STUDY OF INTERNAL BALLISTICS OF HEPTA-TUBULAR POWDERS

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ABSTRACT

In this paper a solution of the system of equations in Internal Ballistics of a conventional gun has been discussed for the hepta-tubular powder. The shot-start pressure is taken different from zero and the covolume correction also is taken into account and thus some of the results of Tavernier[1] who has taken zero shot-start pressure and has neglected the covolume correction, have been generalised. It has been found out that the maximum pressure in the case of hepta-tubular powders always occurs before the all-burnt position.

Introduction

In the present paper an attempt has been made to present the solution of the system of equations in Internal Ballistics of a conventional gun by using hepta-tubular powders in a modified form over that given by Tavernier[1]. Tavernier has assumed that the shot-start pressure is zero and the covolume is equal to the specific volume. But in the present paper the shot-start pressure is not neglected, and covolume correction is also taken into account. As shown by Tavernier[2] in the case of hepta-tubular powders the form function $\varphi(z)$ before the rupture of the grains takes the form

$$\varphi(z)=\alpha-\beta y-\gamma y^2,$$

where y is the fraction of thickness remaining at time 't', and α, β, γ are constants depending on the characteristics of the hepta-tubular powders used. Also in this case the form function when expressed as a relation between z and y is

$$z = (1-y)(a-by-cy^2),$$

which is the most general cubic form-function and therefore includes the cubic form function discussed by Venkatesan[3] when

$$a = 1, b = -\theta, \text{ and } c = -\psi.$$

After the rupture of the grains, these form functions take very complicated forms in terms of a certain angle ω which in turn depends on y.

However when $a=1$, there is no second phase of combustion, the point of rupture coinciding with the point of all-burnt. Hence the case considered in this paper includes as stated above the cubic form function

$$z = (1-y)(1+\theta y+\theta' y^2)$$

and all its particular cases, the quadratic form function

$$z = (1-y)(1+\theta y),$$

and the linear form function

$$z = (1-y).$$

Assumptions regarding combustion

The following assumptions regarding combustion have been made in this paper in order to get a tangible solution :—

(*i*) That all the grains of the charge are of the same geometrical shape, of the same dimensions and are homogeneous in character,

(*ii*) That burning takes place in parallel layers,

(*iii*) That the linear velocity of the combustion at any instant is proportional to the pressure at that instant.

Principal Notations

The following principal notations have been adopted in the present paper:—

μ = The fictitious mass of the projectile, taking into consideration the different passive resistance as well the differences of pressure between the base and the breech.

v = Velocity of the projectile.

σ = Cross section of the arm taking rifling into consideration.

P = Mean-pressure under which the powder burns.

z = Fraction of the powder burnt upto the time 't'.

y = Fraction of the thickness (web-size) of the powder remaining at time 't'.

f = The force of the powder.

n = Ratio of the two specific heats of gases.

$$\nu = \frac{n-1}{2}$$

$\bar{\omega}$ = Weight of the charge.

$$r = \frac{\nu \mu v^2}{f \bar{\omega}}$$

η = Covolume of the gases.

$\frac{1}{\delta}$ = Specific volume of the gases.

$$B = \frac{\left(\eta - \frac{1}{\delta}\right)\bar{\omega}}{\left(C' - \frac{\bar{\omega}}{\delta}\right)}$$

x = Shot-travel at any instant 't'.

C^* = Internal volume of the gun up to the base of the projectile at any instant t, $= C' + \sigma x$.

C' = Internal volume at the instant when $t = 0$.

D' = web-size.

A = Vivacity of the powder.

β' = Linear velocity of the combustion of the powder under unit pressure.

S_0 = Surface of the powder initially exposed to the combustion.

W_0 = Initial volume of the powder.

S = Surface of the combustion at the instant t.

$\varphi(z) = \frac{S}{S_0}$ = function of the progressivity of the powder.

The classical Equations of Internal Ballistics

The classical equations of Internal Ballistics after some modifications are given as :—

1. Resal's Equation of Energy :—

$$z - r = \frac{P}{f}\left[\left(\frac{c^*}{\bar{\omega}} - \frac{1}{\delta}\right) - \left(\eta - \frac{1}{\delta}\right)z\right] \quad \ldots \quad (1)$$

which reduces to equation (5) of Tavernier[1] when co-volume correction is neglected, i.e. when $\eta = \frac{1}{\delta}$ is taken.

2. Equation of Inertia :—

$$\mu \frac{dv}{dt} = \sigma P \quad \ldots \quad (2)$$

3. Equation of combustion :—

$$\frac{dz}{dt} = AP\varphi(z). \quad \ldots \quad (3)$$

Other Notations

Initially when $t = 0$, $z = z_0$, $y = y_0$, $P = P_0$, and $v = 0$.

For the sake of convenience in writing, we may introduce the following symbols as done by Tavernier[1].

$$P_0 = \frac{f\bar{\omega}}{C' - \frac{\bar{\omega}}{\delta}} \quad \ldots \quad (4)$$

$$V(z) = \int_{z_0}^{z} \frac{dz}{\varphi(z)} \quad \ldots \quad (5)$$

$$\xi = \frac{\nu\sigma^2}{A^2 f\bar{\omega}\mu} \quad \ldots \quad (6)$$

$$Y = \frac{C^* - \frac{\bar{\omega}}{\delta}}{C' - \frac{\omega}{\delta}} \quad \ldots \quad (7)$$

$$D(z) = \xi \int_{z_0}^{z} \frac{V}{\phi} \frac{dz}{z - \xi V^2} \quad \ldots \quad (8)$$

y = Fraction of the thickness of the powder remaining at time 't'.

m = Ratio of the exterior diameter of the grain to the interior diameter of the hole of the grain ($m \geqslant 3$).

ρ = Ratio of the length of the grain to the exterior diameter of the grain

$$\left(\rho > \frac{m-3}{4m}\right)$$

Case I:—During the first phase of the combustion of the hepta-tubular powder (before the rupture of the grains, i.e., before the web is consumed).

Following Tavernier[2], we have

$$z = (1-y)\,(a-by-cy^2) \qquad \text{(9)}$$

with

$$a = \frac{m-3}{8m\rho\,(m^2-7)}\left[7m^2\rho+19m\rho+\left(\frac{m+1}{2}\right)^2\right] \qquad \text{(9A)}$$

$$b = \frac{(m-3)^2}{16m\rho\,(m^2-7)}\left[6m\rho-5m-5\right] \qquad \text{(9B)}$$

$$c = \frac{3(m-3)^3}{32m\rho(m^2-7)} \qquad \text{(9C)}$$

$$a-b-c = \frac{m-3}{4m\rho\,(m^2-7)}\left[2m^2\rho+14m\rho+m^2-7\right] \qquad \text{(9D)}$$

$$\varphi(z) = \frac{S}{S_0} = \alpha-\beta y-\gamma y^2 \qquad \text{(10)}$$

Where

$$\alpha = \frac{a+b}{a-b-c} \qquad \text{(10A)}$$

$$\beta = \frac{2(b-c)}{a-b-c} \qquad \text{(10B)}$$

$$\gamma = \frac{3c}{a-b-c} \qquad \text{(10C)}$$

Case II:—During the second phase of the combustion (after the rupture of the grains i.e. after the web is consumed), we have

$$z = 1 - \frac{3(m+1)^2}{8\pi(m^2-7)}\left[1+\frac{1}{\rho m} - \frac{m+1}{4\rho m \,Cos\,\omega}\right]H(\omega) \qquad \text{(11)}$$

with

$$H(\omega)=F(\omega)+G(\omega) \qquad \text{(11A)}$$

$$F(\omega)=\sqrt{3}-3\tan\omega-3\left(\frac{\pi}{6}-\omega\right)sec^2\omega \qquad \text{(11B)}$$

$$G(\omega)=\left(\frac{\pi}{6}-\psi\right)(4-sec\omega)^2-\frac{\left[2\,sin\left(\frac{\pi}{3}+\omega\right)\right]\left[\sqrt{3}+tan\omega\right]}{sin\left(\frac{\pi}{3}+\omega+\psi\right)}\,sin\left(\frac{\pi}{6}-\psi\right)$$

$$-\chi\, sec^2\,\omega+\frac{2\,sin\,\psi\,sin\,\chi}{sin\left(\frac{\pi}{3}+\omega+\psi\right)Cos\,\omega} \quad .. \quad (11C)$$

$$Cos\,\psi=\frac{5\,Cos\,\omega-2}{4\,Cos\,\omega-1} \quad .. \quad (11D)$$

$$Cos\left(\frac{\pi}{3}+\omega+\chi\right)=2-3\,Cos\omega \quad .. \quad (11E)$$

where ω varies from 0° to 42° 25′ (the values corresponding to the rupture of the grains and the complete combustion respectively) and accordingly ψ varies from 0° to 30° and χ from 120° to 0° as given in the following table [2,4] :—

ω Degrees	ω Radians	χ Degrees	χ Radians	ψ Degrees	ψ Radians
0°	0·0000	120°	2·0944	0°·00′	0·00
5°	0·0873	106° 20′	1·8559	2° 54′	0·0506
10°	0·1745	92° 38′	1·6168	5° 50′	0·1018
15°	0·2618	78° 52′	1·3765	8° 51′	0·1545
20°	0·3491	65° 00′	1·1345	12° 00′	0·2094
25°	0·4363	50° 58′	0·8896	15° 21′	0·2679
30°	0·5236	36° 44′	0·6411	18° 59′	0·3314
32° 30′	0·5672	29° 31′	0·5152	20° 56′	0·3654
35°	0·6109	22° 13′	0·3878	23° 00′	0·4014
37° 30′	0·6545	14° 50′	0·2592	25° 11′	0·4395
40°	0·6981	7° 21′	0·1283	27° 32′	0·4805
42° 25′	0·7403	0° 00′	0·0000	30° 00′	0·5236

$$y=\frac{m+1}{m-3}(1-sec\,\omega) \quad .. \quad (12)$$

$$\varphi(z)=\frac{3(m+1)}{8\pi(2m^2\rho+14m\rho+m^2-7)}\left[4(\rho m+1)K(\omega)-(m+1)\,I(\omega)\right] \quad .. \quad (13)$$

with

$$I(\omega)=H'(\omega)\,Cot\,(\omega)+H(\omega) \quad .. \quad (13A)$$

$$K(\omega)=H'(\omega)\frac{Cos^2\omega}{sin\,\omega} \qquad \text{.. (13B)}$$

$H'(\omega)$ being the differential coefficient of $H(\omega)$ with respect to ω.

Also for *both phases of the combustion*, we have

$$V(z)=(a-b-c)(y_o-y). \qquad \text{.. (14)}$$

Clearly at the point of rupture of the grains of the powder

$$\left.\begin{array}{l} y=o,\ z=a\ ,\quad \varphi(z)=\alpha\ , \\ \qquad V(z)=(a-b-c)y_o \\ \omega=o, F(\omega)=\sqrt{3}-\frac{\pi}{2},\quad G(\omega)=\frac{5\pi}{6}-\sqrt{3}\ . \\ H(\omega)=\frac{\pi}{3},\ K(\omega)=-\frac{10\pi}{3}\ \text{and}\ I(\omega)=-3\pi\ . \end{array}\right\} \text{.. (15)}$$

At the end of combustion of the powder,

$$\omega=42^\circ\,25',\ z=1,\ H(\omega)=0.$$
$$H'(\omega)=0 \text{ and } \varphi(z)=0.$$

Also

$$y=y_{min}=-\frac{15-6\sqrt{3}}{13}.\frac{m+1}{m-3}\sim-.3545\frac{m+1}{m-3} \qquad \text{(16)}$$

and

$$V(z)=V(1)=(a-b-c)(y_o-y_{min}).$$

Solution of the Equations

On dividing (2) by (3), we get

$$\mu\frac{dv}{dz}=\frac{\sigma}{A\varphi(z)} \qquad \text{.. (17)}$$

$$\therefore\ v=\frac{\sigma}{A\mu}\int_a^z\frac{dz}{\varphi(z)}=\frac{\sigma}{A\mu}V(z) \qquad \text{.. (18)}$$

or

$$V(z)=\frac{A\mu}{\sigma}v \qquad \text{.. (19)}$$

so that

$$\gamma=\frac{\nu\mu v^2}{f\bar{\omega}}=\frac{\nu\mu\sigma^2}{f\bar{\omega}A^2\mu^2}V^2(z)$$
$$=\xi V^2. \qquad \text{.. (20)}$$

$\therefore$ the equation of energy (1) can be written as :—

$$z-\xi V^2(z)=\frac{P}{f}\left[\left(\frac{C^*}{\omega}-\frac{1}{\delta}\right)-\left(\eta-\frac{1}{\delta}\right)z\right] \qquad \text{.. (21)}$$

$$\text{or}\ \frac{P}{f}=\frac{z-\xi V^2}{\left(\frac{C^*}{\bar{\omega}}-\frac{1}{\delta}\right)-\left(\eta-\frac{1}{\delta}\right)z} \qquad \text{.. (21A)}$$

giving P as a function of z, z_o and shot-travel $x\left(=\frac{C^*-C'}{\sigma}\right)$

The equation (21) can also be written as :—

$$\frac{P}{f\omega} = \frac{z-\xi V^2}{\left(C^* - \frac{\bar{\omega}}{\delta}\right) - \left(\eta - \frac{1}{\delta}\right)\bar{\omega} z}$$

or
$$P = \frac{P_c\,(z-\xi V^2)}{Y-Bz} \qquad \ldots \quad (21A)$$

with

$$B = \frac{\left(\eta - \frac{1}{\delta}\right)\bar{\omega}}{\left(C' - \frac{\bar{\omega}}{\delta}\right)} \qquad \ldots \quad (21B)$$

From equation (2), we get

$$\mu v \frac{dv}{dx} = \sigma P.$$

Substituting the values of v and P from (18) and (21A), we get the following differential equation of the first order

$$\frac{dY}{dz} - \frac{\xi}{\nu}.\frac{V}{\varphi}\,\frac{1}{z-\xi V^2}\,Y = -\,\frac{\xi}{\nu}.\,\frac{V}{\varphi}\,\frac{Bz}{z-\xi V^2} \qquad \ldots \quad (22)$$

$$\text{or} \quad \frac{d}{dz}\Big[Y-Bz\Big] - \frac{\xi}{\nu}.\frac{V}{\varphi}.\frac{Y-Bz}{z-\xi V^2} = -B$$

The Integrating factor

$$H = e^{-\int_{z_o}^{z} \frac{\xi}{\nu}.\frac{V}{\varphi}.\frac{1}{z-\xi V^2}dz}$$

i.e.

$$\log H = -\frac{\xi}{\nu}\int_{z_o}^{z} \frac{V}{\varphi} \cdot \frac{1}{z-\xi V^2}\,dz \qquad \ldots \quad (23A)$$

$$= -\frac{D}{\nu}, \qquad \text{from (8)}$$

i.e.

$$H = e^{-D/\nu} \qquad \ldots \quad (23B)$$

∴ the solution of (22) is

$$\Big[(Y-Bz)H\Big]_{z_o}^{z} = -\,B\int_{z_o}^{z} H dz$$

since initially when $z = z_o$, $H = 1$ and $Y = 1$.

$$\therefore\ (Y-Bz)\,H - (1-Bz_o) = -\,BZ \qquad \ldots \quad (24)$$

where

$$Z = \int_{z_0}^{z} H\,dz \qquad \text{.. .. .. ..} \qquad (25)$$

$$\therefore\ Y - Bz = \frac{1 - B(Z + z_0)}{H} \qquad \text{.. .. .. ..} \qquad (26)$$

Also from (18) and (14), we get

$$\left.\begin{aligned} v &= \frac{\sigma}{A\mu} \cdot V(z) \\ &= \frac{\sigma}{A\mu}\,(a-b-c)(y_0 - y) \end{aligned}\right\} \qquad \text{.. .. ..} \qquad (27)$$

and from (21A) and (26)

$$\left.\begin{aligned} P &= \frac{P_c\,(z - \xi V^2)}{Y - Bz} \\ &= \frac{P_c\,(z - \xi V^2) H}{1 - B(Z + z_0)} \end{aligned}\right\} \qquad \text{.. .. .. ..} \qquad (28)$$

The equations (26), (27) and (28) give Y (defining the shot travel), velocity v and pressure P at any instant.

It is evident that H and Z are functions of z and therefore functions of y. These equations (26), (27) and (28) closely resemble with (17), (18) and (19) of Patni[5].

Initially when shot starts,

$$P_0 = \frac{P_c\, z_0}{1 - Bz_0} \qquad \text{.. .. ..} \qquad (29)$$

or

$$\begin{aligned} P_0 &= P_c\, z_0\, (1 - Bz_0)^{-1} \\ &= P_c\, z_0\, [1 + Bz_0 + B^2 z_0^2 + \ldots\ldots] \\ &= P_c\, z_0\, [1 + Bz_0] \quad \text{nearly} \end{aligned} \qquad \text{.. ..} \qquad (30)$$

and

$$\begin{aligned} z_0 &= \frac{P_0}{P_c}\left[1 + \frac{BP_0}{P_c}\right]^{-1} \\ &= \frac{P_0}{P_c} - B\left(\frac{P_0}{P_c}\right)^2 \quad \text{approximately} \end{aligned} \qquad \text{.. ..} \qquad (30A)$$

Again the pressure cannot be negative, also the quantity involving the covolume correction in (28) is small in comparison with Y, so that the quantity $(Y - Bz)$ is always positive and hence the equation (28) is valid only

if $\quad z - \xi\, V^2(z) \geqslant 0$

i.e. if

$$\xi \leqslant \frac{z}{V^2(z)}, \qquad \text{.. .. .. ..} \qquad (31)$$

for all values of z from 0 to 1.

Maximum Pressure

Denoting the values of the various variables at the point of maximum pressure by the suffix 'm', we have from equation (28),

$$P_m = \frac{P_c\,(z_m - \xi\,V^2_m)}{Y_m - Bz}$$

$$= \frac{P_c\,(z_m - \xi\,V^2_m)\,H_m}{1 - B\,(Z_m + z_o)} \qquad \text{.. .. .. (32)}$$

But at the point of maximum pressure, $dP = 0$.

Therefore differentiating (28) and putting $dP=0$, we get with the help of (22),

$$\xi\,\frac{V_m}{\varphi_m} = \frac{\nu}{2\nu+1}\left[1 + \frac{B\,(z_m - \xi\,V^2_m)\,H_m}{1 - B\,(Z_m + z_o)}\right] \qquad \text{.. .. (33)}$$

This equation when solved will give us the value z_m of z at which the maximum pressure occurs. For this purpose a graphical method can be used if necessary. This equation reduces to (46) of Tavernie[1] when $B=0$, *i.e.*, when covolume correction is neglected.

From (32) and (33) we also get

$$\xi\,\frac{V_m}{\varphi_m} = \frac{\nu}{2\nu+1}\left[1 + B\,\frac{P_m}{P_c}\right] \qquad \text{.. .. .. (34)}$$

Position of All-Burnt

At the position of complete combustion, $z=1$.

If at this position the values of the variables be denoted by using the suffix B, the equations (23 B), (26), (7), (27) and (28) give

$$H_B = e^{-D_B/\nu} \qquad \text{.. .. .. (35)}$$

with

$$D_B = \xi \int_{z_o}^{1} \frac{V\,dz}{\varphi\,.\,z\left(1 - \frac{\xi V^2}{z}\right)} \qquad \text{.. .. (35A)}$$

Also

$$Y_B - B = \frac{1 - B\,(Z_B + z_o)}{H_B} \qquad \text{.. .. .. .. (36)}$$

$$C^*_B = Y_B\left(C' - \frac{\bar{w}}{\delta}\right) + \frac{\bar{w}}{\delta} \qquad \text{.. .. .. (37)}$$

$$v_B = \frac{\sigma}{A\mu}\,V_B \qquad \text{.. .. .. (38)}$$

with

$$V_B = V(1) = \int_{z_o}^{1} \frac{dz}{\varphi(z)} \qquad \text{.. .. .. (38A)}$$

and

$$\left.\begin{aligned} P_B &= \frac{P_c\,[1-\xi V^2_B]}{[Y_B - B]} \\ &= \frac{P_c\,[1-\xi V^2_B]\,H_B}{1 - B\,(Z_B + z_0)} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

After the All-Burnt position

Since after the all-burnt position $z=1$, the equation (28) becomes

$$P = \frac{P_c\,(1-\xi V^2)}{Y - B} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

∴ from (39), we have

$$\frac{P}{P_B} = \left(\frac{1-\xi V^2}{1-\xi V^2_B}\right)\left(\frac{Y_B - B}{Y - B}\right) \quad \ldots \quad \ldots \quad (41)$$

Also as after the position of all-burnt, the expansion may be regarded as very nearly adiabatic, we have

$$P\left(\frac{C^*}{\omega} - \eta\right)^n = P_B\left(\frac{C^*_B}{\omega} - \eta\right)^n \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

i.e.

$$\frac{P}{P_B} = \left\{\frac{\dfrac{C^*_B}{\omega} - \eta}{\dfrac{C^*}{\omega} - \eta}\right\}^n = \left[\frac{Y_B - B}{Y - B}\right]^n \quad \ldots \quad \ldots \quad (43)$$

since

$$Y - B = \frac{\dfrac{C^*}{\omega} - \eta}{\dfrac{C'}{\omega} - \eta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43A)$$

∴ from (41) and (43), we have

$$\frac{1-\xi V^2}{1-\xi V^2_B} = \left[\frac{Y_B - B}{Y - B}\right]^{n-1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

Also from (6) and (19), we have

$$1 - \xi V^2 = 1 - \frac{\nu \mu}{f \omega}\, v^2 \quad \ldots \quad \ldots \quad (45)$$

∴ from (44), we have

$$1 - \frac{\nu \mu v^2}{f \omega} = (1 - \xi V^2_B)\left[\frac{Y_B - B}{Y - B}\right]^{n-1}$$

i.e.

$$v^2 = \frac{f \bar{\omega}}{\nu \mu}\left[1 - (1 - \xi V^2_B)\left(\frac{Y_B - B}{Y - B}\right)^{n-1}\right] \quad \ldots \quad (46)$$

Also from equation (40) and (44), we have

$$P = P_c \left(1 - \xi V^2_B\right) \left(\frac{Y_B - B}{Y - B}\right)^{n-1} \quad \ldots \quad \ldots \quad (47)$$

The equations (46) and (47) give the values of the velocity of the projectile and the pressure P at any instant after the all-burnt position.

In particular the muzzle velocity v_M is given as

$$v^2_M = \frac{f \bar{w}}{\nu \mu} \left[1 - (1 - \xi v^2_B) \left(\frac{Y_B - B}{Y_M - B} \right)^{n-1} \right] \quad \ldots \quad (48)$$

where suffix '$_M$' denotes the value of the variables at the muzzle.

Application to Hepta-Tubular Powders

In the case of Hepta-Tubular Powders, we have the following three cases:—

I. When the powder is burning before the rupture of the grains, *i.e.* during the first phase of combustion.

II. When the powder is burning after the rupture of the grains, *i.e.*, during the second phase of combustion.

III. After the all-burnt position, *i.e.*, during the phase of 'detente'.

Case I—In this case we get with the help of (8), (14), and (10) after simplification,

$$D(z) = \xi \int_{z_0}^{z} \frac{V(z)}{z - \xi V^2} \cdot \frac{dz}{\varphi(z)}$$

$$= -(a - b - c)^2 \xi$$

$$\int_{y_0}^{y} \frac{(y_0 - y)\, dy}{\left[\begin{array}{l} \{a - \xi (a - b - c)^2 y^2_0\} - y\, \{b + a - 2\xi (a - b - c)^2 y_0\} \\ + y^2 \{b - c - \xi (a - b - c)^2\} + c\, y^3 \end{array} \right]}$$

$$= -\frac{(a - b - c)^2}{c}\, \xi \int_{y_0}^{y} \frac{(y_0 - y)\, dy}{(A_1 - y)(A_2 + y)(A_3 - y)} \quad \ldots \quad (49)$$

with

$$A_2 - A_3 - A_1 = \frac{b - c - \xi (a - b - c)^2}{c} \quad \ldots \quad (49A)$$

$$A_2 A_3 + A_1 A_2 - A_1 A_3 = \frac{b + a - 2 \xi y_0 (a - b - c)^2}{c} \quad \ldots \quad (49B)$$

$$A_1 A_2 A_3 = \frac{a - \xi y^2_0 (a - b - c)^2}{c} \quad \ldots \quad \ldots \quad (49C)$$

$$\therefore D(z) = -\frac{(a - b - c)^2 \xi}{c} \times \left[\int_{y_0}^{y} \left(\frac{y_0 - A_1}{(A_2 + A_1)(A_3 - A_1)(A_1 - y)} + \frac{y_0 + A_2}{(A_1 + A_2)(A_3 + A_2)(A_2 + y)} + \frac{y_0 - A_3}{(A_1 - A_3)(A_2 + A_3)(A_3 - y)} \right) dy \right]$$

$$= - \frac{(a-b-c)^2 \xi}{c} \left[\log \left(\frac{A_1 - y}{A_1 - y_0} \right)^{\frac{A_1 - y_0}{(A_2 + A_1)(A_3 - A_1)}} \left(\frac{A_2 + y}{A_2 + y_0} \right)^{\frac{y_0 + A_2}{(A_1 + A_2)(A_3 + A_2)}} \left(\frac{A_3 - y}{A_3 - y_0} \right)^{\frac{A_3 - y_0}{(A_1 - A_3)(A_2 + A_3)}} \right] \quad \text{.. .. ..} \quad (50)$$

Hence ..

$$\log H = \frac{\xi (a-b-c)^2}{\nu c} \log \left[\left(\frac{A_1 - y}{A_1 - y_0} \right)^{\frac{A_1 - y_0}{(A_2 + A_1)(A_3 - A_1)}} \left(\frac{A_2 + y}{A_2 + y_0} \right)^{\frac{A_2 + y_0}{(A_1 + A_2)(A_3 + A_2)}} \left(\frac{A_3 - y}{A_3 - y_0} \right)^{\frac{A_3 - y_0}{(A_1 - A_3)(A_2 + A_3)}} \right] \quad \text{.. .. ..} \quad (51)$$

$$\text{Or } H = \lambda (A_1 - y)^L \; (A_2 + y)^M \; (A_3 - y)^N \quad \text{.. .. ..} \quad (52)$$

where

$$\lambda = (A_1 - y_0)^{-L} \; (A_2 + y_0)^{-M} \; (A_3 - y_0)^{-N} \quad \text{.. ..} \quad (52A)$$

$$L = \frac{A_1 - y_0}{(A_2 + A_1)(A_3 - A_1)} \cdot \frac{\xi}{\nu} \; \frac{(a-b-c)^2}{c} \quad \text{..} \quad (52B)$$

$$M = \frac{A_2 + y_0}{(A_1 + A_2)(A_3 + A_2)} \cdot \frac{\xi}{\nu} \; \frac{(a-b-c)^2}{c} \quad \text{..} \quad (52C)$$

$$N = \frac{A_3 - y_0}{(A_1 - A_3)(A_2 + A_3)} \cdot \frac{\xi}{\nu} \; \frac{(a-b-c)^2}{c} \quad \text{..} \quad (52D)$$

Also

$$Z = \int_{z_0}^{z} H dz = \lambda (a-b-c) \int_{y}^{y_0} (A_1 - y)^L \; (A_2 + y)^M \; (A_3 - y)^N \; (\alpha - \rho y - \gamma y^2) \, dy \quad \text{..} \quad (53)$$

which can be evaluated by numerical integration.

The equations (52) and (53) give the values of H and Z in terms of y at any instant during the first phase of combustion *i.e.*, before the rupture of the grains.

Hence from the equations (26), (27) and (28), we can get Y (and hence the shot-travel), the velocity v and the pressure P at any instant in terms of y.

Case II—At the point of rupture of grains, we have

$$z = a, \; y = o, \text{ and } D(z) = D(a)$$

∴ from (56), we have

$$D(a) = \log \left(\frac{A_1 - y_0}{A_1} \right)^{\nu L} \left(\frac{A_2 + y_0}{A_2} \right)^{\nu M} \left(\frac{A_3 - y_0}{A_3} \right)^{\nu N} \quad \text{..} \quad (54)$$

and from (52) and (54)

$$H = H_r = \left(\frac{A_1 - y_o}{A_1}\right)^{-L} \left(\frac{A_2 + y_o}{A_2}\right)^{-M} \left(\frac{A_3 - y_o}{A_3}\right)^{-N}$$

$$Z = Z_r = \lambda (a - b - c) \int_o^{y_o} (A_1 - y)^L (A_2 + y)^M (A_3 - y)^N (\alpha - \beta y - \gamma y^2) dy \quad .. \quad (55)$$

Hence the values of Y_r, C_r^* can be obtained from (26) and (7).

Now at any moment after the rupture of the grains i.e., when $z > a$,

$$\begin{aligned} D(z) &= \int_{z_o}^{z} \frac{V dz}{\varphi \cdot (z - \xi V^2)} \\ &= \int_{z_o}^{a} \frac{V dz}{\varphi \cdot (z - \xi V^2)} + \int_{a}^{z} \frac{V dz}{\varphi \cdot (z - \xi V^2)} \\ &= D(a) + \int_{a}^{z} \frac{V dz}{\varphi \cdot (z - \xi V^2)} \quad .. \quad .. \quad .. \quad (56) \end{aligned}$$

For the calculation of the integral of the second member of (56), we have from (14) and (12),

$$V(z) = \frac{a - b - c}{(m - 3)\cos\omega}\left[(m + 1) - \cos\omega \left\{m(1 - y_o) + (1 + 3y_o)\right\}\right]$$

$$\frac{dz}{\varphi} = dv = \frac{a - b - c}{m - 3}\left[\frac{(m + 1)\sin\omega}{\cos^2\omega}\right] d\omega$$

$$\begin{aligned} \therefore z - \xi v^2 = 1 &- \frac{3(m + 1)^2}{8\pi(m^2 - 7)}\left[1 + \frac{1}{\rho m} - \frac{m + 1}{4\rho m \cos\omega}\right] H(\omega) \\ &- \frac{\xi(a - b - c)^2}{(m - 3)^2 \cos^2\omega}\left[(m + 1) - \cos\omega\left\{m(1 - y_o) + (1 + 3y_o)\right\}\right]^2 \end{aligned}$$

$$\therefore \int_a^z \frac{V dz}{\varphi (z - \xi v^2)} = \frac{(a - b - c)^2}{(m - 3)^2} \times$$

$$\int_o^{\omega} \frac{\dfrac{(m + 1)(m + 1 - 4\cos\omega)\sin\omega}{\cos^3\omega}\, d\omega}{\left\{\begin{aligned} &1 - \frac{3(m + 1)^2}{8\pi(m^2 - 7)}\left\{1 + \frac{1}{\rho m} - \frac{m + 1}{4\rho m \cos\omega}\right\} H(\omega) \\ &- \frac{\xi(a - b - c)^2}{(m - 3)^2\cos^2\omega}\left[(m + 1) - \cos\omega\left\{m(1 - y_o) + (1 + 3y_o)\right\}\right]^2 \end{aligned}\right\}} \quad .. \quad .. \quad .. \quad (57)$$

This integration can be evaluated numerically and the value of $D(z)$ at any moment in this case obtained accordingly. Having obtained the value of $D(z)$, equations (23B) and (25) can give us the values of H and Z either as functions of ω or of z where z is given in terms of ω by (11).

Then equations (26), (27) and (28) will give values of Y (defining shot-travel) velocity v and pressure P at any instant after rupture in terms of ω.

Case III—At the all-burnt position, we have

$$z = 1, \quad y_{min} = \frac{6\sqrt{3} - 15}{13} \cdot \frac{m+1}{m-3}$$

$$D(z) = D_B(z) = D(1)$$

We have from (56)

$$D(1) = D(a) + \int_a^1 \frac{V dz}{\varphi \cdot (z - \xi V^2)}$$

where the latter integral is obtained from (57), when the upper limit is $\omega = (42° \, 25')$.

Then H_B is given by (23B) or (35). Also Z_B is given by (25) when $z=1$, and Y_B and V_B by (36) and (38A).

After the all-burnt position, the value of Y, the velocity v and the pressure P at any point will be given by (7), (46) and (47).

In particular the muzzle velocity v_M shall be given by (48).

Conditions for Maximum Pressure

From (21), (9) or (11) and (14), we have

$$P = \frac{f\bar{\omega}\left[\zeta(y) - \xi(a-b-c)^2(y_o - y)^2\right]}{\left[\left(c^* - \frac{\bar{\omega}}{\delta}\right) - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\,\zeta(y)\right]}$$

Differentiating with respect to y, we have

$$\left[\left(c^* - \frac{\bar{\omega}}{\delta}\right) - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\,\zeta(y)\right]^2 \left(\frac{dP}{dy}\right)$$

$$= \left\{\zeta'(y) + 2\xi(a-b-c)^2(y_o - y)\right\}\left\{c^* - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\,\zeta(y)\right\}$$

$$+ \frac{\xi}{\nu}(a-b-c)^2(y_o - y)\left\{c^* - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\,\zeta(y)\right\}$$

$$+ \left\{\zeta(y) - \xi V^2\right\}\left\{\eta - \frac{1}{\delta}\right\}\bar{\omega}\,\zeta'(y) \quad \cdots \quad \cdots \quad (58)$$

Initially $y = 1$, $\zeta(y) = 0$, $\zeta'(y) = b + c - a$,

$z = o$, $c^* = c'$.

$\therefore$ equations (58) becomes

$$\left(c' - \frac{\bar{\omega}}{\delta}\right)^2 \left(\frac{dP}{dy}\right)_{y=1}$$

$$= \left\{(b+c-a) + 2\,\xi\,(a-b-c)^2(y_o - 1)\right\}\left\{c' - \frac{\bar{\omega}}{\delta}\right\}$$

$$+ \frac{\xi}{\nu}(a-b-c)^2(y_o - 1)\left(c' - \frac{\bar{\omega}}{\delta}\right)$$

$$- \frac{\xi}{\nu}(a-b-c)^2(y_o - 1)^2\left(\eta - \frac{1}{\delta}\right)\bar{\omega}\,(b+c-a) \quad \cdots \quad (59)$$

But $\left(\frac{dP}{dy}\right)_{y=1}$ should be negative since initially P increases as y decreases.

$$\therefore \left[(b+c-a)+2\xi(a-b-c)^2(y_o-1)+\frac{\xi}{\nu}(a-b-c)^2(y_o-1)\right]\left(c'-\frac{\bar{\omega}}{\delta}\right)$$

$$<\xi\,\bar{\omega}(a-b-c)^2(y_o-1)^2\left(\eta-\frac{1}{\delta}\right)(b+c-a)$$

Or $$(a-b-c)\left(c'-\frac{\bar{\omega}}{\delta}\right)\left[1+\xi(a-b-c)(1-y_o)\left(2+\frac{1}{V}\right)\right]$$

$$>\xi\,\bar{\omega}\,(a-b-c)^3(1-y_o)^2\left(\eta-\frac{1}{\delta}\right)$$

Or $$1+\xi(a-b-c)(1-y_o)\left(2+\frac{1}{\nu}\right)$$

$$>\xi(a-b-c)^2(1-y_o)^2\frac{\left(\eta-\frac{1}{\delta}\right)\bar{\omega}}{c'-\frac{\bar{\omega}}{\delta}}$$

Or $$1+\xi(a-b-c)(1-y_o)\left[2+\frac{1}{\nu}-(a-b-c)(1-y_o)B\right]>o. \qquad (60)$$

Also at the point of rupture, we have

$$y=o,\ \zeta(y)=a,\ \zeta'(y)=-(a+b)$$
$$\text{and } c^*=c_r^*,$$

$\therefore$ equation (58) gives

$$\left[\left(c_r^*-\frac{\bar{\omega}}{\delta}\right)-\left(\eta-\frac{1}{\delta}\right)\bar{\omega}\,a\right]^2\left(\frac{dP}{dy}\right)_{y=o}$$

$$=\left[-(a+b)+2\xi(a-b-c)^2y_o\right]\left[c_r^*-\frac{\bar{\omega}}{\delta}-\left(\eta-\frac{1}{\delta}\right)\bar{\omega}\,a\right]$$

$$+\frac{\xi}{\nu}(a-b-c)^2y_o\left[c_r^*-\frac{\bar{\omega}}{\delta}-\left(\eta-\frac{1}{\delta}\right)\bar{\omega}\,a\right]$$

$$-\bar{\omega}(a+b)\left(\eta-\frac{1}{\delta}\right)\left[a-\xi(a-b-c)^2y_o^2\right]$$

Or $$\left[\left(c_r^*-\frac{\bar{\omega}}{\delta}\right)-\left(\eta-\frac{1}{\delta}\right)\bar{\omega}\,a\right]^2\left(\frac{dP}{dy}\right)_{y=o}$$

$$=\left[c_r^*-\frac{\bar{\omega}}{\delta}-\left(\eta-\frac{1}{\delta}\right)\bar{\omega}\,a\right]\left[\xi(a-b-c)^2y\left(2+\frac{1}{\nu}\right)-(a+b)\right]$$

$$-(a+b)\left[a-\xi(a-b-c)^2y_o^2\right]\left(\eta-\frac{1}{\delta}\right)\bar{\omega} \qquad (61)$$

At the point of all-burnt, we have

$$z = 1, \zeta(y) = 1, \zeta'(y) = o,$$

$$y = y_{min}, \ c^* = c^*_B,$$

$\therefore$ equation (58) gives

$$\left[c^*_B - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\right]^2 \left(\frac{dP}{dy}\right)_{y = y_{min}}$$

$$= [2\,\xi\,(a - b - c)^2\,(y_o - y_{min})] \left[c^*_B - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\right]$$

Or $\left[c^*_B - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\right]^2 \left(\frac{dP}{dy}\right)_{y = y_{min}}$

$$= \xi\,(a - b - c)^2 (y_o - y_{min}) \left(2 + \frac{1}{\nu}\right) \left[c^*_B - \frac{\bar{\omega}}{\delta} - \left(\eta - \frac{1}{\delta}\right)\bar{\omega}\right] \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

a positive quantity since $\frac{c'}{\bar{\omega}} > \eta$ and $c^*_B > c'$.

$\therefore \left(\frac{dP}{dy}\right)_{y = y_{min}}$ is always positive at the all-burnt position, and the maximum pressure therefore occurs before the all-burnt.

Thus the maximum pressure may occur:—

(*i*) before the rupture of the grains,

if $\left(\frac{dP}{dy}\right)$ is positive at the point of rupture.

(*ii*) at the rupture,

if $\left(\frac{dP}{dy}\right)$ is zero at rupture.

(*iii*) after the rupture,

if $\left(\frac{dP}{dy}\right)$ is still negative at rupture.

The condition so that maximum pressure may occur before the rupture of the grains is

$$\left(\frac{dP}{dy}\right)_{y = \rho} > o,$$

i.e., from (61), (7) and (21B),

$$[Y_r - B_a] \left[\xi\,(a - b - c)^2\,y_o \left(2 + \frac{1}{\nu}\right) - (a + b)\right]$$

$$> (a + b)\,B\,[a - \xi\,(a - b - c)^2\,y^2_o\,]$$

Or $\xi (a-b-c)^2 y_o \left[Y_r \left(2+\frac{1}{\nu}\right) - B\left\{a\left(2+\frac{1}{\nu}\right)-(a+b)y_o\right\}\right]$
$> Y_r\,(a+b)$

$$\text{Or } \xi > \frac{Y_r\,(a+b)}{(a-b-c)^2 y_o \left[\left(2+\frac{1}{\nu}\right)Y_r - B\left\{\left(2+\frac{1}{\nu}\right)a-(a+b)y_o\right\}\right]} \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

This condition reduces to (64) of Tavernier[1] when $B=0$ and $y_o=1$, *i.e.*, when the covolume correction is neglected and the shot-start pressure is also taken as zero.

Again the condition for the maximum pressure to occur after or at the rupture is

$$\left(\frac{dP}{dy}\right)_{y=o} \leqslant o$$

$$i.e.\ \xi < \frac{Y_r\,(a+b)}{(a-b-c)^2 y_o \left[\left(2+\frac{1}{\nu}\right)Y_r - B\left\{\left(2+\frac{1}{\nu}\right)a-(a+b)y_o\right\}\right]} \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

$$\text{and } \xi = \frac{Y_r\,(a+b)}{(a-b-c)^2 y_o \left[\left(2+\frac{1}{\nu}\right)Y_r - B\left\{\left(2+\frac{1}{\nu}\right)a-(a+b)y_o\right\}\right]} \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

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