$\frac{\partial}{\partial r}(r\,v_r) = 0. \quad (2\cdot3)$$

On integration, (2·3) gives rv_r = constant, but $v_r = 0$ at $r=a$ and hence v_r must be zero everywhere.

Equation (2·2) gives

$$\rho\frac{\partial v_\theta}{\partial t} = \Theta\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2}\right) + \frac{\partial\Theta}{\partial r}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)$$

and $$\rho\frac{v^2_\theta}{r} = \frac{\partial p}{\partial r} - \frac{1}{2}\frac{\partial}{\partial r}\left\{\psi\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2\right\}$$

the extraneous forces being assumed to be absent. We replace v_θ by Ω for convenience. We thus have

$$\rho\frac{\partial\Omega}{\partial t} = \Theta\left(\frac{\partial^2\Omega}{\partial r^2} + \frac{1}{r}\frac{\partial\Omega}{\partial r} - \frac{\Omega}{r^2}\right) + \frac{\partial\Theta}{\partial r}\left(\frac{\partial\Omega}{\partial r} - \frac{\Omega}{r}\right) \quad \ldots \quad (2\cdot4)$$

$$\text{and } \rho \frac{\Omega^2}{r} = \frac{\partial p}{\partial r} - \frac{1}{2}\frac{\partial}{\partial r}\left\{\psi\left(\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r}\right)^2\right\} \quad \text{(2·5)}$$

Θ and ψ are functions of K_1, K_2, and K_3. In the present case we find that $K_1 = K_3 = 0$ and K_2 is $-\frac{1}{4}\left(\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r}\right)^2$.

Initially when the inner cylinder is at rest and the liquid is rotating in a steady state, from (2·4) we have

$$\Theta\left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r^2}\right) + \frac{\partial \Theta}{\partial r}\left(\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r}\right) = 0 \quad \text{(2·6)}$$

with the boundary conditions

$$\left.\begin{aligned} \Omega &= 0 \quad \text{at} \quad r = a \\ \Omega &= b\Omega_2 \text{ at} \quad r = b \end{aligned}\right\} \quad \text{(2·7)}$$

Integrating (2·6) by taking $\Theta = \mu r^n$ and making use of the boundary conditions (2·7), we have

$$\Omega = \frac{b^{n+2}\,\Omega_2}{b^{n+2} - a^{n+2}}\, r\left(1 - \frac{a^{n+2}}{r^{n+2}}\right)$$

This agrees with Rivlin's result for flow between rotating infinite cylinders when Θ is given our value (Rivlin[3]).

Again, when an impulsive twist has been given to the inner cylinder and it begins to rotate with a constant angular velocity Ω_1, the equation of motion (2·4) with this value of Θ becomes

$$\frac{\rho}{\mu}\frac{\partial \Omega}{\partial t} = r^n \frac{\partial^2 \Omega}{\partial r^2} + (n+1)\, r^{n-1}\frac{\partial \Omega}{\partial r} - (n+1)\, r^{n-2}\,\Omega \quad \text{(2·8)}$$

and the boundary conditions now are

$$\left.\begin{aligned} &\Omega = 0 && \text{at} \quad r = a && \text{for} \quad t = 0 \\ &\Omega = a\,\Omega_1 && \text{at} \quad r = a && \text{for} \quad t > 0 \\ &\Omega = b\,\Omega_2 && \text{at} \quad r = b && \text{for} \quad t \geqslant 0 \\ &\Omega = \frac{b^{n+2}\,\Omega_2}{b^{n+2} - a^{n+2}}\, r\left(1 - \frac{a^{n+2}}{r^{n+2}}\right) && \text{for any } r \text{ at } t = 0 \end{aligned}\right\} \quad \text{(2·9)}$$

Applying Laplace transform

$$\bar{\Omega}(r, s) = \int_0^{\infty} e^{-st}\, \Omega(r, t)\, dt$$

in (2·8), we have

$$r^n \frac{d^2 \bar{\Omega}}{dr^2} + (n+1)\, r^{n-1} \frac{d \bar{\Omega}}{d r} - \left\{ (n+1)\, r^{n-2} + s/\nu \right\} \bar{\Omega}$$

$$= \frac{1}{\nu} \frac{b^{n+2}\, \Omega_2}{(a^{n+2} - b^{n+2})}\, r \left(1 - \frac{a^{n+2}}{r^{n+2}}\right) \quad .. \quad (2\cdot10)$$

We have made use of boundary conditions (2·9) in obtaining the equation (2·10). In equation (2·10) $\nu = \mu/\rho$ may be compared with kinematic coefficient of viscosity when $n=0$. For $n = 0$ the equation reduces to Mallick's equation.

Solution of the problem

In what follows we discuss a particular solution of equation (2·10) when $n=2$.

Equation (2·10) now becomes

$$r^2 \frac{d^2 \bar{\Omega}}{dr^2} + 3r \frac{d \bar{\Omega}}{dr} - \left(3 + \frac{s}{\nu}\right) \bar{\Omega} = \frac{1}{\nu} \frac{b^4\, \Omega_2}{(a^4 - b^4)}\, r \times$$

$$\left(1 - \frac{a^4}{r^4}\right) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3\cdot1)$$

Solving equation (3·1) for $\bar{\Omega}$, we get

$$\bar{\Omega} = \frac{\Omega_2\, b}{s} \left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \frac{1}{r} \left(C_1\, r^{\sqrt{4 + s/\nu}}\right.$$

$$\left. + C_2\, r^{\sqrt{4 + s/\nu}} \right) \quad .. \quad .. \quad .. \quad .. \quad (3\cdot2)$$

Making use of the boundary conditions (2·9) we find

$$C_1 = \frac{\Omega_1\, a^2\, a^{\sqrt{4 + s/\nu}}}{s\left(a^{2\sqrt{4 + s/\nu}} - b^{2\sqrt{4 + s/\nu}} \right)},$$

and

$$C_2 = \frac{\Omega_1\, a^2\, b^{2\sqrt{4 + s/\nu}}\, a^{\sqrt{4 + s/\nu}}}{s\left(b^{2\sqrt{4 + s/\nu}} - a^{2\sqrt{4 + s/\nu}} \right)}.$$

Substituting the values of C_1 and C_2 in (3·2), we get

$$\bar{\Omega} = \frac{\Omega_2\, b}{s} \left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \frac{\Omega_1\, a^2}{rs} \times$$

$$\frac{\sinh\left\{ (4 + s/\nu)^{\frac{1}{2}} \log (r/b) \right\}}{\sinh\left\{ (4 + s/\nu)^{\frac{1}{2}} \log (a/b) \right\}} \quad .. \quad (3\cdot3)$$

The value of Ω is obtained by inverting the Laplace transform, which gives

$$\Omega = \Omega_2 b \left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \frac{\Omega_1 a^2}{r} \frac{1}{2\pi i} \times$$

$$\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds \quad .. \quad (3\cdot4)$$

We now evaluate the integral appearing in (3·4).

For this we make use of Jordan's Lemma[4], which states that if $\bar{f}(s)$ is analytic and converges uniformly to zero as s increases indefinitely, then for $t > 0$

$$\lim_{R \to \infty} \int e^{st} \bar{f}(s)\, ds = 0$$

where R is the radius of the semicircle which forms the contour of integration.

The function

$$\frac{1}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}}, \quad b \geqslant r \geqslant a$$

satisfies the conditions of Jordan's Lemma. We take our contour to be an infinite semicircle bounded by the straight line $x=c$, which runs from $-\infty$ to $+\infty$.

Integrating along this contour,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds$$

$$+ \frac{1}{2\pi i} \int \frac{e^{st}}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(\gamma/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds$$

= Sum of the residues at the poles inside the contour.

But by Jordan's Lemma

$$\frac{1}{2\pi i}\int \frac{e^{st}}{s} \cdot \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds = 0.$$

Thus we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds$$

= Sum of the residues at the poles inside the contour, namely at

$$s = 0, -4\nu - \frac{m^2\pi^2\nu}{\log^2 a/b}$$

$$m(=1, 2, 3, \ldots.)$$

or

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{st}}{s} \frac{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(r/b)\right\}}{\sinh\left\{(4+s/\nu)^{\frac{1}{2}} \log(a/b)\right\}} ds$$

$$= \frac{a^2}{r^2} \; \frac{b^4 - r^4}{b^4 - a^4} + \frac{2}{\pi} e^{-4\nu t} \sum_{m=1}^{\infty} \frac{(-1)^m m e^{-\frac{4m^2}{B^2}\nu t}}{m^2 + B^2} \sin\left(\frac{2m}{B} \log r/b\right)$$

where $B = \frac{2}{\pi} \log a/b.$

Substituting the value of the integral in (3·4),

$$\Omega = b\Omega_2\left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \frac{\Omega_1 a^2}{r}\left\{\frac{a^2}{r^2} \; \frac{b^4 - r^4}{b^4 - a^4} + \frac{2}{\pi} e^{-4t\nu} \times \sum_{m=1}^{\infty} \frac{(-1)^m m e^{-\frac{4m^2}{B^2}\nu t}}{m^2 + B^2} \sin\left(\frac{2m}{B} \log r/b\right)\right\}$$

or

$$\Omega = b\Omega_2\left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \frac{\Omega_1 a^2}{r}\left\{\frac{a^2}{r^2} \cdot \frac{b^4 - r^4}{b^4 - a^4} - \frac{2}{\pi} e^{-4t\nu} \times \sum_{m=1}^{\infty} \frac{m e^{-\frac{4m^2}{B^2}\nu t}}{m^2 + B^2} \sin\left(\frac{2m}{B} \log a/r\right)\right\} \quad \ldots (3.5)$$

Frictional couples

In this section we find the frictional couples on the two cylinders per units of their lengths.

If F_1 and F_2 denote the frictional couples at the inner and outer cylinder per unit of their lengths, we have

$$F_1 = \int_0^{2\pi} (t_{r\theta})_{r=a} \, a^2 d\theta$$

Substituting the value of $t_{r\theta}$ and integrating, we get

$$F_1 = 8\pi\mu a^4 \left\{ \frac{b^4(\Omega_1 - \Omega_2)}{a^4 - b^4} + \frac{\Omega_1}{B\pi} \sum_{m=1}^{\infty} \frac{m^2 e^{-\left(\frac{4m^2}{B^2}+4\right)t\nu}}{m^2 + B^2} \right\} \quad \ldots \quad (4\cdot1)$$

Further

$$F_2 = \int_0^{2\pi} (t_{r\theta})_{r=b} \, b^2 d\theta$$

or

$$F_2 = 8\pi\mu b^4 \left\{ \frac{a^4(\Omega_1 - \Omega_2)}{a^4 - b^4} + \frac{\Omega_1}{B\pi} \frac{a^2}{b^2} \sum_{m=1}^{\infty} \frac{(-1)^m m^2 e^{-\left(\frac{4m^2}{B^2}+4\right)\nu t}}{m^2 + B^2} \right\} \quad \ldots \quad (4\cdot2)$$

When $t \to \infty$ we find that the motion again becomes steady and the value of the velocity at that time is

$$\Omega_{t=\infty} = \Omega_2 b \left(\frac{b}{r}\right)^3 \frac{a^4 - r^4}{a^4 - b^4} + \Omega_1 a \left(\frac{a}{r}\right)^3 \cdot \frac{b^4 - r^4}{b^4 - a^4} \quad \ldots \quad (4\cdot3)$$

The values of F_1 and F_2 as $t \to \infty$ are

$$\left.\begin{aligned} F_{1\,t=\infty} &= \frac{8\pi\mu a^4 b^4(\Omega_1 - \Omega_2)}{a^4 - b^4} \\ F_{2\,t=\infty} &= \frac{8\pi\mu a^4 b^4(\Omega_1 - \Omega_2)}{a^4 - b^4} \end{aligned}\right\} \quad \ldots \quad (4\cdot4)$$

Pressure at any point of the liquid

Integrating equation (2·5),

$$p = \rho \int \frac{\Omega^2}{r}\, dr + \tfrac{1}{2}\left\{ \psi \left(\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r} \right)^2 \right\} + M$$

where M is the constant of integration.

An Identity

The last boundary condition in (2·9) when $n = 2$ is

$$\Omega = \Omega_2\, b \left(\frac{b}{r} \right)^3 \cdot \frac{a^4 - r^4}{a^4 - b^4}$$

Comparing this value of Ω with the value of Ω obtained in (3·5) at $t = 0$, we have the identity

$$\frac{a^2}{r^2} \cdot \frac{b^4 - r^4}{b^4 - a^4} = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{m}{B^2 + m^2}\, sin \left(\frac{2m}{B}\, log\, a/r \right)$$

which is satisfied under the condition

$$0 < \pi\, \frac{log\ a/r}{log\ a/b} < 2\pi$$

We know that this is always true in our case.

Discussion

The value of velocity as given in (3·5) is

$$\Omega = b\, \Omega_2 \left(\frac{b}{r} \right)^3 \cdot \frac{a^4 - r^4}{a^4 - b^4} + \Omega_1\, a \left(\frac{a}{r} \right)^3 \cdot \frac{b^4 - r^4}{b^4 - a^4}$$

$$- \frac{\Omega_1\, a^2}{r} \cdot \frac{2}{\pi}\, e^{-4t\nu} \sum_{m=1}^{\infty} \frac{m\, e^{-\frac{4m^2}{B^2} t\nu}}{m^2 + B^2}\, sin \left(\frac{2m}{B}\, log\ a/r \right).$$

This we may write as

$$\Omega = I + A + T$$

where I is the initial velocity, A is the part added to I when the steady state is attained and T is the transient part of velocity, which is negative except for very small values of νt. This transient part T gradually decreases with time, but for a given t it oscillates with r.

In particular if we take $\nu t = \cdot 1$, $a = 1$, $b = 2$, then T is

$$T = - \frac{.4267}{r} \sum_{m=1}^{\infty} \frac{m\, e^{-22\cdot 7272 m^2}}{m^2 + \cdot 0176}\, sin \left(m\pi\, \frac{log\ r}{log 2} \right).$$

In this series the first term is the dominating term.
This gives

$$T = -\frac{\cdot 4267}{r}\,\frac{e^{-22\cdot 7272}}{1\cdot 0176}\,\sin\left(\pi\,\frac{\log r}{\log 2}\right).$$

This value of T increases with r as r increases from 1 to $\sqrt{2}$ and then it begins to decrease as r increases from $\sqrt{2}$ to 2.

From the above discussion one may see that for a fixed t, not very small, unsteadiness increases with r, takes a maximum value and ultimately dies out as r approaches its maximum value.

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