AN EXACT ANALYTICAL SOLUTION OF THE EQUATIONS OF INTERNAL BALLISTICS FOR THE PRESSURE-INDEX LAW OF BURNING

by

J. N. Kapur and B. S. Jain

Delhi University, Delhi

ABSTRACT

In the present paper, an exact analytical solution of the equations of internal ballistics, for the specific case of a tubular charge has been given. This solution applies to some particular values of the pressure-index α greater than unity, and for these values, the function $G(\gamma, \alpha)$ of Clemmow has also been explicitly determined.

Introduction

The solution of the equations of internal ballistics for the pressure-index law of burning was first discussed by Clemmow[1, 2, 3] for the case when the pressure-index α was necessarily less than unity. Later he[4] extended his theory to the case when α could be greater than unity, as in the intervening period, the search for propellants possessing less erosive and flash-producing qualities than those in service use led to the discovery, by a greatly-improved closed-vessel technique, of propellants which burn according to pressure-laws for which the index α exceeds unity and may be as high as 1·25. Clemmow, in this report, confined himself to the *isothermal* model only and gave a series solution for $1 < \alpha < 3/2$ and an exact solution for $\alpha = 3/2$.

Recently Kapur[5] has given a number of alternative methods for solving the equations of internal ballistics for the pressure-index law of burning, but in the most general case, his methods, like Clemmows, depend on the solution of non-linear differential equations, though his equations apply equally to the cases α less than or greater than unity.

In the present paper, an exact analytical solution has been obtained for the *non-isothermal* model for particular values of the pressure-index α greater than unity for the specific case of a tubular propellant. In particular, tables have been given for $\alpha = \frac{7}{6}$, $\frac{6}{5}$ and $\frac{5}{4}$. The basic differential equation has also been integrated for the case $\alpha = 1$ to give explicitly the equation of the pressure-space curve for the case of finite shot-start pressure.

For zero shot-start pressure, the expression for the maximum pressure has been obtained and this enables us to tabulate the functions $G(\gamma, \alpha)$ of Clemmow tabulated earlier by him for $\gamma = 1$, $\alpha \leqslant 1$[4,7] for some values of α greater than unity.

The Basic Equations

Neglecting co-volume correction terms, the fundamental equations are (Clemmow[3], page 117),

$$z = \zeta\xi + \tfrac{1}{2}(\gamma - 1)\frac{\eta^2}{M} \quad \ldots \quad (1)$$

$$\eta\frac{d\eta}{d\xi} = M\zeta \quad \ldots \quad (2)$$

$$z = (1-f)(1+\theta f) \quad \ldots \quad (3)$$

$$\eta\frac{df}{d\xi} = -\zeta^{\alpha} \quad \ldots \quad (4)$$

Where the dimensionless variables ξ, η, ζ corresponding to shot-travel, velocity and pressure respectively, and the central ballistic parameter M are given by

$$\xi = 1 + \frac{x}{l} \quad \ldots \quad (5)$$

$$\eta = \frac{AD}{FC\beta}\left(\frac{FC}{Al}\right)^{1-\alpha} v \quad \ldots \quad (6)$$

$$\zeta = \frac{Al}{FC}\, p \quad \ldots \quad (7)$$

$$M = \frac{A^2D^2}{FC\beta^2 w_1}\left(\frac{FC}{Al}\right)^{2-2\alpha} \quad \ldots \quad (8)$$

For the specific case of a tubular propellant, Kapur[5], has deduced from equations (1)—(4), the following differential equations:

$$X\zeta\zeta'' + (\alpha - 2)X\zeta'^2 + (1 - \gamma\alpha + \gamma)\zeta\zeta' = -M\gamma^3\zeta^{5-2\alpha-\frac{1}{\gamma}} \quad \ldots \quad (9)$$

and

$$Y\xi\xi'' + (\gamma - 1 - \gamma\alpha)Y\xi'^2 + \alpha\xi\xi' = MY^{2-2\alpha}\,\xi^{2\gamma\alpha - 3\gamma + 3} \quad \ldots \quad (10)$$

where

$$Y = \zeta\xi^{\gamma}, \quad X = \zeta^{1/\gamma}\xi \quad \ldots \quad (11)$$

It has also been proved that

$$\frac{dY}{dz} = \xi^{\frac{\gamma-1}{\gamma}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

For finite shot-start pressure, the internal conditions for the integration of (9) and (10) are:

$$\xi = 1, \eta = 0, \zeta = \zeta_o, Y = \zeta_o, X = \zeta_o^{\frac{1}{\gamma}}$$

$$\frac{d\zeta}{dY} = 1, \frac{d\zeta}{dX} = \gamma\zeta_o^{\frac{\gamma-1}{\gamma}} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Integration of the Equations

We integrate (10) subject to (13). Let $\xi = Z^m$, then (10) becomes

$$mYZZ'' + m[(m-1) + m(\gamma - 1 - \gamma\alpha)]YZ'^2 + m\alpha ZZ'$$

$$= MY^{2-2\alpha} z^{2m\gamma\alpha - 3m\gamma + m + 2} \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Now we choose m so that the coefficient of Z'^2 vanishes *i.e.*

$$m = \frac{1}{\gamma(1-\alpha)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

then (14) becomes

$$YZ'' + \alpha Z' = M\gamma(1-\alpha)Y^{2-2\alpha} Z^{\frac{\gamma\alpha - 2\gamma + 1}{\gamma(1-\alpha)}} \quad \ldots \quad \ldots \quad (16)$$

The equation is atonce integrable if

$$\alpha = \frac{2\gamma - 1}{\gamma} \neq 1 \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

In this case (16) becomes

$$YZ'' + \alpha Z' = M\gamma(1-\alpha)Y^{2-2\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Integrating

$$Z = \xi^{\gamma(1-\alpha)} = A + BY^{1-\alpha} + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Y^{3-2\alpha}, \quad \ldots \quad (19)$$

where the constants A and B are determined from (13) as

$$A = 1 + \frac{M}{(2-\alpha)(3-2\alpha)}\zeta_o^{3-2\alpha}, \quad B = -\frac{M}{(2-\alpha)^2}\zeta_o^{2-\alpha} \quad \ldots \quad (20)$$

Also from (11) and (19)

$$\zeta^{\alpha-1} = AY^{\alpha-1} + B + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Y^{2-\alpha} \quad \ldots \quad (21)$$

From (12), (17), (19) and (21),

$$z - \zeta_\circ = \int_{\zeta_\circ}^{y} \xi^{1-\gamma}\, dY = \int_{\zeta_\circ}^{y} \xi^{\left(\frac{1-\alpha}{2-\alpha}\right)} dY$$

$$= \int_{\zeta_\circ}^{y} \left[A + BY^{1-\alpha} + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Y^{3-2\alpha}\right] d$$

or

$$z - \zeta_\circ = A(Y - \zeta_\circ) + \frac{B}{2-\alpha}\left(Y^{2-\alpha} - \zeta_\circ^{\;2-\alpha}\right) +$$

$$\frac{M(1-\alpha)\left(Y^{4-2\alpha} - \zeta_\circ^{\;4-2\alpha}\right)}{(2-\alpha)^2(3-2\alpha)(4-2\alpha)} \quad .. \quad .. \quad (22)$$

For maximum pressure $\frac{d\zeta}{dY} = 0$ and (21) gives

$$Y_1 = \left[\frac{A}{M}(2-\alpha)(3-2\alpha)\right]^{\frac{1}{3-2\alpha}} \quad .. \quad .. \quad (23)$$

and

$$\zeta_{max}^{\alpha-1} = A\left[\frac{A}{M}(2-\alpha)(3-2\alpha)\right]^{\left(\frac{\alpha-1}{3-2\alpha}\right)} + B + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} \times$$

$$\left[\frac{A}{M}(2-\alpha)(3-2\alpha)\right]^{\left(\frac{2-\alpha}{3-2\alpha}\right)} \quad .. \quad .. \quad (24)$$

For given values of α, $\zeta_\circ$ and M, (19), (21) and (22) enable us to tabulate ζ, ξ and z as functions of Y and this tabulation is to be continued till $z = 1$, when the shot-travel and the pressure at all-burnt will be determined.

Special Cases

The case $\alpha = 1$

In this case $m = \infty$ and the above solution fails if

$\gamma = 1 + \frac{1}{q}$ and we substitute $\xi = Z^q$, Eqn(10) becomes

$$YZZ'' - YZ'^2 + ZZ' - \frac{M}{q} Z = 0 \quad .. \quad .. \quad .. \quad (25)$$

putting $Y = e^{v}$, $Z = e^{v-w}$, we get

$$\frac{d^2w}{dv^2} = -\frac{M}{q}\, e^{w} \qquad \ldots \quad (26)$$

Integrating, using the internal conditions (26), we get after some simplification.

$$\xi^{\frac{1}{q}} = \frac{MY}{2c^2q}\left[\left(\frac{Y}{\zeta_\circ}\right)^{\frac{c}{2}}\sqrt{\frac{c-1}{c+1}} + \left(\frac{Y}{\zeta_\circ}\right)^{-\frac{c}{2}}\sqrt{\frac{c+1}{c-1}}\right]^2, \qquad \ldots \quad (27)$$

where

$$c^2 = 1 + \frac{2M}{q}\zeta_o \qquad \ldots \quad (28)$$

(27) gives the explicit equation of the pressure-space curve. It can also be deduced from the results obtained by Kapur[6].

The case $\alpha = \frac{3}{2}$.

In this case also the above solution has to be modified.
(18) becomes

$$Y^2Z'' + \tfrac{3}{2}YZ' = -M \qquad \ldots \quad (29)$$

Integrating and proceeding as previously we get

$$\zeta^{-1} = c + DY^{-\frac{1}{2}} - 2M \log Y \qquad \ldots \quad (30)$$

$$\zeta^{\frac{1}{2}} = cY^{\frac{1}{2}} + D - 2MY^{\frac{1}{2}} \log Y \qquad \ldots \quad (31)$$

$$\zeta^{\frac{1}{2}}_{\max} = ce^{\left(\frac{c}{4M} - 1\right)} + D - 2Me^{\left(\frac{c}{4M} - 1\right)}\left(\frac{c}{2M} - 2\right) \qquad \ldots \quad (32)$$

$$z - \zeta_\circ = c(Y - \zeta_\circ) + 2D(Y^{\frac{1}{2}} - \zeta_\circ^{\frac{1}{2}}) - 2M \times (Y \log Y - \zeta_\circ \log \zeta_\circ - Y + \zeta_\circ) \qquad \ldots \quad (33)$$

where

$$C = 2M \log \zeta_\circ + 4M + 1 \qquad \ldots \quad (34)$$

$$D = -4M\, \zeta_\circ^{\frac{1}{2}} \qquad \ldots \quad (35)$$

TABLES FOR PARTICULAR VALUES OF α

TABLE I

$$\alpha = \frac{7}{6}, \gamma = \frac{6}{5}, M = 1, \zeta_o = 0{\cdot}1.$$

Y	0·1	0·3	0·5	0·7	0·9	1·0	1·0976
z	0·1	0·297,40	0·486·62	0·666,94	0·838,59	0·921,26	1·00,000
ξ	1·0	1·17,57	1·48,66	1·89,84	2·43,27	2·75,66	3·11,69
ζ	0·1	0·247,03	0·310,71	0·324,37	0·309,70	0·296,17	0·280,53

TABLE II

$$\alpha = 6/5, \gamma = 5/4, M = 1, \zeta_o = 0{\cdot}1$$

Y	0·1	0·3	0·5	0·7	0·9	1·0	1·1	1·1456
z	0·1	0·296,33	0·481·39	0·654,55	0·816,33	0·893,15	0·967·34	1·00000
ξ	1·0	1·2005	1·5570	2·0388	2·6813	3·0802	3·5434	3·7791
ζ	0·1	0·238,73	0·287,49	0·287,33	0·262,30	0·245,06	0·226,26	0·217,42

TABLE III

$$\alpha = 5/4, \gamma = 4/3, M = 2, \zeta_o = 0{\cdot}1.$$

Y	0·1	0·3	0·5	0·7	0·9
z	0·1	0·288,22	0·441,72	0·560,04	0·645,71
ξ	1·0	2·29,17	3·21,51	7·63,25	23·14,46
ζ	0·1	0·099,294	0·105,368	0·046,581	0·013,645

Variation of Maximum Pressure

For zero shot-start pressure, (19), (21) and (22) become

$$\xi^{\left(\frac{1-\alpha}{2-\alpha}\right)} = 1 + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Y^{3-2\alpha} \quad .. \quad (36)$$

$$\zeta^{\alpha-1} = Y^{\alpha-1} + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Y^{2-\alpha} \quad .. \quad (37)$$

$$z = Y + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2a)(4-2\alpha)} Y^{4-2\alpha} \quad .. \quad (38)$$

Also

$$\eta^2 = \frac{2M}{\gamma-1}(z-\zeta\xi) = \left(\frac{M}{2-\alpha}\right)^2 Y^{4-2\alpha}$$

so that

$$\eta = \frac{M}{2-\alpha} Y^{2-\alpha} \quad .. \quad .. \quad .. \quad .. \quad (39)$$

The all-burnt position is given by

$$\xi_B^{\left(\frac{1-\alpha}{2-\alpha}\right)} = (2\alpha-3) + \frac{4-2\alpha}{Y_B} \quad .. \quad .. \quad (40)$$

where Y_B is given by

$$1 = Y_B + \frac{M(1-\alpha)}{(2-\alpha)^2(3-2\alpha)(4-2\alpha)} Y_B^{4-2\alpha} \quad .. \quad (41)$$

The maximum pressure occurs when $Y = Y_m$ where

$$Y_m = \left[\frac{M}{(2-\alpha)(3-2\alpha)}\right]^{\frac{1}{2\alpha-3}} \quad .. \quad .. \quad (42)$$

so that

$$\zeta_{max} = \left(\frac{3-2\alpha}{2-\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{1}{3-2\alpha}\right)^{\frac{1}{2\alpha-3}} \left(\frac{M}{2-\alpha}\right)^{\frac{1}{2\alpha-3}} \quad .. \quad (43)$$

Clemmows [3,4] formula is

$$\zeta_{max} = G(\gamma, \alpha)\, M^{\frac{1}{2\alpha-3}} \quad .. \quad .. \quad .. \quad (44)$$

comparing and remembering (17), we get

$$G\left(\frac{1}{2-\alpha}, \alpha\right) = \frac{(3-2\alpha)^{\frac{(\alpha-2)}{(\alpha-1)(2\alpha-3)}}}{(2-\alpha)^{\frac{(3\alpha-4)}{(\alpha-1)(2\alpha-3)}}} \quad .. \quad .. \quad (45)$$

from where we get the following table:

TABLE IV

α	1	$\frac{7}{6}$	$\frac{6}{5}$	$\frac{5}{4}$
γ	1	$\frac{6}{5}$	$\frac{5}{4}$	$\frac{4}{3}$
$G(\gamma, \alpha)$	0·367,88	0·108,55	0·069,83	0·027,78

Clemmows tables refer to $\gamma = 1$, and $\alpha \leqslant 1$.

For any given value of α to get a single—entry table, similar to Clemmows[3] (page 233), we make the substitution

$$MY^{3-2\alpha} = Z, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (46)$$

so that (36), (37), (38) and (39) give

$$\xi = \left[1 + \frac{(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Z\right]^{\left(\frac{2-\alpha}{1-\alpha}\right)} \quad \ldots \quad (47)$$

$$M^{\frac{1}{3-2\alpha}} \zeta = Z^{\frac{1}{3-2\alpha}} \left[1 + \frac{(1-\alpha)}{(2-\alpha)^2(3-2\alpha)} Z\right]^{\frac{1}{\alpha-1}} \quad \ldots \quad (48)$$

$$M^{\frac{1}{3-2\alpha}} z = Z^{\frac{1}{3-2\alpha}} \left[1 + \frac{(1-\alpha)}{(2-\alpha)^2(3-2\alpha)(4-2\alpha)} Z\right] \quad \ldots \quad (49)$$

$$M^{\left(\frac{\alpha-1}{3-2\alpha}\right)} \eta = \frac{1}{2-\alpha} Z^{\left(\frac{2-\alpha}{3-2\alpha}\right)} \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

The R.H.S. of these equations depend on Z and α only. The results have been tabulated for $\alpha = \frac{11}{10}, \frac{10}{9}, \frac{9}{8}, \frac{8}{7}, \frac{6}{5}, \frac{5}{4}$. We reproduce the table for $\alpha = \frac{6}{5}$, $\gamma = \frac{5}{4}$

TABLE V

Z	$M^{\frac{1}{3}}\eta$	ξ	$M^{\frac{5}{3}}\zeta$	$M^{\frac{5}{3}}z$
0	0	1·0	0	0
·1	·058,02	1·2386	·016,489	·020,84
·2	·146,20	1·5527	·039,462	·063,95
·3	·251,04	1·9731	·057,492	·121,31
·4	·368,40	2·5458	·067,529	·188,88
·5	·496,06	3·3422	·069,701	·263,71
·6	·632,57	4·4762	·065,556	·343,46
·7	·776,92	6·1342	·057,166	·426,11
·8	·928,32	8·6366	·046,565	·509,88
·9	1·086,18	12·5546	·035,500	·593,17
1·0	1·250,00	18·9689	·025,261	·674,48

Acknowledgements

The authors are grateful to Prof. B. R. Seth for pointing out the possibility of the integration of their basic differential equations for special cases, and to Dr. J. Corner and Chief Defence Scientist U. K. for help in getting Dr. Clemmows report[4]. They further express their gratitude to Dr. D. S. Kothari and Dr. R. S. Verma for their interest and encouragement.

References

1. Clemmows, C.A., *Phil. Trans. Roy. Soc.* **227A**, 345, 1928.
2. Clemmows, C.A., *Phil. Trans. Roy. Soc.* **231-A**, 265 1933.
3. Clemmows, C.A., '*Internal Ballistics*' H.M.S.O., London, 1951.
4. Clemmows, C.A., *A.R.D.E. Report (P) 11/56*, Ministry of Supply, U.K.
5. Kapur, J.N., *Proc. Nat. Inst. Sci. Ind.* **24A**, 15, 1958.
6. Corner, J., *Theory of Interior Ballistics of Guns.* John Wiley & Sons, New York, 1950.
7. Kapur, J.N., *Mem. Art. Franc. Paris* (in press).