USE OF GENERALIZED BIRTH AND DEATH PROCESS IN SOLVING QUEUING PROBLEMS

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ABSTRACT

The general procedure for determining the fluctuations in different phases of a population in which the changes take place either as functions of time or some parameter of the population is considered in this paper. It is indicated that the solutions for most of the queuing problems can be obtained as special cases of the above procedure. Following this, the exact solutions for stable or unstable conditions arising for some simple situations have been obtained.

Introduction

The problem of birth and death process has been considered in some great detail by Physicists, Statisticians and Mathematicians. Notable contributions on this topic have been made during the past two decades by Lotka[1], Feller[2], Arley[3], Moyal[4], Bartlett[5], Kendall[6] and others[7]. The main purpose of this note is to indicate briefly that the solutions of many of the queuing problems can be obtained as special cases of the generalized birth and death process referred to above.

Some Fundamental Results

(*a*) *One dimensional process*—Let multiplication and death in a given population take place according to the following rule :—

(*i*) The populations generated by two co-existing individuals develop completely independent of one another.

(*ii*) An individual existing at time t has a chance $\lambda dt + 0\ (dt)$ of multiplying and $\mu dt + 0\ (dt)$ of dying during the interval t and $t + dt$.

(*iii*) The birth and death rates are the same in the population for any value of t.

Assuming $p_n\ (t)$ to be the probability that the size of the population at time t is n, it can be seen that

$$p_n\ (t + \delta t) = \{1 - n\ (\lambda + \mu)\ \delta t\ \} p_n\ (t) + (n - 1)\ \lambda \delta t\ p_{n-1}(t) + (n + 1)\ \mu \delta t\ p_{n+1}(t) \quad \ldots \quad (1)$$

Representing $p_n\ (t)$ by p_n and proceeding to the limit $\delta t \geqslant 0$ equation (1) reduces to

$$\frac{dp_n}{dt} = -(\lambda + \mu)\ np_n + (n - 1)\ \lambda\ p_{n-1} + (n + 1)\ \mu p_{n+1} \quad \ldots \quad (2)$$

Defining $\phi(z) = \sum_{n=0}^{\infty} z^n p_n(t)$ .. .. (3)

as the probability generating function (p.g.f.) for p_n it can be seen that equation (2) reduces to

$$\frac{d\phi}{dt} = (\lambda z - \mu)(z-1)\frac{d\phi}{dz} \quad .. \quad .. \quad .. \quad .. \quad (4)$$

The solution of (4) has been obtained by Kendall (1948) and is given below for reference

$$p_o(t) = \xi_t\,, \quad p_n(t) = \left\{1 - p_o(t)\right\}(1-\eta_t)\,\eta_t^{\,n-1} \quad .. \quad .. \quad (5)$$

where ξ_t and η_t are functions of t and are given by

$$\xi_t = 1 - \frac{e^{-\rho}}{W}\,, \quad \eta_t = 1 - \frac{8}{W}$$

$$W = \tfrac{1}{2}(1+e^{-\rho}) + \tfrac{1}{2}e^{-\rho}\int_o^t e^{\rho(\tau)}\left\{\lambda(\tau) + \mu(\tau)\right\} dT$$

$$\text{and} \quad P(t) = \int_o^t \left\{\mu(\tau) - \lambda(\tau)\right\} dT$$

As regards the first two moments of the distribution, it can be deduced from equation (2) and leads to the following differential equations

$$\left.\begin{aligned} \frac{d\bar{n}}{dt} &= (\lambda - \mu)\,\bar{n} \\ \frac{d\overline{n^2}}{dt} &= 2(\lambda - \mu)\,\overline{n^2} + (\lambda + \mu)\,\bar{n} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad (6)$$

where $\bar{n}$ and $\overline{n^2}$ are the expected values of n and n^2 respectively. Solving (6) we note

$$\bar{n} = e^{\int(\lambda-\mu)\,dt} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (7)$$

$$V(n_t) = e^{-\rho}\left\{1 - e^{-\rho} + 2e^{-\rho}\int_o^t e^{\rho(T)}\lambda(\tau)\,dT\right\} \quad .. \quad .. \quad (8)$$

In the derivation of (7) the initial population is taken to be unity and λ and μ are taken to be functions of t.

When λ and μ are constants.

$$\bar{n} = e^{(\lambda-\mu)t}, \; V(\eta_t) = \frac{\lambda+\mu}{\lambda-\mu}\, e^{(\lambda-\mu)t}\left\{e^{(\lambda-\mu)t} - 1\right\} \quad (9)$$

For $\lambda = \mu$, $\bar{n} = 1$, $V(\eta_t) = 2\lambda t$ .. .. (10)

(b) *Two dimensional process*—Suppose a process involves two kinds of populations say, *A* and *B*, such that when *A* or *B* dies they produce k individuals of the opposite kind with probability $\lambda\, dt$. Further assume that the chance that any of the individuals die without producing anything is $\mu\, dt$.

Taking $p(n, m; t)$ to be the probability that at time t, the number of A's and B's in the population is n and m respectively, it can be seen that

$$\frac{dp(n, m)}{dt} = (n+1)\,\mu\, p(n+1, m) + (m+1)\,\mu\, p(n, m+1) - (n+m) \times (\lambda+\mu)\; p(n, m) + (n+1)\,\lambda\, p(n+1, m-k) + (m+1) \times \lambda\, p(n-k, m+1) \qquad (11)$$

Defining as before, $\phi(z_1, z_2) = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} p(n, m, t)\, z_1^n\, z_2^m$ as the joint probability generating fnnction for $p(n, m)$ we note

$$\frac{\partial\phi}{\partial t} = \left\{ \mu - (\lambda+\mu)\; z_1 + \lambda\, z_2^k \right\} \frac{\partial\phi}{\partial z_1} + \left\{ \mu - (\lambda+\mu)\, z_2 + \lambda\, z_1^k \right\} \frac{\partial\phi}{\partial z_2} \qquad (12)$$

The solution of this equation cannot be obtained so readily. However the first two moments can be evaluated from the following differential equations obtained from (11)

$$\left.\begin{aligned} \frac{\partial \bar{n}}{\partial t} &= -(\lambda+\mu)\,\bar{n} + k\lambda\bar{m}, \\ \frac{\partial \bar{m}}{\partial t} &= -(\lambda+\mu)\;\bar{m} + k\lambda\,\bar{n}, \end{aligned}\right\} \qquad (13)$$

$$\left.\begin{aligned} \frac{\partial \overline{n^2}}{\partial t} &= -2(\lambda+\mu)\,\overline{n^2} + 2k\lambda\,\overline{nm} + (\lambda+\mu)\,\bar{n} + k^2\lambda\,\bar{m}; \\ \frac{\partial \overline{m^2}}{\partial t} &= -2(\lambda+\mu)\,\overline{m^2} + 2\,k\lambda\,\overline{nm} + (\lambda+\mu)\,\bar{m} + k^2\lambda\,\bar{n}; \\ \frac{\partial \overline{nm}}{\partial t} &= -2(\lambda+\mu)\,\overline{nm} + k\lambda\,\overline{n^2} + k\lambda\,\overline{m^2} - k\lambda\,\bar{n} - k\lambda\,\bar{m}; \end{aligned}\right\} \qquad (14)$$

From (13) we get

$$\bar{n} = \alpha e^{-(\lambda+\mu-k\lambda)t} + \beta e^{-(\lambda+\mu+k\lambda)t} \qquad (15)$$

The initial conditions $\bar{n}(o) = o$ and $\bar{m}(o) = o$ give $\alpha = \beta = \frac{1}{2}$.

$$\text{Thus } \bar{n} = \tfrac{1}{2}\left(e^{-at} + e^{-bt}\right)$$

$$\bar{m} = \tfrac{1}{2}\left(e^{-at} - e^{-bt}\right) \qquad (16)$$

where $a = \lambda(1-k) + \mu$; $b = \lambda(1+k) + \mu$.

As regards the evaluation of the variance and covariance of n and m, following Arley, we note from (14) that

$$\frac{d\,\overline{(n+m)^2}}{dt} = -2a\,\overline{(n+m)^2} + \left\{ \lambda + \mu + k\lambda\,(k-2) \right\} \overline{n+m} \quad (17)$$

Solving (17),

$$\overline{(n+m)^2} = \frac{e^{-at}}{a} \left\{ \lambda + \mu + k\lambda\,(k-2) + k\lambda\,(1-k)\,e^{-at} \right\} \quad (18)$$

Substituting (16) and (18) in (14) and solving the resulting equations, we obtain

$$\overline{} = \xi + \frac{p}{2b}\,e^{-at} + \frac{qe^{-bt}}{2a} + \left(1 - \eta - \frac{pa+qb}{2ab}\right) e^{-ct};$$

$$\overline{m^2} = \xi + \frac{p}{2b}\,e^{-at} - \frac{qe^{-bt}}{2a} + \left(- \quad - \frac{pa-qb}{2ab}\right) e^{-ct};$$

$$\overline{mn} = \frac{(k\lambda)^2\,(k-1)}{a} \left\{ \frac{e^{-at}}{2b-a} - \frac{e^{-2at}}{2\,(b-a)} + \frac{ae^{-2bt}}{2\,(b-a)\,(2b-a)} \right\}$$

where $c = 2\,(\lambda+\mu)$; $p = \lambda + \mu + k^2\,\lambda$; $q = \lambda + \mu - k^2\lambda$;

$$\xi = k\lambda\,\frac{(k-1)}{a} \left\{ \frac{2\,k^2\,\lambda^2\,e^{-at}}{b\,(2b-a)} - \frac{e^{-2at}}{4} - \frac{ae^{-2bt}}{4\,(2b-a)} \right\};$$

$$\eta = \frac{(k\lambda)^2\,(k-1)}{a\,(2b-a)} \left(\frac{2k\lambda}{b} - \frac{b}{b-a} \right) \quad \ldots \quad \ldots \quad (19)$$

The variances and covariances for n and m can be calculated after making the necessary corrections.

Queuing problems in relation to Birth and Death Process

We shall now examine the solutions discussed above with reference to some simple queuing problems. Taking the simplest case where the queue is formed and cleared with probabilities λdt and $\mu\,dt$ respectively, the differential equation corresponding to (1) for the probability $p_n\,(t)$ for the length of the queue to be n at time t is given by

$$\frac{dpn}{dt} = -(\lambda+\mu)\,p_n + \lambda\;p_{n-1} + \mu p_{n+1} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

In terms of the $p.\,g.\,f.\ \phi$, this reduces to the form

$$\frac{d\phi}{dt} = (\lambda - \mu/z)\,(z-1)\,\phi \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$\phi = 2e^{(\lambda-\mu/z)\,(z-1)t} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

It has been assumed that when $t = o$, the size of the queue is 1.

The expected value and the variance of the length of the queue are

$$\left.\begin{aligned} \overline{n} &= (\lambda-\mu)t + 1 \\ V\,(n) &= (\lambda+\mu)t \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Suppose we are concerned with the length of the queue and also the number of people served in a system where the probability of arrival is λdt and the probability for service is μdt. Let $p(n_1, n_2\, t)$ be the probability that there are n_1, people in the queue and n_2 under service at time t. The differential equation for this state is given by

$$\begin{aligned} p(n_1, n_2, t + dt) &= \left\{ 1 - (\lambda + 2\mu)\, dt \right\} P(n_1, n_2, t) \\ &+ \lambda dt\, p(n_1 - 1, n_2, t) + \mu dt\, p(n_1 + 1, n_2 - 1, t) \\ &+ \mu dt\, p(n_1, n_2 + 1, t) \qquad \ldots \quad (24) \end{aligned}$$

$$\frac{dp(n_1, n_2)}{dt} = -(\lambda + 2\mu)\, p(n_1, n_2) + \lambda\, p(n_1 - 1, n_2) + \mu\, p(n_1 + 1, n_2 - 1) + \mu\, p(n_1, n_2 + 1) \qquad \ldots \quad (25)$$

In terms of *p.g.f.*, the above equation reduces to

$$\frac{d\phi}{dt} = \left\{ -(\lambda + 2\mu) + \lambda z_1 + \mu/z_2 + \frac{\mu z_2}{z_1} \right\} \phi \qquad \ldots \quad (26)$$

$$\therefore \quad \phi = k e^{-\left\{ (\lambda + 2\mu) + \lambda z_1 + \mu/z_2 + \frac{\mu z_2}{z_1} \right\} t} \qquad \ldots \quad (27)$$

Suppose when $t = o$, $n_1 = 1$ and $n_2 = o$

$$\phi = z_1\, e^{-\left\{ (\lambda + 2\mu) + \lambda z_1 + \mu/z_2 + \frac{\mu z_2}{z_1} \right\} t} \qquad \ldots \quad (28)$$

$$\left. \begin{aligned} \bar{n}_1 &= (\lambda - \mu)t + 1 \\ \bar{n}_2 &= o \end{aligned} \right\} \qquad \ldots \quad (29)$$

$$\left. \begin{aligned} V(n_1) &= (\lambda + \mu)t \\ V(n_2) &= \mu t \\ Cov(n_1, n_2) &= -\mu t \end{aligned} \right\} \qquad \ldots \quad (30)$$

For stationary conditions we assume that $\frac{dp_n}{dt} = o$ for different values of n. The solutions of these have been considered by Feller[2] Benson & Cox[8] and others[9].

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