SOME CASES OF VERTICAL ASCENT OF A ROCKET

by

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ABSTRACT

The differential equations of motion of vertically ascending rocket are integrated in closed form in terms of Bessel functions. During burning the drag co-efficient and acceleration due to gravity are assumed to be constant. Four different cases of motion are treated under different assumptions regarding the law of mass-variations of the rocket.

Introduction

We consider in this paper some cases of the vertical ascent of a rocket under different assumptions regarding the law of mass-variation of the rocket. During burning the drag-coefficient and the acceleration due to gravity are assumed to be constant. In each of the cases considered it turns out that the equation of the motion can be integrated in closed form in terms of Bessel functions. The four cases considered here may be described as follows :—

Let M be the mass of the rocket at any time during the burning, v its velocity (upwards) at that instant. Then the equation of motion is

$$\frac{dv}{dt} = -\frac{c\dot{M}}{M} - g - \frac{K v^2}{2M} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Here g is the acceleration due to gravity, K is the drag coefficient and c is the exhaust velocity. Regarding M we consider two assumptions :

(A) $\dot{M} = \text{constant} = -a$

(B) $\dfrac{\dot{M}}{M} = \text{constant} = -a$

In regard to c we consider the two cases:

(1) $c = \text{constant} = c_o$

(2) $c = c_o + v$

We thus get the following four cases for consideration

(1) Case A_1, $\dot{M} = \text{constant} = -a$
$c = \text{constant} = -c_o$

(2) Case B_1, $\dfrac{\dot{M}}{M} = \text{constant} = -a$

$c = \text{constant} = c_o$

(3) Case A_2, $\dot{M} = \text{constant} = -a$

$$c = c_0 + v$$

(4) Case B_2, $\frac{\dot{M}}{M}$ constant $= -a$

$$c = c_0 + v$$

It is shown that in each of these cases the equation of the motion may be integrated in terms of Bessel functions so as to yield an explicit expression for the velocity during the burning. The vertical distance travelled during the burning is given by the integral of this expression and in any particular case this integral may be evaluated by a simple numerical quadrature. After the charge is all burnt the motion will be described by equation (1) with the first term on the right omitted. This equation may be readily integrated to obtain the maximum height reached. For the three of the cases mentioned above we have also worked out a numerical example and calculated the all-burnt velocity as well as the maximum height reached. The results are shown in Table 1.

Case A_1 : Since here $\dot{M} = -a$

$$\therefore \quad M = M_0 - at$$

where M_0 is the initial mass of the rocket. The equation of motion will be

$$\frac{dv}{dt} = \frac{ac}{M_0 - at} - g - \frac{Kv^2}{2(M_0 - at)}$$

$$i.e. \quad \frac{dv}{dM} = -\frac{c}{M} + \frac{g}{a} + \frac{Kv^2}{2aM}$$

$$\text{or} \quad \frac{dv}{dM} = \frac{\gamma v^2}{M} - \frac{c}{M} + \beta \qquad \ldots \quad (2)$$

where

$$\beta = g/a$$

$$\gamma = \frac{K}{2a}$$

$$\text{Let} \quad v = -\frac{M}{\gamma u} \cdot \frac{du}{dM}$$

Then equation (2) goes over into

$$M^2 \frac{d^2u}{dM^2} + M\frac{du}{dM} + \gamma(\beta M - c)u = 0 \qquad \ldots \quad (3)$$

If we write

$$x = 2\sqrt{\beta \gamma M}$$

then (3) becomes

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + (x^2 - \delta^2) = 0 \qquad \ldots \quad (4)$$

Where $\delta^2 = 4\,\gamma\, c$

The solution of this is

$$u = \lambda_1 J_\delta (2\sqrt{\beta\gamma M}) + \lambda_2 Y_\delta (2\sqrt{\beta\gamma M})$$

where λ_1, λ_2 are constants.

Then

$$v = -\frac{M}{\gamma u}\,\frac{du}{dM}$$

$$\text{i.e.}\quad v = -\frac{\sqrt{\beta\gamma M}}{\gamma}\left[\frac{J'_\delta (2\sqrt{\beta\gamma M}) + \lambda Y'_\delta (2\sqrt{\beta\gamma M})}{J_\delta (2\sqrt{\beta\gamma M}) + \lambda Y_\delta (2\sqrt{\beta\gamma M})}\right]$$

$$\text{with}\quad \lambda = \frac{\lambda_2}{\lambda_1} \qquad \ldots \qquad (5)$$

The constant λ is determined by the condition $v = 0, t = 0$

Thus

$$\lambda = -\left[\frac{J_{\delta-1}(2\sqrt{\beta\gamma M}) - J_{\delta+1}(2\sqrt{\beta\gamma M})}{Y_{\delta-1}(2\sqrt{\beta\gamma M}) - Y_{\delta+1}(2\sqrt{\beta\gamma M})}\right]_{M = M_\circ}$$

The velocity when the charge is all burnt is

$$v_B = -\frac{\sqrt{\beta\gamma M}}{\gamma}\left[\frac{J'_\delta (2\sqrt{\beta\gamma M_B}) + \lambda Y'_\delta (2\sqrt{\beta\gamma M_B})}{J_\delta (2\sqrt{\beta\gamma M_B}) + \lambda Y_\delta (2\sqrt{\beta\gamma M_B})}\right] \qquad (7$$

where M_B is the mass of the empty rocket. The vertical distance traversed up to the instant of " all burnt" is

$$s_1 = \int_\circ^{t_B} v.\, dt = \int_{M_\circ}^{M_B} v.\frac{dt}{dM}.dM$$

$$= \frac{1}{a}\int_{M_B}^{M_\circ} v.\, dM \qquad (8)$$

After the charge is all burnt the equation of motion is

$$v\frac{dv}{ds} = \frac{dv}{dt} = -g - \frac{K}{2M_B}v^2 \qquad \ldots \qquad (9)$$

whence the travel up to the maximum height is given by

$$s_2 = \int_0^{v_B} \frac{v\,dv}{g + \frac{K}{2M_B} v^2} = \frac{M_B}{K} \log \left(1 + \frac{K v^2_B}{2gM_B}\right)$$

The total height reached is then

$$H = s_1 + s_2$$

Case B_1 : Here

$$\frac{\dot{M}}{M} = -a$$

$$\therefore\ M = M_\circ e^{-at}$$

The equation of motion becomes

$$\frac{dv}{dt} = ac - g - \gamma\, e^{at}\ v^2$$

$$= \alpha - \gamma\, e^{at}\ v^2$$

where $\alpha = ac - g$

$$\gamma = \frac{K}{2M_\circ}$$

Let $v = \frac{1}{\gamma u e^{at}} \cdot \frac{du}{dt}$ . . . (12)

Then the equation (11) reduces to

$$\ddot{u} - a\dot{u} - \alpha\gamma e^{at}\, u = o$$

bstitute again

$$u = yx$$
$$x = e^{at/2}$$

Then the new equation in y and x will be

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (1 + A^2x^2)\, y = o$$

where $A^2 = \frac{4\alpha\gamma}{a^2}$

of which the solution is

$$y = \lambda_1 I_1(Ax) + \lambda_2 K_1(Ax)$$

Hence

$$u = e^{at/2}\left[\lambda_1 I_1\left(Ae^{at/2}\right) + \lambda_2 K_1\left(Ae^{at/2}\right)\right] \quad \ldots \quad \ldots \quad (13)$$

and from (12) we have

$$v = \frac{a}{2\gamma} e^{-at}\left[\frac{\left[I_1(Ae^{at/2}) + \lambda K_1(Ae^{at/2})\right] + Ae^{at/2}\left[I'_1(Ae^{at/2}) + \lambda k'_1(Ae^{at/2})\right]}{I_1(Ae^{at/2}) + \lambda K_1(Ae^{at/2})}\right] \quad (14)$$

where

$$\lambda = \frac{\lambda_2}{\lambda_1}$$

The constant is determined by the conditon $v=o$, $t=o$

so that

$$\lambda = -\left[\frac{I_1\left(Ae^{at/2}\right) + Ae^{at/2} I'_1\left(Ae^{at/2}\right)}{K_1\left(Ae^{at/2}\right) + Ae^{at/2} K_1'\left(Ae^{at/2}\right)}\right]_{t=o} \quad \ldots \quad \ldots \quad (15)$$

The velocity at all burnt is given by $v_B = \dfrac{a \cdot e^{-at_B}}{2\gamma} \times$

$$\left\{\frac{\left[I_1\left(Ae^{at/2}\right) + \lambda K_1\left(Ae^{at/2}\right)\right] + Ae^{at/2}\left[I'_1\left(Ae^{at/2}\right) + \lambda K'_1\left(Ae^{at/2}\right)\right]}{I_1\left(Ae^{at/2}\right) + \lambda K_1\left(Ae^{at/2}\right)}\right\}_{t=t_B} \quad \ldots \quad \ldots \quad (16)$$

The maximum vertical distance obtained as before is given by

$$H = s_1 + s_2$$

where

$$s_1 = \int_o^{t_B} v\, dt$$

and s_2 is obtained by (10) with v_B given by (16).

Case A_2 :

In this case we have

$$\dot{M} = -a$$

so that

$$M = M_o - at$$

and

$$c = c_o + v$$

Then the equation (1) becomes

$$\frac{dv}{dt} = \frac{ac_o}{M_o - at} + \frac{av}{M_o - at} - g - \frac{Kv^2}{2(M_o - at)}$$

or

$$\frac{dv}{dM} = \frac{\gamma v^2}{M} - \frac{v}{M} + \left(\frac{\beta M - c_o}{M}\right) \quad .. \quad .. \quad (17)$$

where $\beta = g/a$

$$\gamma = \frac{K}{2a}$$

Let

$$v = u + \frac{1}{2\gamma}$$

Then the above equation goes over to

$$\frac{du}{dM} = \frac{\gamma}{M} u^2 - \frac{c'}{M} + \beta \quad .. \quad .. \quad (18)$$

where $\beta = g/a$

$$\gamma = \frac{K}{2a}$$

$$c' = \left(c_o + \frac{1}{4\gamma}\right)$$

The equation (18) is the same as equation (2).

∴ Solution of this is

$$u = -\frac{\sqrt{\beta\gamma M}}{\gamma}\left[\frac{J'_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y'_\mu\,(2\sqrt{\beta\gamma M})}{J_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y_\mu\,(2\sqrt{\beta\gamma M})}\right]$$

Where $\mu^2 = 4\gamma c' = 4\gamma c_o + 1$

$$\therefore v = \frac{1}{2\gamma} - \frac{\sqrt{\beta\gamma M}}{\gamma}\left[\frac{J'_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y'_\mu(2\sqrt{\beta\gamma M})}{J_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y_\mu\,(2\sqrt{\beta\gamma M})}\right] \quad .. \quad (19)$$

The velocity at all burnt is

$$v_B = -\frac{1}{2\gamma}\left[\frac{2\sqrt{\beta\gamma M}\,[J'_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y'_\mu\,(2\sqrt{\beta\gamma M})] - [(J_\mu\, 2\sqrt{\beta\gamma M}) + \lambda Y_\mu\,(2\sqrt{\beta\gamma M})]}{J_\mu\,(2\sqrt{\beta\gamma M}) + \lambda Y_\mu\,(2\sqrt{\beta\gamma M})}\right]_{M = M_B}$$

$$.. \quad .. \quad .. \quad (20)$$

The maximum height is obtained as in the previous case, using the value of v_B given by (20).

Case B_2:

Here we have $\dot{M}/M = -a$

and $$c = c_o + v$$

So that the equation of motion becomes

$$\frac{dv}{dt} = a\left(c_o + v\right) - g - \frac{K}{2M} v^2$$

where $M = M_o e^{-at}$

$$\text{or} \quad \frac{dv}{dt} = \alpha + av - \gamma e^{at} v^2 \qquad \ldots \quad (21)$$

where $\alpha = ac_o - g$

$$\gamma = \frac{K}{2M_o}$$

$$\text{Put} \quad v = \frac{1}{u} \cdot \frac{1}{\gamma e^{at}} \cdot \frac{du}{dt} \qquad \ldots \quad (22)$$

Then the above equation will become

$$\ddot{u} - 2a\dot{u} - \alpha\gamma e^{at} u = o \qquad \ldots \quad (23)$$

Let again

$$u = yx^2$$

$$x = e^{at/2}$$

Then the equation (21) reduces to

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (4 + A^2x^2)y = o$$

$$\text{where } A^2 = \frac{4\alpha\gamma}{a^2} \qquad \ldots \quad (24)$$

of which the solution is

$$y = \lambda_1 I_2 (Ax) + \lambda_2 K_2 (Ax) \qquad \ldots \quad (25)$$

and hence

$$\therefore u = e^{at}\left[\lambda_1 I_2\left(Ae^{at/2}\right) + \lambda_1 K_2\left(Ae^{at/2}\right)\right]$$

$$\text{From } v = \frac{1}{u} \cdot \frac{1}{\gamma e^{at}} \cdot \frac{du}{dt}$$

we have

$$v = \frac{ae}{\gamma}^{-at} \left[\frac{I_2\left(Ae^{at/2}\right) + \lambda K_2\left(Ae^{at/2}\right) + \frac{Ae^{at/2}}{2}\left(I'_2(Ae^{at/2}) + \lambda K'_2(Ae^{at/2})\right)}{I_2(Ae^{at/2}) + \lambda K_2(Ae^{at/2})} \right] \quad \ldots \quad \ldots \quad (26)$$

Where $\lambda = \frac{\lambda_2}{\lambda_1}$.

From $v = o$, at $t = o$, we obtain

$$\lambda = -\left[\frac{I_2\left(Ae^{at/2}\right) + \frac{A}{2} e^{at/2} I'_2\left(Ae^{at/2}\right)}{K_2\left(Ae^{at/2}\right) + \frac{A}{2} e^{at/2} K'_2\left(Ae^{at/2}\right)} \right]_{t=o} \quad \ldots \quad \ldots \quad (27)$$

The all burnt velocity is

$$v_B = \frac{ae^{-at/B}}{\gamma} \left[\frac{I_2\left(Ae^{at/2}\right) + \lambda K_2\left(Ae^{at/2}\right) + \frac{A}{2} e^{at/2}\left(I'_2(Ae^{at/2}) + \lambda K'_2\left(Ae^{at/2}\right)\right)}{I_2\left(Ae^{at/2}\right) + \lambda K_2\left(Ae^{at/2}\right)} \right]_{t=t_B} \quad (28)$$

The maximum height is calculated as in the preceding case, using the value of v_B given by (28).

Some numerical results:

To illustrate the above formulæ, the following cases were calculated numerically.

	Initial mass (lb)	Final mass (lb)	Burning time (seconds)	c_o (ft/sec).	γ	All burnt velocity v_B (ft/sec)	Maximum height H (ft)
Case A_1	21	12	1·2	4550	$5{\cdot}5\times10^{-5}$	2180	12523
Case B_1	21	12	1·2	6400	$2{\cdot}13\times10^{-5}$	3532	17627
Case A_2	81·0	72·4 lb	0·89	6400	$31{\cdot}4\times10^{-5}$	1544	6743

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