INTERNAL BALLISTICS OF RECOILLESS HIGH-LOW PRESSURE GUNS USING HEPTA-TUBULAR POWDERS

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ABSTRACT

In this paper, the theory of recoilless high-low pressure guns has been discussed by taking the form function and the results have been applied to the case of hepta-tubular powders. The results for high-low pressure guns follow as a particular case.

Introduction

The theory of leaking guns has been studied by Corner [1,2]. This theory is applicable to worn orthodox guns, smooth bore mortars, recoilless guns and high-low pressure guns. The form function for the propellant was taken by him as $z=(1-f)(1+\theta f)$. The system of equations obtained by him can be integrated numerically only. He also gave a simple solution for the case of linear law of burning. Thiruvenkatachar and Venkatesan[3] discussed a method of successive approximation to solve the equations for the leaking guns for the tubular propellant, *i.e.* when $\theta = 0$ only. A simple theory of the internal ballistics of high-low pressure guns for tubular propellant (*i.e.* $\theta = 0$) was given by Corner[4]. Aggarwal [5,6] has extended these results for tubular propellants to the case of charges, having the form function $z = (1-f)(1+\theta f)$. Kapur [7] has combined the features of the two guns—high-low pressure guns and recoilless guns—and has given a simple theory of a recoilless high-low pressure gun of the same order of accuracy as Crow's theory of orthodox guns and Corner's theory for high-low pressure guns. In his paper, Kapur has started by taking the form function as $z = \varphi(f)$ and has found out the conditions for maximum pressure in the first chamber, and obtained some results for the particular cases of the form function, *viz.* $z = (1-f)(1+\theta f)$ and $z = (1-f)$.

In the present paper, we have discussed the theory of recoilless high-low pressure guns by taking the general form function $z = \zeta(y)$ for the propellants and have applied the theory to the case of hepta-tubular powders, the use of which is becoming more and more prominent now-a-days.

In the case of hepta-tubular powders, there are two phases of combustion—

(*i*) before the rupture of the grains, when the form function takes the general cubic form

$$z = (1-y)(a-by-cy^2),$$

and

(*ii*) after the rupture of the grains, when the form function of the propellants is a very complicated function of y.

It is clear that if $a = 1$, the point of rupture coincides with the point of all-burnt, and therefore the first phase of combustion in the case of hepta-tubular powders will correspond to the complete phase of burning for other ordinary types of propellants *viz.*, tubular, cylindrical (cord), slotted, square, etc. Further if $c = o$ and $b = -\theta$, the form function of the first phase of combustion takes the general quadratic form

$$z = (1-y)(1+\theta y),$$

which includes, as a particular case, that of cylindrical propellants for which the form-coefficient is $\theta = 1$. When $b = o$, $c = o$, $(a = 1)$, we get the form function for the tubular propellants. Thus the cubic form function for the first phase of hepta-tubular powders includes the form function for all shapes in general use.

Principal Notations and Assumptions

A recoilless high-low pressure gun can be represented in form by the following diagram—

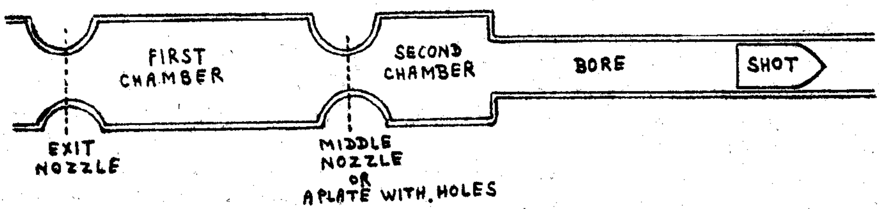

Fig 1: A recoilless high-low pressure gun

The following principal notations have been used here:—

μ_1 = The effective mass of the projectile.

μ = The fictitious mass of the projectile.

$= \mu_1 + \dfrac{\bar{\omega}}{3}$

v = Velocity of the projectile.

σ = Cross-section of the arm taking into consideration the rifling.

P = Mean pressure in the first chamber.

p = Mean pressure in the second chamber.

c^* = Internal volume of the first chamber.

c' = Internal volume of the second chamber.

f = Mean value of RT *i.e.*, force constant.

$\bar{\omega}$ = Mass of the charge.

δ = Density of the charge.

z = Fraction of the charge burnt at time 't'.

N_1 = Fraction of the charge remaining at time 't' in first chamber in the gaseous state.

N_2 = Fraction of the charge remaining at time 't' in second chamber in the gaseous state,

y = Fraction of the thickness (web-size) of the charge remaining at time 't'.

D' = Web-size.

A = Vivacity of the powder.

β' = Linear velocity of the combustion of the powder under the unit pressure.

n = Ratio of the two specific heats of gases.

$\bar{n}$ = $1 + (n-1)(1+k)$, where k is the ratio of heat losses up to time 't' to the kinetic energy of the shot at that time and assumed constant.

$$\varepsilon = \frac{\eta\rho'}{1-\eta\rho'}$$

η = Covolume of the propellant gases.

ρ' = Density of the gases.

$$\Psi' = n^{\frac{1}{2}}\left(\frac{2}{n+1}\right)^{\frac{n+1}{2(n-1)}}\left[1-0\cdot224\,\epsilon+0\cdot104\,\epsilon^2+o\,(\epsilon^3)\right]$$

S_0 = Surface of the powder initially exposed to the combustion.

S = Surface of the combustion at the instant 't'.

$\varphi(z) = \dfrac{S}{So}$ = The Charbonnier's Form-Function of the progressivity of the powder.

x = Shot travel at time 't'.

S_1^* = Area of the exit nozzle.

S_2^* = Area of the middle nozzle.

$$B = \frac{\bar{\omega}\left(\eta-\frac{1}{\delta}\right)}{\left(C^*-\frac{\bar{\omega}}{\delta}\right)}$$

We have assumed that

$$\frac{P}{p} \leqslant \left(\frac{2}{n+1}\right)^{\frac{n}{n-1}}\left[1-0\cdot248\,\epsilon+0\cdot117\,\epsilon^2+o\,(\varepsilon^3)\right],$$

so that the rate of flow is settled by 'P' alone— a familiar result in the one-dimensional theory of nozzles.

If P/p is greater than the above limit, we shall have to introduce a correction in the equation of continuity given afterwards by taking a suitable back pressure factor as explained by Corner[2].

Basic Equations of Internal Ballistics

The basic equations for a recoilless high-low pressure gun for an isothermal model can be written as :—

Equation of state for the first chamber :—

$$P\left[C^{*}-\frac{\varpi\,(1-z)}{\delta}-\varpi\,N_{1}\,\eta\right]=\varpi\,N_{1}f \qquad \ldots \qquad \ldots \qquad (1)$$

Equation of state for the second chamber :—

$$p\,[\,c'+\sigma x-\varpi\,N_{2}\,\eta\,]=\varpi\,N_{2}f \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Equation of continuity for the first chamber :—

$$\varpi\,\frac{dz}{dt}=\varpi\frac{dN_{1}}{dt}+\frac{\psi'\,S_{1}{}^{*}P}{\sqrt{f}}+\frac{\psi'\,S_{2}{}^{*}\,P}{\sqrt{f}} \qquad \ldots \qquad \ldots \qquad (3)$$

Equation of continuity for the second chamber :—

$$\varpi\,\frac{dN_{2}}{dt}=\frac{\psi'\,S_{2}{}^{*}\,P}{\sqrt{f}} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (4)$$

Equation of Inertia :—

$$\mu\,\frac{d^{2}x}{dt^{2}}=\mu\,\frac{dv}{dt}=\mu v\,\frac{dv}{dx}=\sigma p \qquad \ldots \qquad \ldots \qquad \ldots \qquad (5)$$

Equation of combustion :—

$$\frac{dz}{dt}=A\,P\,\varphi\,(z) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (6)$$

The equation of combustion can also be written as :—

$$D'\,\frac{dy}{dt}=-\,\beta'\,P \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (7)$$

and the $(\,z,\,y\,)$ form function in the form ,

$$z=\zeta\,(y) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (8)$$

Clearly $A\,\varphi\,(z)=-\,\zeta'\,(y)\,\dfrac{\beta'}{D'}=-\,\dfrac{\beta'}{D'}\,\dfrac{dz}{dy}$, $\ldots \qquad \ldots \qquad (9)$

and for hepta-tubular powders,

$$A=\frac{(a-b-c)\,\beta'}{D'} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9A)$$

Solution of the Equations

For the first chamber :—

From (3) and (6), we get

$$\frac{dz}{dt}=\frac{dN_{1}}{dt}+\frac{\lambda}{\varphi}\,\frac{dz}{dt}+\frac{\lambda'}{\varphi}\,\frac{dz}{dt} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (10)$$

with

$$\lambda=\frac{\psi'\,S_{1}{}^{*}}{\varpi\,\sqrt{f}\,A} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (10A)$$

$$\lambda' = \frac{\Psi' S_2 *}{\bar{\omega} \sqrt{f} A} \quad \ldots \quad (10B)$$

Integrating (10) and putting the initial conditions

$$t = 0 , z = 0 , N_1 = 0 ,$$

we get

$$z - (\lambda + \lambda') V = N_1 \quad \ldots \quad (11)$$

with

$$V (z) = \int_0^z \frac{dz}{\varphi (z)} \quad \ldots \quad 11(A)$$

But for both phases of combustion of the hepta-tubular powders, we have [Tavernier[8] or Gupta[9]]

$$V (z) = (a - b - c) \ (1 - y) \quad \ldots \quad (11B)$$

$$\therefore N_1 = z - (\lambda + \lambda') (a - b - c) (1 - y)$$

$$= z - (\Psi_1 + \Psi_2) (1 - y) \quad \ldots \quad (12)$$

with

$$\Psi_1 = \lambda (a - b - c) \quad \ldots \quad (12A)$$

$$\Psi_2 = \lambda' (a - b - c)$$

Clearly from (12A), (10A), (10B) and (9A), we have

$$\Psi_1 = \frac{\Psi' S_1 *}{\bar{\omega} \sqrt{f}} \ \frac{D'}{\beta'} \quad \ldots \quad (13)$$

$$\Psi_2 = \frac{\Psi' S_2 *}{\bar{\omega} \sqrt{f}} \ \frac{D'}{\beta'}$$

Also from (4), (6), (11A), (11B), and (13) we have on integration

$$N_2 = \Psi_2 (1 - y) \quad \ldots \quad (14)$$

Here the quantities Ψ_1 and Ψ_2 are the two dimensionless leakage parameters.

Now from (1), we have

$$P = \frac{\bar{\omega} N_1 f}{\left[C * - \dfrac{\bar{\omega} (1 - z)}{\delta} - \bar{\omega} N_1 \eta \right]} \quad \ldots \quad (15)$$

Therefore, with the help of (12) and (8), equation (15) becomes

$$P = \frac{\bar{\omega} f [\zeta (y) - (\Psi_1 + \Psi_2) (1 - y)]}{\left[C* - \dfrac{\bar{\omega}}{\delta} + \dfrac{\bar{\omega}}{\delta} \zeta (y) - \bar{\omega} \eta \left\{ \zeta (y) - (\Psi_1 + \Psi_2)(1 - y) \right\} \right]} \quad (16)$$

$$= \frac{B_1 f \delta [\zeta (y) - (\Psi_1 + \Psi_2) (1 - y)]}{[1 - B \zeta (y) + B_2 (1 - y)]} \quad \ldots \quad (17)$$

with

$$B = \frac{\bar{\omega}\left(\eta - \frac{1}{\delta}\right)}{C^* - \frac{\bar{\omega}}{\delta}} \quad \ldots \quad (17A)$$

$$B_1 = \frac{\bar{\omega}}{\delta\left(C^* - \frac{\bar{\omega}}{\delta}\right)} \quad \ldots \quad (17B)$$

$$B_2 = B_1 \delta \eta (\Psi_1 + \Psi_2) = \frac{\bar{\omega}\, \eta (\Psi_1 + \Psi_2)}{C^* - \frac{\bar{\omega}}{\delta}} \quad \ldots \quad (17C)$$

Differentiating (17) and on simplification we get,

$$\left[1 - B\zeta(y) + B_2(1 - y) \right]^2 \left(\frac{dp}{dy}\right)$$

$$= B_1 f\delta \left[\left\{ \zeta(y) + \Psi_1 + \Psi_2 \right\} \left\{ 1 - B\zeta(y) + B_2(1 - y) \right\} + \left\{ B\zeta'(y) + B_2 \right\} \left\{ \zeta(y) - (\Psi_1 + \Psi_2)(1 - y) \right\} \right] \quad (18)$$

Now for the hepta-tubular powders, we have

(*i*) initially when $z = o,\ y = 1,\ V = o.$

$$\zeta(y) = o,\ \zeta'(y) = \zeta'(1) = b + c - a.$$

$\therefore$ Equation (18) gives

$$\left(\frac{dP}{dy}\right)_{y=1} = B_1 f \delta [\, b + c - a + \Psi_1 + \Psi_2 \,] \quad \ldots \quad (19)$$

(*ii*) at the point of rupture of the grains, when

$$z = a, \quad y = 0,$$

$$\zeta(y) = a \quad \zeta'(y) = -(a + b),$$

$\therefore$ Equation (18) gives after simplification,

$$\left[1 - Ba + B_2\right]^2 \left(\frac{dP}{dy}\right)_{y=0}$$

$$= B_1 f \delta \left[(\Psi_1 + \Psi_2 - a - b) - b \left\{ B_2 - (\Psi_1 + \Psi_2) B \right\} \right] \quad \ldots \quad (20)$$

(*iii*) at the all-burnt position, we have

$$z = 1, \quad y = y_{min.} \quad \zeta(y) = 1 \quad \zeta'(y) = 0.$$

$\therefore$ Equation (18) gives

$$\left[1 - B + B_2 (1 - y_{min}) \right]^2 \left(\frac{dP}{dy} \right)_{y = y_{min}}$$

$$= B_1 f \delta \left[(\Psi_1 + \Psi_2)(1 - B) + B_1 \delta \eta (\Psi_1 + \Psi_2) \right]$$

$$= \frac{\bar{\omega} f C^*}{\left(C^* - \frac{\bar{\omega}}{\delta} \right)^2} [\Psi_1 + \Psi_2] \quad \ldots \quad (21)$$

which is a positive quantity.

Initially $\frac{dP}{dy}$ is negative, since P increases and y decreases; we therefore, have the condition, which holds good throughout the combustion as :—

$$b + c - a + \Psi_1 + \Psi_2 < o \quad \ldots \quad (21A)$$

Again at all-burnt $\frac{dP}{dy}$ is positive, therefore the maximum pressure in the first chamber should occur before the all-burnt position. Thus the maximum pressure in the first chamber may occur :—

(*i*) before the rupture,

if $\frac{dP}{dy}$ is positive at rupture;

(*ii*) at the rupture,

if $\frac{dP}{dy}$ is zero at rupture;

(*iii*) after the rupture,

if $\frac{dP}{dy}$ is still negative at rupture.

It can easily be seen from (18) that the expression on the right-hand side is an increasing function of y as y decreases, so that it passes from negative to positive and once it becomes positive, it cannot be negative again. Hence the uniqueness of maximum pressure is established.

Now the condition that the maximum pressure in the first chamber may occur before the rupture is

$$\left(\frac{dP}{dy} \right)_{y = o} > o$$

i. e.

$$\Psi_1 + \Psi_2 - a > b \left[1 + (\Psi_1 + \Psi_2) B_1 \right]$$

$$\text{or} \qquad b < \frac{\Psi_1 + \Psi_2 - a}{1 + B_1 (\Psi + \Psi_2)} \quad \ldots \quad (22)$$

While the condition for the maximum pressure in the first chamber to occur after or at the rupture is

$$\left(\frac{dP}{dy}\right)_{y=o} \leqslant 0$$

i.e. $$b > \frac{\Psi_1 + \Psi_2 - a}{1 + B_1(\Psi_1 + \Psi_2)} \qquad (23)$$

and $$b = \frac{\Psi_1 + \Psi_2 - a}{1 + B_1(\Psi_1 + \Psi_2)} \qquad (23A)$$

These conditions can easily be seen to cover the conditions which may be obtained for propellants of other shapes also. Thus for constant burning surfaces, *viz.*, tubular, the (z, y) relation is

$$z = 1 - y, \qquad (24)$$

so that we should have $a = 1$, $b=0$, $c = 0$, and the point of rupture as the point of all-burnt.

Thus in this case at all-burnt $\left(\frac{dP}{dy}\right)_{y=o}$ has the same sign as

$\Psi_1 + \Psi_2 - 1$, which is negative from the condition (21A).

$\therefore$ in this case the maximum pressure in the first chamber would occur at all-burnt position.

For the cord having the (z, y) relation as

$$z = (1-y)\ (1+y), \qquad (25)$$

we have $a = 1$, $b = -1$, $c = 0$ and the point of rupture coincides with the point of all-burnt.

In this case $\left(\frac{dP}{dy}\right)_{y=o}$ has the same sign as

$$\Psi_1 + \Psi_2 + B_2 - (\Psi_1 + \Psi_2)\, B,$$

i.e. as that of $\dfrac{(\Psi_1 + \Psi_2)}{\left(C^* - \dfrac{\varpi}{\delta}\right)^2} C^*$ which is always positive.

Therefore in this case the maximum pressure in the first chamber would always occur before the all-burnt position.

For the sphere having the (z, y) relation as

$$\begin{aligned} z &= 1 - y^3 \\ &= (1 - y)\ (1 + y + y^2), \end{aligned} \qquad (26)$$

we have $a = 1$, $b = -1$, $c = -1$ and the point of rupture as the point of all-burnt.

In this case also $\left(\frac{dP}{dy}\right)_{y=o}$ has the same sign as

$$\Psi_1 + \Psi_2 + B_2 - (\Psi_1 + \Psi_2)B,$$

which is also always positive as in the last case.

Therefore in this case also the maximum pressure occurs before the all-burnt position.

Taking the case of hepta-tubular powders both the conditions (22) and (23) may be satisfied.

(*i*) When the condition (22) is satisfied, the maximum pressure in the first chamber would occur before the rupture of the grains and at that moment as $\frac{dP}{dy} = 0$, we should have from (18) after simplification,

$$\zeta'(y)\,[1 + (\Psi_1 + \Psi_2)(1 - y)\,B_1] + (\Psi_1 + \Psi_2)\left\{1 + B_1\,\zeta(y)\right\} = 0. \quad (27)$$

But before the rupture, we have (Tavernier [8])

$$\zeta(y) = (1 - y)(a - by - cy^2)$$

$$\zeta'(y) = -[(a + b) - 2(b - c)\,y - 3\,cy^2]$$

and also

$$\alpha = \frac{a + b}{a - b - c}$$

$$\beta = \frac{2(b - c)}{a - b - c}$$

$$\gamma = \frac{3c}{a - b - c}.$$

$\therefore$ equation (27) becomes after simplification

$$\begin{aligned} F_1(y) \equiv\; & 4(\Psi_1 + \Psi_2)\,B_1\,\gamma\,y^3 \\ & - \Big[6\,\gamma\left\{1 + B_1(\Psi_1 + \Psi_2)\right\} - 3\,B_1\,\beta\,(\Psi_1 + \Psi_2)\Big]\,y^2 \\ & - 6\,\beta\,[1 + B_1(\Psi_1 + \Psi_2)]\,y \\ & + \left[6\,\alpha - (\Psi_1 + \Psi_2)\left\{\frac{6}{a - b - c} - B_1(2\gamma + 3\beta)\right\}\right] = 0. \quad (28) \end{aligned}$$

The equation (28) will give a root between 0 and 1, say y_m which determines the value of y at which the maximum pressure occurs in the first chamber before the rupture of the grains.

In that case the maximum pressure P in the first chamber is given by (17), as

$$P_m = \frac{B_1 f \delta\,(1 - y_m)\,[a - b\,y_m - c\,y^2_m - \psi_1 - \psi_2]}{[1 - (1 - y_m)\left\{B\,(a - b\,y_m - c\,y^2_m) - B_2\right\}]} \quad \ldots \quad (29)$$

(*ii*) But if the condition (23A)

$$b = \frac{\psi_1 + \psi_2 - a}{1 + B_1(\psi_1 + \psi_2)}$$

is satisfied, the maximum pressure would occur at rupture and in that case maximum pressure, say P_{mr} is given by

$$P_{mr} = \frac{B_1 f \delta\,[a - \psi_1 - \psi_2]}{1 - Ba + B_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

(*iii*) Again when the condition (23)

$$b > \frac{\psi_1 + \psi_2 - a}{1 + B_1(\psi_1 + \psi_2)}$$

is satisfied, the maximum pressure would occur after rupture of the grains.

Since $\frac{dP}{dy} = 0$ at the point of maximum pressure, the equation (27) becomes with the help of standard results for hepta-tubular powders [Tavernier[8]]

$$\frac{3(m+1)(m-3)}{32\pi\rho m(m^2-7)}\left\{\left(4m + 4 - \frac{m+1}{\cos\omega}\right) H'(\omega)\,\frac{\cos^9\omega}{\sin\omega} - (m+1)H(\omega)\right\}$$
$$\times\left[1 + B_1(\psi_1+\psi_2)\left\{1 - \frac{m+1}{m-3}(1-\sec\omega)\right\}\right]$$
$$+ (\psi_1+\psi_2)\left[1 + B_1\left\{1 - \frac{3(m+1)^2}{8\pi(m^2-7)}\right.\right.$$
$$\left.\left.\left(1 + \frac{1}{m\rho} - \frac{m+1}{4m\rho\cos\omega}\right)H(\omega)\right]= 0 \quad .. \quad (31)$$

Thus corresponding to any hepta-tubular powder (*i.e.* with the values of m and ρ given) and with the necessary gun and propellant data, this equation when solved would give the value ω_m of ω for which the pressure would be maximum.

Then the maximum pressure in the first chamber will be given by (17), after changing $\zeta(y)$ and y in terms of ω.

Again at all-burnt, we have

$$z = \zeta(y) = 1, \qquad y = y_{min}$$

∴ the pressure P_B at the all-burnt position is given as

$$P_B = \frac{B_1\delta f[1 - (\Psi_1+\Psi_2)(1 - y_{min})]}{1 - B + B_2(1 - y_{min})} \quad .. \quad .. \quad (32)$$

Again from (6) and (7) we get

$$\frac{1}{A\varphi(z)}\,\frac{dz}{dt} = \frac{B_1 f\delta\,[\zeta(y) - (\Psi_1+\Psi_2)(1-y)]}{[1 - B\zeta(y) + B_2(1-y)]} \quad .. \quad .. \quad (33)$$

∴ from (9), we have

$$\frac{dy}{dt} = \frac{A}{b+c-a}\left[\frac{B_1 f\delta\,[\zeta(y) - (\Psi_1+\Psi_2)(1-y)]}{[1 - B\zeta(y) + B_2(1-y)]}\right] \quad .. \quad (34)$$

Integrating and taking the point of rupture as the origin of time, we get

$$\int_0^y \frac{1 - B\zeta(y) + B_2(1-y)}{\zeta(y) - (\Psi_1+\Psi_2)(1-y)}\,dy = \frac{A B_1 f\delta}{b+c-a}\,t \quad .. \quad (35)$$

or

$$\int_0^y F^*(y)\,dy = -B_3\,t \quad .. \quad .. \quad (36)$$

with

$$F^*(y) = \frac{1 - B\,\zeta(y) + B_2(1-y)}{\zeta(y) - (\Psi_1 + \Psi_2)(1-y)} \qquad \text{(36A)}$$

and

$$B_3 = \frac{A\,B_1 f\,\delta}{a - b - c} \qquad \text{(36B)}$$

The equation (35) or (36) holds good for both the phases of combustion and gives the relation between 'y' and 't'.

The equations (29), (30), (31), (32) and (35) are similar to the equations (25), (26), (27), (29) and (32) respectively of H/L guns obtained by the present author in his paper "Internal Ballistics of High low pressure guns with hepta-tubular powders", with the difference that in place of $(\Psi_1 + \Psi_2)$ occurring in the above equations of the present paper we have only Ψ in the paper on 'Internal Ballistics of H/L pressure guns with hepta-tubular powders.' The shot travel at any instant, (in particular the position of all-burnt), the pressure in the second chamber, and the velocity of the shot at any instant, (in particular the muzzzle velocity) can therefore, be obtained in a similar way.

We however, give here the main results which are obtained for the Recoilless High-low pressure guns. Thus for the first phase of combustion, we have,

$$F^*(y) = \frac{1 + N_1 Z + N_2 Z^2 + N_3 Z^3}{L_1 Z + L_2 Z^2 + L_3 Z^3} = F^*(Z)^* \qquad \text{(37)}$$

with

$$Z = 1 - y \qquad \text{(37A)}$$

$$N_1 = B_2 - B\,(a - b - c) \qquad \text{(37B)}$$

$$N_2 = -B\,(b + 2c) \qquad \text{(37C)}$$

$$N_3 = Bc \qquad \text{(37D)}$$

$$L_1 = a - b - c - \Psi_1 - \Psi_2 \qquad \text{(37E)}$$

$$L_2 = b + 2c \qquad \text{(37F)}$$

$$L_3 = -c \qquad \text{(37G)}$$

so that the equation (36) becomes

$$\int_1^z F^{**}(Z)\,dz = B_3 t \qquad \text{(38)}$$

which on integration gives

$$B_3 t = \frac{N_3}{L_3}(Z - 1) + l_1 \log Z + \frac{l_2}{2L_3} \log \frac{L_3 Z^2 + L_2 Z + L_1}{L_3 + L_2 + L_1}$$

$$+ \frac{2L_3 l_3 - L_2 l_2}{2L_3\sqrt{L^2_2 - 4 L_1 L_3}} \log L_4 \left[\frac{2L_3 Z + L_2 - \sqrt{L^2_2 - 4L_1 L_3}}{2L_3 Z + L_2 + \sqrt{L^2_2 - 4L_1 L_3}}\right] \qquad \text{(39)}$$

with

$$l_1 = \frac{1}{L_1} \quad \ldots \quad (39A)$$

$$l_2 = \frac{L_1 L_3 N_2 - L_1 L_2 N_3 - L_3^2}{L_1 L_3} \quad \ldots \quad (39B)$$

$$l_3 = \frac{L_1 L_3 N_1 - L_1^2 N_3 - L_2 L_3}{L_1 L_3} \quad \ldots \quad (39C)$$

$$L_4 = \frac{2L_3 + L_2 + \sqrt{L_2^2 - 4L_1 L_3}}{2L_3 + L_2 - \sqrt{L_2^2 - 4L_1 L_3}} \quad \ldots \quad (39D)$$

while for the second phase of combustion, we have

$$F^*(y) = \frac{1 - BH^*(\omega) + B_2 \left[1 - \frac{m+1}{m-3}(1 - sec\ \omega) \right]}{H^*(\omega) - (\Psi_1 + \Psi_2) \left[1 - \frac{m+1}{m-3}(1 - sec\ \omega) \right]} \quad \ldots \quad (40)$$

$$= \varphi^*(\omega), \quad \text{say} \quad \ldots \quad (41)$$

where

$$H^*(\omega) = \zeta(y) = 1 - \frac{3(m+1)^2}{8\pi(m^2-7)} \left[1 + \frac{1}{\rho m} - \frac{m+1}{4\rho m\ cos\ \omega} \right] H(\omega) \quad (41A)$$

so that for any instant during the second phase of combustion, we have

$$B_3 t = \int_o^w \varphi^{**}(\omega)\, dw \quad \ldots \quad (42)$$

with

$$\varphi^{**}(\omega) = \frac{1 - BH^*(\omega) + B_2 \left[1 - \frac{m+1}{m-3}(1 - sec\ \omega) \right]}{H^*(\omega) - (\Psi_1 + \Psi_2) \left[1 - \frac{m+1}{m-3}(1 - sec\ \omega) \right]} \times \frac{m+1}{m-3}(-\, sec\ \omega\ tan\ \omega) \quad \ldots \quad (43)$$

In particular

$$B_3 t_B = \int_o^{\omega = 42^\circ 25'} \varphi^{**}(\omega)\, dw \quad \ldots \quad (44)$$

where suffix 'B' denotes the values at all-burnt position.

For the second chamber :—

We get from (5), 2) and (14)

$$\mu \frac{dy}{dt} \frac{d}{dy} \left\{ \frac{dx}{dy} \frac{dy}{dt} \right\} = \frac{\sigma \bar{\omega} f \Psi_2 (1-y)}{c' + \sigma x - \omega \eta \Psi_2 (1-y)} \quad \ldots \quad (45)$$

which with the help of (34) becomes

$$\mu \frac{A^2 B_1^2 \delta^2 f^2}{(b+c-a)^2} \frac{\zeta(y)-(\Psi_1+\Psi_2)(1-y)}{1-B\zeta(y)+B_2(1-y)} \frac{d}{dy}\left[\frac{dx}{dy} \times \frac{\zeta(y)-(\Psi_1+\Psi_2)(1-y)}{1-B\zeta(y)+B_2(1-y)}\right] = \frac{\sigma \bar{\omega} f \Psi_2 (1-y)}{c' + \sigma x - \bar{\omega}\eta \Psi_2 (1-y)} \quad (46)$$

or

$$\frac{d}{dy}\left[\frac{\zeta(y)-(\Psi_1+\Psi_2)(1-y)}{1-B\zeta(y)+B_2(1-y)} \frac{dX}{dy}\right] = \frac{L_1^2(1-y)}{X-B_4(1-y)} \times \frac{1-B\zeta(y)+B_2(1-y)}{\zeta(y)-(\Psi_1+\Psi_2)(1-y)} \quad .. \quad (47)$$

with

$$X = \frac{B_3 L_1}{\sigma}\left(\frac{\mu}{\bar{\omega} f \Psi_2}\right)^{\frac{1}{2}} (c' + \sigma x) \quad .. \quad .. \quad .. \quad (47A)$$

$$= X_o \left(1 + \frac{\sigma}{c'} x\right) \quad .. \quad .. \quad .. \quad .. \quad (47B)$$

$$X_o = \frac{B_3 L_1}{\sigma}\left(\frac{\mu}{\bar{\omega} f \Psi_2}\right)^{\frac{1}{2}} c' \quad .. \quad .. \quad .. \quad .. \quad (47C)$$

$$B_4 = \frac{B_3 L_1}{\sigma} \eta \left(\frac{\mu \bar{\omega} \Psi_2}{f}\right)^{\frac{1}{2}} \quad .. \quad .. \quad .. \quad .. \quad (47D)$$

The equation (47) is similar to equation (49) of the paper "Internal Ballistics of high-low pressure guns with hepta-tubular powders" by Gupta[10] with the difference that in place of $(\Psi_1 + \Psi_2)$, there occurs only Ψ. This equation (47) holds good for both phases of combustion *i.e.* before the rupture as well as after the rupture of the grains and gives a relation between y and X (defining shot-travel x at any instant during both phases of combustion).

(*i*) For the first phase of combustion, equation (47) becomes

$$\frac{d}{dZ}\left[\frac{L_1 Z + L_2 Z^2 + L_3 Z^3}{1 + N_1 Z + N_2 Z^2 + N_3 Z^3} \frac{dX}{dZ}\right] = \frac{Z L_1^2}{X - B_4 Z} \times \left[\frac{1 + N_1 Z + N_2 Z^2 + N_3 Z^3}{L_1 Z + L_2 Z^2 + L_3 Z^3}\right] \quad .. \quad .. \quad (48)$$

which is the same as equation (51) of Gupta[9] and hence the series solution (53) of that paper holds good in this case also. Thus the series solution of (48) is given as

$$X = X_o + q_1 Z + q_2 Z^2 + q_3 Z^3 + \quad .. \quad .. \quad .. \quad (49)$$

where

$$q_1 = \frac{1}{X_o} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (49A)$$

$$q_2 = \frac{3N_1}{4X_o} - \frac{3L_2}{4L_1 X_o} + \frac{B_4}{4X_o^2} - \frac{1}{4X_o^3} \quad .. \quad .. \quad (49B)$$

$$q_3 = \left[\frac{N_1^2}{6} + \frac{4\,N_2}{9} - \frac{4\,L_3}{9\,L_1} + \frac{11\,L_2^2}{18\,L_1^2} - \frac{7\,N_1\,L_2}{9\,L_1} \right] \frac{1}{X_o}$$
$$+ \left[\frac{5\,B_4\,N_1}{18} - \frac{5\,B_4\,L_2}{18\,L_1} \right] \frac{1}{X^2_o} + \left[\frac{13\,L_2}{36\,L_1} - \frac{13\,N_1}{36} + \frac{B^2_4}{9} \right]$$
$$\times \frac{1}{X^3_o} - \frac{B_4}{4\,X^4_o} + \frac{5}{36\,X^5_o} \quad \ldots \quad \ldots \quad \ldots \quad (49C)$$

The equation (49) holds good only upto the rupture of the grains, *i.e.* upto $Z = 1$.

The convergence of the series (49) can be established in the same way as done in the case of high-low pressure guns, (Gupta[10]). The condition that L_2/L_1 is less than 1, is here found to be

$$\rho \left[(8\,m^3 - 56\,m)\,(\Psi_1 + \Psi_2) - m^3 - 34\,m^2 + 111\,m \right]$$
$$< 3\,m^3 - 5\,m^2 - 47\,m + 105$$

(*ii*) For the second phase of combustion, the equation (47) becomes

$$\frac{d}{dw}\left[\frac{1}{\varphi^*(w)} \frac{dX}{dw} \frac{\cos^2 w}{\sin w} \right] = \frac{L^2_1\,(m+1)^2}{(m-3)^2} \times$$
$$\left[\frac{[\,(m-3) - (m+1)\,(1 - \sec w)\,]}{(m-3)\,X - B_4\,[\,(m-3) - (m+1)\,(1 - \sec w)\,]} \right] \times$$
$$\frac{\sin w}{\cos^2 w}\,\varphi^*(w). \quad \ldots \quad (50)$$

This equation (50) can be integrated numerically only, to give in terms of w, the initial conditions being

$w = o$,

$X = X_r$ = value of X at the rupture obtained from (49) when $Z = 1$.

and $\left(\frac{dX}{dw} \right)_{w=o} = o.$

Pressure in the second chamber :—

From (2) and (14), we have

$$p = \frac{\bar{\omega}\,f\,\Psi_2\,(1-y)}{C' + \sigma x - \bar{\omega}\,\eta\,\Psi_2\,(1-y)} \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

which with the help of (37A), (47A) and (47D) becomes

$$p = \frac{f\,B_4}{\eta}\;\frac{1}{\left(\frac{X}{Z} - B_4 \right)} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

This equation (52) gives pressure p in the second chamber at any instant in terms of X and Z and ultimately in terms of Z alone, since X can be given in terms of Z for the first phase of combustion by equation (49) and for the second phase of combustion by equation (50) and the relation

$$Z = (1-y) = 1 - \frac{m+1}{m-3}(1 - \sec w) \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

The velocity v of the shot is given from (47A), (37A), (34) and (36B) as

$$v = \frac{db}{dt} = \frac{dX}{dX}\,\frac{dX}{dZ}\,\frac{dZ}{dY}\,\frac{dY}{dt} = \frac{1}{L_1}\left(\frac{\bar{\omega} f \Psi_2}{\mu}\right)^{\frac{1}{2}} \times \frac{[\zeta(y) - (\Psi_1 + \Psi_2) Z]}{[1 - B\zeta(y) + B_2 Z]}\,\frac{dX}{dZ} \quad \ldots \quad \ldots \quad (54)$$

Since X and $\zeta(y)$ can be expressed in terms of Z alone for both phases of combustions, equation (54) gives the velocity v at any instant in terms of Z.

Also the shot-travel x is given by (47B) as

$$x = \frac{C'(X - X_o)}{\sigma X_o} \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

Thus the equations (55), (52) and (54) give the shot-travel, pressure in the second chamber and the velocity of the shot for a given value of Z, at any instant.

After All-Burnt (**On Non-Isothermal Assumptions**)

Since after all-burnt the gases cool considerably by expansion, we have a non-isothermal model*. Let T_1 and T_2 be the temperatures in the two chambers.

For the first chamber, the various equations can, therefore, be written as :—

Energy Equation of the first chamber :—

$$\frac{dT_1}{T_1} = \frac{n-1}{1-\eta\rho'}\,\frac{d\rho'}{\rho'}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

Also $$\frac{d\rho'}{\rho'} = \frac{dN_1}{N_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

i. e. $$\rho' = \rho'_B\,\frac{N_1}{N_{1B}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

so that $$\frac{dT_1}{T_1} = \frac{n-1}{1-\eta\rho'}.\,\frac{dN_1}{N_1} = \frac{n-1}{1-\eta\rho'_B\,\dfrac{N_1}{N_{1B}}}.\,\frac{dN_1}{N_1} \quad (59)$$

Energy Equation of the second chamber :—

$$\frac{d}{dt}(N_2 T_2) = -(\bar{n}-1)\,\frac{\sigma p}{\bar{\omega} R}\,\frac{dn}{dt} + \left\{1 + \frac{n-1}{1-\eta\rho'}\right\} T_1\,\frac{dN_2}{dt} \quad (60)$$

where

$$(\bar{n}-1) = (n-1)(1+k). \quad \ldots \quad \ldots \quad (60A)$$

Equation of state (after all-burnt):—

$$P\,[C^* - \bar{\omega} N_1 \eta] = \bar{\omega} N_1 R T_1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

$$p\,[c' + \sigma x - \bar{\omega} N_2 \eta] = \bar{\omega} N_2 R T_2 \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

*The case of isothermal model can be discussed in a similar way.

Equations of continuity for the two chambers:—

$$O = \bar{\omega}\,\frac{dN_1}{dt} + \frac{\psi' S_1^* P}{\sqrt{RT_1}} + \frac{\psi' S_2^* P}{\sqrt{R\,T_1}} \quad \ldots \quad (63)$$

$$\bar{\omega}\,\frac{dN_2}{dt} = \frac{\psi' S_2^* P}{\sqrt{R\,T_1}} \quad \ldots \quad (64)$$

Equation of Inertia—

$$\mu\,\frac{d^2x}{dt^2} = \sigma\,p \quad \ldots \quad (65)$$

Now from (59) and (57), we have

$$\frac{T_1}{T_{1B}} = \left(\frac{N_1}{N_{1B}}\right)^{n-1} \left[\frac{1 - \eta\rho'_B}{1 - \eta\rho'_B\,\dfrac{N_1}{N_{1B}}}\right]^{n-1} \quad \ldots \quad (66)$$

where the suffix 'B' denotes the values at all-burnt position. Neglecting squares and higher powers of η, we get

$$\frac{T_1}{T_{1B}} = \left(\frac{N_1}{N_{1B}}\right)^{n-1} \left[1 - (n-1)\;\eta\rho'_B\left(\frac{N_{1B} - N_1}{N_{1B}}\right)\right] \quad \ldots \quad (67)$$

so that from (61), we have

$$\frac{P}{P_B} = \frac{C^* - \bar{\omega}N_{1B}\eta}{C^* - \bar{\omega}\,N_1\,\eta}\left(\frac{N_1}{N_{1B}}\right)^{n} \left(\frac{1 - \rho'_B\,\eta}{1 - \eta\rho'_B\,\dfrac{N_1}{N\rho'_B}}\right)^{n-1} \quad \ldots \quad (68)$$

Again neglecting squares and higher powers of η, we have

$$\frac{P}{P_B} = \left[1 - \left(1 - \frac{N_1}{N_{1B}}\right)\eta\,N_{1B}\left\{\frac{\bar{\omega}}{C^*} + (n-1)\frac{\rho^1_B}{N_{1B}}\right\}\right]\left(\frac{N_1}{N_{1B}}\right)^{n} \quad (69)$$

where P_B is given by (32).

Again from (63), (69), and (67) on simplification after neglecting squares and higher powers of η, we have

$$\frac{dN_1}{dt} = -\,m_B\,(K_B + Q_B\,N_1)\;N_1^{\frac{n+1}{2}} \quad \ldots \quad (70)$$

with

$$m_B = \frac{\psi'(S_1^* + S_2^*)\,\rho_B}{\bar{\omega}\sqrt{R}\;\;\sqrt{X_{1B}}}\cdot\frac{1}{(N_{1B})^{\frac{n+1}{2}}} \quad \ldots \quad (70A)$$

$$Q_B = \eta\left[\frac{\bar{\omega}}{C^*} + \frac{n-1}{2}\;\frac{\rho'_B}{N_{1B}}\right] \quad \ldots \quad (70B)$$

$$K_B = 1 - N_{1B}\,Q_B \quad \ldots \quad (70C)$$

Integrating (70) and applying conditions at the all-burnt, we get

$$\left(\frac{N_{1B}}{N_1}\right)^{\frac{n-1}{2}} + A_B\left(\frac{N_{1B}}{N_1}\right)^{\frac{n-3}{2}} = 1 + A_B + C_B\,(t - t_B) \quad \ldots \quad (71)$$

with

$$A_B = \frac{Q_B}{K_B} \cdot \frac{n-1}{3-n} \cdot N_{1B} \quad \ldots \quad (71A)$$

$$C_B = m_B\,K_B\left(\frac{n-1}{2}\right)N_{1B}^{\frac{n-1}{2}} \quad \ldots \quad (71B)$$

When $\eta = 0$, $Q_B = 0$, $A_B = 0$, and $K_B = 1$.

$$\therefore \quad \frac{N_{1B}}{N_1} = [1 + C_B\,(t - t_B)]^{\frac{2}{n-1}} \quad \ldots \quad (72)$$

Now when $\eta \neq 0$, as a first approximation let us assume that

$$\frac{N_{1B}}{N_1} = [1 + C_B\,(t - t_B)]^{\frac{2}{n-1}} + h \quad \ldots \quad (73)$$

where h is a very small quantity.

Substituting this in (71), we get

$$\left[\left\{1 + C_B\,(t - t_B)\right\}^{\frac{2}{n-1}} + h\right]^{\frac{n-1}{2}} + A_B\left[\left\{1 + C_B\,(t - t_B)\right\}^{\frac{2}{n-1}} + h\right]^{\frac{n-3}{2}} = 1 + A_B + C_B\,(t - t_B).$$

$$\therefore h \doteq \frac{2\,A_B\left[1 - \left\{1 + C_B\,(t - t_B)\right\}^{\frac{n-3}{n-1}}\right]}{\left\{1 + C_B\,(t - t_B)\right\}^{\frac{n-5}{n-1}}\left[(n-1)\left\{1 + C_B\,(t - t_B)\right\}^{\frac{2}{n-1}} + (n-3)A_B\right]} \quad (74)$$

(neglecting squares and higher powers of η).

$$\therefore \frac{N_{1B}}{N_1} = \left[1 + C_B\,(t - t_B)\right]^{\frac{2}{n-1}} + \frac{2\,A_B\left[1 - \left\{1 + C_B\,(t - t_B)\right\}^{\frac{n-3}{n-1}}\right]}{\left\{1 + C_B\,(t - t_B)\right\}^{\frac{n-5}{n-1}}\left[(n-1)\left\{1 + C_B\,(t - t_B)\right\}^{\frac{2}{n-1}} + (n-3)A_B\right]}$$

Hence

$$\frac{N_{1B}}{N_1} = \frac{\left\{J(t)\right\}^{\frac{n-5}{n-1}}\left[(n-1)\left\{J(t)\right\}^{\frac{2}{n-1}} + (n-3)A_B\right]}{(n-1)J(t) + 2A_B + (n-5)A_B\left\{J(t)\right\}^{\frac{n-3}{n-1}}} \quad .. \quad (75)$$

with

$$J(t) = 1 + C_B(t - t_B) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (75A)$$

$$\therefore \frac{N_1}{N_{1B}} = J^{-\frac{2}{n-1}}\left[1 + \frac{2A_B}{(n-1)J^{\frac{2}{n-1}}}\left\{1 - J^{\frac{3-n}{n-1}}\right\}\right]$$

$$\text{or } \frac{N_1}{N_{1B}} = \left\{1 + C_B(t - t_B)\right\}^{-\frac{2}{n-1}}\left[1 + \frac{2A_B}{(n-1)\left\{1+C_B(t-t_B)\right\}^{\frac{2}{n-1}}} \times \right.$$

$$\left.\left\{1 - \left\{1+C_B(t-t_B)\right\}^{\frac{3-n}{n-1}}\right\}\right] \quad .. \quad .. \quad (76)$$

(neglecting squares and higher powers of A_B).

Therefore we have from (67) and (76),

$$\frac{T_1}{T_{1_B}} = \left[J^{-\frac{2}{n-1}}\left\{1 + \frac{2A_B}{(n-1)J^{\frac{2}{n-1}}}\left(1 - J^{\frac{3-n}{n-1}}\right)\right\}\right]^{n-1} \times$$

$$\left[1 - \eta\rho'_B(n-1)\left[1 - \left\{J^{-\frac{2}{n-1}}\left(1 + \frac{2A_B}{(n-1)J^{\frac{2}{n-1}}}\left(1 - J^{\frac{3-n}{n-1}}\right)\right)\right\}\right]\right]$$

$$\text{or } \frac{T_1}{T_{1B}} = J^{-2}\left[1 + \frac{2A_B}{J^{\frac{2}{n-1}}}\left(1 - J^{\frac{3-n}{n-1}}\right) - \eta\rho'_B(n-1)\left(1 - J^{-\frac{2}{n-1}}\right)\right] \quad (77)$$

(neglecting squares and higher powers of η, A_B and also the multiple of η and A_B and its higher powers).

$$\text{or } \frac{T_1}{T_{1B}} = \left\{1 + C_B\,(t - t_B)\right\}^{-2} \times$$

$$\left\{1 + \frac{2 A_B}{\left\{1 + C_B (t - t_B)\right\}^{\frac{2}{n-1}}} \left[1 - \left\{1 + C_B (t - t_B)\right\}^{\frac{3-n}{n-1}}\right] - \eta\, \rho'_B\,(n-1)\left[1 - \left\{1 + C_B (t - t_B)\right\}^{-\frac{2}{n-1}}\right]\right\} \quad (78)$$

Again from (64), (67) and (69), we get

$$\frac{dN_2}{dt} = \frac{\Psi'\, S_2^*}{\sqrt{R}\ \bar{\omega}}\ \frac{P_B}{\sqrt{T_{1B}}} \times \frac{\left[1 - \left(1 - \frac{N_1}{N_{1B}}\right) \eta\, N_{1B} \left\{\frac{\bar{\omega}}{C^*} + (n-1)\frac{\rho'_B}{N_{1B}}\right\}\right]\left(\frac{N_1}{N_{1B}}\right)^n}{\left(\frac{N_1}{N_{1B}}\right)^{\frac{n-1}{2}}\left[1 - \eta\,\rho'_B\left(1 - \frac{N_1}{N_{1B}}\right)\right]^{\frac{1}{2}}}$$

$$\text{or } \frac{dN_2}{dt} = \frac{\Psi'\, S_2^*}{\sqrt{R}\ \bar{\omega}}\ \frac{P_B}{\sqrt{T_{1B}}}\left(\frac{N_1}{N_{1B}}\right)^{\frac{n+1}{2}} \times \left[1 + \eta\left\{\frac{\rho'_B}{2}\,(3 - 2n) - N_{1B}\,\frac{\bar{\omega}}{C^*}\right\}\left(1 - \frac{N_1}{N_{1B}}\right)\right] \quad .. \quad (79)$$

(neglecting squares and higher powers of η).

$$\text{or } \frac{dN_2}{dt} = U_1\left[J^{-\frac{2}{n-1}}\left\{1 + \frac{2 A_B}{(n-1)\, J^{\frac{2}{n-1}}} \times \left(1 - J^{\frac{3-n}{n-1}}\right)\right\}\right]^{\frac{n+1}{2}}\left[1 + \eta\, U_2\left(1 - \frac{N_1}{N_{1B}}\right)\right] \quad .. \quad (80)$$

with

$$U_1 = \frac{\Psi'\, S_2^*\, P_B}{\sqrt{R}\ \bar{\omega}\ \sqrt{T_{1B}}} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (80A)$$

$$U_2 = \left\{\frac{\rho'_B\,(3 - 2n)}{2} - N_{1B}\,\frac{\bar{\omega}}{C^*}\right\} \quad .. \quad .. \quad .. \quad (80B)$$

$$\text{or}\quad \frac{dN_2}{dt} = U_1 J^{-\frac{n+1}{n-1}} \left[1 + \eta\, U_2 \left\{1 - J^{-\frac{2}{n-1}}\right\} + \frac{A_B (n+1)}{(n-1)\, J^{\frac{2}{n-1}}} \left(1 - J^{\frac{3-n}{n-1}}\right)\right] \quad \ldots \quad \ldots \quad (81)$$

(neglecting $A_B \,.\, \eta$ and squares and higher powers of A_B and $A_B,\ \eta$).

$$\text{or}\quad \frac{dN_2}{dt} = U_1 \left[\left\{1 + C_B (t - t_B)\right\}^{-\frac{n+1}{n-1}} + \eta\, U_2 \left[\left\{1 + C_B (t - t_B)\right\}^{-\frac{n+1}{n-1}} - \left\{1 + C_B (t - t_B)\right\}^{-\frac{n+3}{n-1}}\right] + \frac{A_B (n+1)}{(n-1)} \left[\left\{1 + C_B (t - t_B)\right\}^{-\frac{n+3}{n-1}} - \left\{1 + C_B (t - t_B)\right\}^{-\frac{2n}{n-1}}\right] \right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

Integrating (82), we get

$$N_2 = \frac{U_1}{C_B} \left[-\frac{(n-1)}{2}\left\{1 + C_B (t - t_B)\right\}^{-\frac{2}{n-1}} + \eta U_2 \left[-\frac{(n-1)}{2}\left\{1 + C_B (t - t_B)\right\}^{-\frac{2}{n-1}} + \frac{n-1}{4}\left\{1 + C_B (t - t_B)\right\}^{-\frac{4}{(n-1)}}\right] + \frac{A_B (n+1)}{(n-1)} \left[-\frac{(n-1)\left\{1 + C_B (t - t_B)\right\}^{-\frac{4}{n-1}}}{4} + \frac{(n-1)\left\{1 + C_B (t - t_B)\right\}^{-\frac{n+1}{n-1}}}{(n+1)}\right] \right] + \text{constant} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83)$$

When $t = t_B$, $\quad N_2 = N_{2B}$.

$\therefore$ we have from (83),

$$N_2 = \frac{U_1(n-1)}{2C_B}\Bigg[-\Big\{1+C_B(t-t_B)\Big\}^{-\frac{2}{n-1}}$$

$$+\ \eta U_2\Bigg[-\Big\{1+C_B(t-t_B)\Big\}^{-\frac{2}{n-1}}+\tfrac{1}{2}\Big\{1+C_B(t-t_B)\Big\}^{-\frac{4}{n-1}}\Bigg]$$

$$+\ \frac{A_B(n+1)}{(n-1)}\Bigg[-\tfrac{1}{2}\Big\{1+C_B(t-t_B)\Big\}^{-\frac{4}{n-1}}$$

$$+\ \frac{2}{(n+1)}\Big\{1+C_B(t-t_B)\Big\}^{-\frac{n+1}{n-1}}\Bigg]\Bigg]$$

$$+\ N_{2B}-\frac{U_1}{2C_B}(n-1)\Bigg[-1-\frac{\eta U_2}{2}-\frac{A_B}{2}\cdot\frac{(n-3)}{(n-1)}\Bigg] \quad \ldots \quad (84)$$

Again from (60), (5) and (58), we get

$$\frac{d}{dt}(N_2T_2) = -\frac{(n-1)\mu}{\bar{\omega}R}\cdot\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}+\Bigg[1+\frac{n-1}{1-\eta\rho'_B\dfrac{N_1}{N_{1B}}}\Bigg]T_1\frac{dN_2}{dt} \quad \ldots \quad (85)$$

$\therefore$ integrating, we get

$$N_2T_2 - N_{2B}T_{2B} = -\frac{(n-1)\mu}{2\bar{\omega}R}\Bigg[\left(\frac{dx}{dt}\right)^2-\left(\frac{dx}{dt}\right)^2_B\Bigg]$$

$$+\int_{t_B}^{t}\Bigg(1+\frac{n-1}{1-\eta\rho'_B\dfrac{N_1}{N_{1B}}}\Bigg)T_1\frac{dN_2}{dt}\cdot dt \quad \ldots \quad (86)$$

But from (67), we get

$$\int_{t_B}^{t}\Bigg(1+\frac{(n-1)}{1-\eta\rho'_B\dfrac{N_1}{N_{1B}}}\Bigg)T_1\frac{dN_2}{dt}\,dt$$

$$=\int_{t_B}^{t}\Bigg[1+(n-1)\left(1+\eta\rho'_B\frac{N_1}{N_{1B}}\right)\Bigg]T_{1B}\left(\frac{N_1}{N_{1B}}\right)^n \times \frac{1}{\Big[1-\eta\rho'_B(n-1)\left(1-\dfrac{N_1}{N_{1B}}\right)\Big]}\frac{dN_2}{dt}\cdot dt.$$

$$= \int_{t_B}^{t} \left[n - n(n-1)\eta \rho'_B + \eta \rho'_B (n^2 - 1) \frac{N_1}{N_{1B}} \right] \times$$

$$\left(\frac{N_1}{N_{1B}} \right)^{n-1} \frac{dN_2}{dt} . dt.$$

(neglecting squares and higher powers of η).

$$\text{or} \int_{t_B}^{t} \left(1 + \frac{n-1}{1 - \eta \rho'_B \frac{N_1}{N_{1B}}} \right) T_1 \frac{dN_2}{dt} . dt$$

$$= T_{1B} U_1 \int_{t_B}^{t} \Bigg[n - n(n-1)\eta \rho'_B + \eta \rho'_B (n^2 - 1) J^{-\frac{2}{n-1}} \times$$

$$\left\{ 1 + \frac{2A_B}{(n-1) J^{\frac{2}{n-1}}} \left(1 - J^{\frac{3-n}{n-1}} \right) \right\} \Bigg]$$

$$J^{-2} \left\{ 1 + \frac{2A_B}{(n-1) J^{\frac{2}{n-1}}} \left(1 - J^{\frac{3-n}{n-1}} \right) \right\}^{n-1} \times J^{-\frac{n+1}{n-1}} \times$$

$$\left[1 + \eta U_2 \left(1 - J^{-\frac{2}{n-1}} \right) + \frac{A_B (n+1)}{(n-1) J^{\frac{2}{n-1}}} \left(1 - J^{\frac{3-n}{n-1}} \right) \right] dt$$

$$= T_{1B} U_1 \int_{t_B}^{t} \Bigg[\left\{ n - n(n-1)\eta \rho'_B + n\eta U_2 \right\} J^{-\frac{3n-1}{n-1}} + \Bigg\{ \eta \rho'_B (n^2 - 1)$$

$$+ \frac{n(3n-1)}{(n-1)} A_B - n\eta U_2 \Bigg\} J^{-\frac{3n+1}{n-1}} - \frac{n(3n-1)}{(n-1)} A_B \, J^{\frac{2-4n}{n-1}} \Bigg] dt$$

$$= T_{1B} U_1 \left[C_1 J^{-\frac{2n}{n-1}} + C_2 J^{-\frac{2(n+1)}{(n-1)}} + C_3 J^{-\frac{(3n-1)}{(n-1)}} \right]_{t_B}^{t}$$

$$= T_{1B} U_1 \left[C_1 J^{-\frac{2n}{n-1}} + C_2 J^{-\frac{2(n+1)}{(n-1)}} + C_3 J^{\frac{-(3n-1)}{(n-1)}} - C_1 - C_2 - C_3 \right] \quad \ldots \ldots \ldots \ldots (87)$$

where

$$C_1 = \frac{(n-1)\left\{ (n-1)\eta \rho'_B - 1 - \eta U_2 \right\}}{2C_B} \quad \ldots \ldots \ldots (87A)$$

$$C_2 = \frac{(n-1)\left\{n\eta U_2 - \eta\rho'_B(n^2-1) - n\left(\frac{3n-1}{n-1}\right)A_B\right\}}{2(n+1)C_B} \quad \text{.. (87B)}$$

$$C_3 = \frac{nA_B}{C_B} \quad \text{.. .. .. (87C)}$$

Hence from (86), we have

$$N_2T_2 - N_{2B}T_{2B} = -\frac{(n-1)\mu}{2\varpi R}\left[\left(\frac{dx}{dt}\right)^2 - \left(\frac{dx}{dt}\right)_B^2\right]$$

$$+ T_{1B}U_1\left[C_1\left\{1 + C_B(t-t_B)\right\}^{-\frac{2n}{n-1}} + C_2\left\{1 + C_B(t-t_B)\right\}^{-\frac{2(n+1)}{(n-1)}}\right.$$

$$\left. + C_3\left\{1 + C_B(t-t_B)\right\}^{-\frac{3n-1}{n-1}} - C_1 - C_2 - C_3\right] \quad \text{.. .. .. .. (88)}$$

where C_1, C_2, C_3 are given by (87A), (87B) and (87C).

From (65), (62), (84) and (88) we have on neglecting squares and higher powers of η and multiple of A_B, η,

$$\frac{\mu}{\varpi\,\sigma\,R}\,\frac{d^2x}{dt^2}\left[C' + \sigma x - \varpi\,\eta\left[-\frac{U_1(n-1)}{2C_B}\left\{1 + C_B(t-t_B)\right\}^{-\frac{2}{n-1}}\right.\right.$$

$$\left.\left. + N_{2B} + \frac{U_1(n-1)}{2C_B}\right]\right] - N_{2B}\,T_{2B}.$$

$$= -\frac{(n-1)\,\mu}{2\,\varpi\,R}\left(\frac{dx}{dt}\right)^2 + \frac{(n-1)\,\mu}{2\,\varpi\,R}\left(\frac{dx}{dt}\right)_B^2 + T_{1B}U_1 \times$$

$$\left[C_1\left\{1 + C_B(t-t_B)\right\}^{-\frac{2n}{n-1}} + C_2\left\{1 + C_B(t-t_B)\right\}^{-\frac{2(n+1)}{(n-1)}}\right.$$

$$\left. + C_3\left\{1 + C_B(t-t_B)\right\}^{-\frac{3n-1}{n-1}} - C_1 - C_2 - C_3\right] \quad \text{.. (89)}$$

where U_1, C_1, C_2 and C_3 are given by (80A), (87A), (87B) and (87C).

The equation (89) is an ordinary differential equation of second order and will give us x and $\frac{dx}{dt}$ as functions of t.

However if η is neglected, the equations (76), (78), (84), (88) and (89) become

$$N_1 = N_{1B}\left\{1 + C_B(t-t_B)\right\}^{-\frac{2}{n-1}} \quad \text{.. .. .. (90)}$$

$$T_1 = T_{1B} \left\{ 1 + C_B (t - t_B) \right\}^{-2} \quad \ldots \quad (91)$$

$$N_2 = N_{2B} + \frac{U_1 (n-1)}{2 C_B} \left[1 - \left\{ 1 + C_B (t - t_B) \right\}^{-\frac{2}{n-1}} \right] \quad (92)$$

$$N_2 T_2 = N_{2B} T_{2B} - \frac{(\bar{n}-1)\mu}{2 \varpi R} \left[\left(\frac{dx}{dt}\right)^2 - \left(\frac{dx}{dt}\right)_B^2 \right]$$

$$+ \frac{T_{1B} U_1 (n-1)}{2 C_B} \left[1 - \left\{ 1 + C_B (t - t_B) \right\}^{-\frac{2n}{n-1}} \right] \quad (93)$$

$$\frac{\mu}{\varpi \sigma R} \frac{d^2 x}{dt^2} \left[C' + \sigma x \right] - N_{2B} T_{2B}$$

$$= - \frac{(\bar{n}-1)\mu}{2 \varpi R} \left(\frac{dx}{dt}\right)^2 + \frac{(\bar{n}-1)\mu}{2 \varpi R} \left(\frac{dx}{dt}\right)_B^2$$

$$+ \frac{T_{1B} U_1 (n-1)}{2 C_B} \left[1 - \left\{ 1 + C_B (t - t_B) \right\}^{-\frac{2n}{n-1}} \right] \quad \ldots \quad (94)$$

These equations (90), (91), (92), (93) and (94) can easily be reduced to (50), (46), (54), (55) and (57) respectively of Kapur [7]*.

If we put $\Psi_1 = 0$, $\Psi_2 = \Psi$, $N_1 = N$, and $N_2 = 1 - N$ in the equations obtained here for recoilless high-low pressure guns, we get the corresponding results for high-low pressure guns as deduced by Gupta [10].

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* There is some mistake due to calculation in equation (55) of Kapur[7]. It should be

$$N_2 T_2 - N_{2B} T_{2B} = - \frac{\bar{\gamma} - 1}{2 C R} W_2 \left\{ \left(\frac{dx}{dt}\right)^2 - \left(\frac{dx}{dt}\right)_B^2 \right\}$$

$$+ \frac{\Psi S_2}{C \sqrt{R}} \sqrt{T_{1B}}\, p_{1B} \frac{\theta_1 (\bar{\gamma} - 1)}{2} \left[1 - \left\{ 1 + \left(\frac{t - t_B}{\theta_1} \right) \right\} \right]^{-\frac{2\gamma}{\gamma - 1}}$$

and hence the equations (56) and (57) of Kapur [7] also require corrections.

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