A THEORY OF ANTI-SUBMARINE SEARCH

by

N. Balasubramanian

Office of the Scientific Research and Development, Naval Headquarters, New Delhi

ABSTRACT

In this paper a methodology has been developed, involving Dirac-δ-like functions, for comparative assessment of efficiency of different types of search manoeuvres in use in anti-submarine warfare.

The motion of the target is treated as a modified 'Random Walk' and the time-dependent probability field of its presence is evaluated.

Introduction

There are set patterns of search manoeuvre adopted by ships to track an enemy submarine whose contact has been momentary. A comparative assessment of the efficiency of different searches is to be done. Towards this end, and towards finally devising an optimum search pattern under given conditions a methodology is developed involving Dirac-δ-like functions defined ad hoc.

The problem is principally divided into two parts, viz. (1) the *a priori* probability field of the target's presence subsequent to the known contact, and (2) the probability of detection arising out of a particular search path across the above field.

The first part concerns the performance characteristics of the target, such as the speed of the target, the minimum diameter of its turn etc. and the second part involves the performance of the searching ship and the sighting instrument, such as the search speed, the rate of training of the scanning instrument etc.

The problem is solved analytically by quantizing the motions of the target and the searcher. The target is considered to move with a constant speed u along its path, which is supposed to be made up of contiguous straight line segments of equal length a. The motion is considered horizontal and the probability density $p(z, t)$ is found for the target to be present, after time t since the known contact, at the position $z=x+iy$ where x and y are the coordinates *w.r.t.* the known position of contact as the origin. Time is expressed as the number of straight hops made by the target.

Random Turn

First suppose that the submarine is equally likely to turn in any direction after every hop. Then, since at a given instant, the presences of the target at Z, for every Z in the plane, constitute a totality of mutually independent events,

$$P(z, t) = \iint P(\zeta, t-1)\, p(\zeta, z)\, d\zeta \quad (t \geqslant 1) \qquad \ldots \quad (1)$$

Where $\iint d\zeta$ denotes double integration over the ζ—plane and $p(\zeta, z)$ is the probability density that the target reaches Z from ζ in one hop of length a. To express $p(\zeta, z)$, define $\triangle(z)$ thus :

$$O \triangle (z) = \underset{\epsilon \to o}{Lt}\, O \triangle_s (z) \;;$$

s is a neighbourhood of Z with maximum width ϵ. O is any operator like integ·ration and multiplication used in this paper, and

$$\triangle_s (Z) = \{1/(\text{Area of } s)\} \text{ or } o \text{ according as } Z \text{ is in } S \text{ or not.} \quad \ldots \quad (2)$$

It is clear that for any integrable function $f(z)$

$$\iint f(Z) \triangle (Z - a)\, dZ = f(a) \text{ or } o \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

according as whether the double integration is over a region including the point a or not. The case of $f(Z)=1$ specifies $\triangle (Z - a)$ as the probability density field representing the certain presence of an object at a. On the same lines

$$\sum_{i=1}^{n} p_i \triangle (Z - Z_i) \quad (\text{where } \Sigma\, p_i = 1) \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

is the probability (per unit area) that the target is at Z based on the certain knowledge that it is at Z_i $(1 \leqslant i \leqslant n)$ with an *a priori* probability P_i. For then the probability of presence of the target in a region S is $\Sigma_j\, p_j$ for those j for which Z_i lies in S and this is the value given by integrating the expression (4) over S. Suppose it to be known that the target is present at any point on

$$\text{an arc} \quad Z = Z(s) \quad , o \leqslant s \leqslant l \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

with *a priori* probability $p(s)$ per unit length, s being the arcual length between $Z(o)$ and $Z(s)$. Then, making use of the $\triangle$-function, the probability (per unit area) of the target's presence at Z is given by

$$E(Z) = \int_o^l p(s) \triangle \Big\{ Z - Z(s) \Big\} ds \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

By the definition in (2), the above integral is zero unless Z lies on the given arc, in which case however it is infinite. But then it is infinite in such a way that further integration w.r.t. Z leads to finite results. In fact, by choosing the neighbourhood S of Z (vide (2)) as an infinitesimal rectangle of one side equal and parallel to the arcual element ds at Z (which is taken on the curve)

$$E(Z) = \underset{\delta \to o}{Lt} \frac{p(s)}{\delta} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where δ is the other side of the rectangle : so that, if A is a region inside which the portion of the given curve between $Z(s_1)$ and $Z(s_2)$ lies, then with proper understanding

$$\iint_A E(Z)\, dz = Lt \iint_B \frac{p(s)}{\delta} dz \quad \text{as } \delta \to o \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

where B is a belt of width δ about the arc between $Z(s_1)$ and $Z(s_2)$ and hence

$$\iint_A E(z)\, dz = \int_{s_1}^{s_2} p(s)\, ds \qquad \ldots \qquad (9)$$

which is the probability of presence of the target inside A. Thus the probability density can be expressed as

$$E(z) = p(s)\, \delta\,(\,|Z - Z(s)|\,) \qquad \ldots \qquad (10)$$

where s is such that $|Z - Z(s)|$ is minimum and δ is the Dirac impulse function.

(10) facilitates calculation of $p(\zeta, z)$ of (2). On the supposition of equal likelihood of the target turning in any direction, it ensues that, after a hop of length a from the position ζ, the target is present, with equal likelihood, anywhere on the circle $|Z - \zeta| = a$. Hence $p(\zeta, Z)$ is the value of $E(Z)$ when the $p(s)$ of (10) is $1/2\pi a$ and the curve considered is the circle. So, by (10)

$$\left.\begin{aligned} & p(\zeta, z) = \frac{1}{2\pi a}\, \delta\,(\,|z - \zeta| - a), \\ & \text{and } P(z, t) = \iint_{\text{entire plane}} P(\zeta, t-1)\, \delta\,(\,|z - \zeta| - a)\,(2\pi a)^{-1}\, d\zeta \\ & \text{with } P(z, o) = \triangle(z) \end{aligned}\right\} \qquad (11)$$

This leads to the result

$$P(z, t) = (2\pi a)^{-t} \iint dz_1 \ldots \iint dz_t \prod_{i=1}^{t} \delta\,(\,|z_{i+1} - z_i| - a)\, \triangle(z_1)$$

$$= (2\pi a)^{-t} \operatorname*{Lt}_{\substack{\epsilon_i \to o \\ o \leqslant i \leqslant t}} V \,/\, \pi \epsilon_o^2\, \epsilon_1\, \epsilon_2 \ldots \epsilon_t \qquad \ldots \qquad (12)$$

where V is the hypervolume of the region

$$\left.\begin{aligned} & |z_1| \leqslant \epsilon_o \\ & \big|\, |z_{i+1} - z_i| - a \,\big| \leqslant \tfrac{1}{2}\epsilon_i \\ & 1 \leqslant i \leqslant t \\ & \text{and } z_{t+1} \text{ stands for } z \end{aligned}\right\} \qquad \ldots \qquad (13)$$

Apart from (12), $P(Z,t)$ can be evaluated through the recurrence relation (11) or its modified form derived below. Remembering how $p(\zeta, z)$ was derived,

$$P(Z, t) = \iint d\zeta\, P(\zeta, t-1) \int_0^{2\pi} \frac{\triangle\,\{z - (\zeta + ae^{i\psi})\}}{2\pi}\, d\psi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\psi \iint P(\zeta, t-1)\, \triangle\,(z - \zeta - ae^{i\psi})\, d\zeta$$

$$= \frac{1}{2\pi} \int_o^{2\pi} P(z - ae^{i\psi}, t-1)\, d\psi$$

$$= \int_o^1 P(z - ae^{2\pi i \lambda}, t-1)\, d\lambda \quad .. \quad .. \quad (14)$$

The last step but one is by virtue of the 'sifting' property of $\triangle(z)$ as expressed in (3). As instances of (14),

$$P(z, 1) = \int_o^1 \triangle (z - ae^{2\pi i \lambda})\, d\lambda$$

$$= \delta(|z| - a)/2\pi a \quad .. \quad .. \quad .. \quad (15)$$

$$P(z,2) = \int_o^1 \frac{\delta(|z - ae^{2\pi i \lambda}| - a)}{2\pi a}\, d\lambda$$

$$= \underset{\epsilon \to o}{Lt} \frac{1}{4\pi a \epsilon} \int d\lambda \quad .. \quad .. \quad .. \quad (16)$$

where the integration is over those ranges of λ for which $|z - ae^{2\pi i \lambda}|$ lies between $a - \epsilon$ and $a + \epsilon$. The geometry of the Argand plane shows that

$$\left.\begin{array}{l} P(z, 2) = o \text{ when } |z| > 2a \\ \quad \text{and } = 1/\pi^2 r \sqrt{4a^2 - r^2} \\ \text{when } r \equiv |z| \leqslant 2a \end{array}\right\} \quad .. \quad .. \quad (17)$$

The same result can also be derived from (11) by finding

$$\frac{1}{4\pi^2 a^2} \iint \delta(|\zeta| - a)\, \delta(|z - \zeta| - a)\, d\zeta$$

$$= \frac{1}{4\pi^2 a^2} \underset{\epsilon_1, \epsilon_2 \to o}{Lt} A/2\epsilon_1 \times 2\epsilon_2 \quad .. \quad .. \quad (18)$$

where A is the area common to the annular belts of width $2\epsilon_1$ and $2\epsilon_2$ about the circles of radius a round zero and Z. Thus from (14), $P(Z, t)$ can be evaluated through successive integrations given by

$$P(z,t) = \int_o^1 \ldots \int_o^1 \triangle \left(z - a \sum_{r=1}^{t} e^{2\pi i \alpha_r}\right) d\alpha_1 \ldots d\alpha_t \quad (19)$$

The validity of this method of $\triangle$ — functions is verified by noting that

$$\iint P(z, t)\, dz = 1 \text{ as it should be} \quad .. \quad .. \quad .. \quad (20)$$

$$P(z, t) = o \text{ when } |z| > ta \quad .. \quad .. \quad .. \quad (21)$$

and $P(z, t) = \iint P(\zeta, \tau)\, P(z - \zeta, t - \tau)\, d\zeta$ .. .. (22)

for $o \leqslant t \leqslant \tau$ .. .. (23)

These can be proved with the use of (11) by induction. The last relation leads to an explicit evaluation of $P(Z, t)$.

Setting $F(\zeta, t) = \iint P(z, t)\, e^{iZ_\wedge\zeta}\, dz$

where $Z_\wedge \zeta$ denotes $\frac{1}{2}(z\bar{\zeta} + \zeta\bar{z})$ .. .. (23)

(22) implies that

$F(\zeta, t_1)\, F(\zeta, t_2) = F(\zeta, t_1 + t_2);\ o \leqslant t_1, t_2$ .. .. .. (24)

$$\text{whence } \left\{ F(\zeta, t) \right\}^{1/t} = \iint \frac{\delta(|z| - a)}{2\pi a}\, e^{iz_\wedge\zeta}\, dz$$

$$= \int_o^{2\pi} d\theta \int_o^{\infty} dr.\, r\, \frac{\delta(r - a)}{2\pi a}\, e^{i r \rho \operatorname{Cos}(\theta - \varphi)}$$

$$= \frac{1}{2\pi} \int_o^{2\pi} exp\,(ia\rho \operatorname{Cos} \varphi)\, d\varphi$$

$$= J_o(a\rho) \quad .. \quad .. \quad .. \quad .. \qquad (25)$$

where $\zeta = \rho e^{i\phi}$

So, inversion of (23) gives

$$P(z, t) = \frac{1}{4\pi^2} \int_o^{2\pi} d\varphi \int_o^{\infty} d\rho.\rho \left\{ J_o(a\rho) \right\}^t exp\,(-i\rho\, |z| \operatorname{Cos}\varphi) \qquad (26)$$

Restricted Rate of Turn

Next a more realistic assumption is made as to the possible directions of turning by the target. Any structure with a steering mechanism (such as a bicycle, automobile or a ship) requires a minimum width of lane to turn right about. To be more precise, there is a maximum to the curvature of the path which the vehicle can trace. This limiting curvature is independent of the speed and is characteristic of the mass and the structure of the vehicle. Let d be the minimum diameter of 'right about turn' of the target, so that the maximum curvature of its path will be $2/d$. In the quantized motion of straight hops of length a , this implies that the angular deviation between any two successive hops is at most

$$\alpha = 2a/d \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (26)$$

The equation (1) fails in this case since the probability of the target reaching Z from ζ in one hop is not now independent of the direction of approach to ζ in the previous hop, and thereby is dependent on the actual path traced up to Z. Deal, then, with the probability (density) $Q(Z, t, \psi)$ of the target reaching Z at time t with the last hop making an angle ψ with the positive direction of X-axis. The first requirements on Q are

$$P(z, t) = \int_{K}^{2\pi + K} Q(z, t, \psi)\, d\psi \qquad (28)$$

and $Q(z, t, \psi + 2n\pi) = Q(z, t, \psi)$ (29)

Assuming equally likely turn, at any stage, within an angle α a recursive relation involving Q is derived, as in the case of (11) or (14). Define a function $\hat{\delta}(\psi)$ by

$$\left.\begin{aligned} &O\,\hat{\delta}(\psi) = \underset{\epsilon \to o}{Lt}\; O\;\hat{\delta}_{\epsilon}(\psi), \\ &O \text{ is any operator used in this context in this paper} \\ &\hat{\delta}_{\epsilon}(\psi) = \frac{1}{2\epsilon} \text{ when } |\psi - 2n\pi| \leqslant \epsilon \quad (n = o, \pm 1, \pm 2, \ldots) \\ &\quad \text{and} = o \text{ otherwise} \end{aligned}\right\} \qquad (30)$$

Relations analogous to (3), (4), (6) can be derived, viz.

$$\int_{K}^{2\pi + K} F(\psi)\,\hat{\delta}(\psi - \theta)\, d\psi = F(\theta + 2\nu\pi) \qquad (31)$$

where ν is the integral part of $\frac{K}{2\pi} + 1$

$$\text{and } \sum_i p_i \Delta(z - z_i)\,\hat{\delta}(\psi - \psi_i)$$

is the probability field for the case where the target is known to have reached one of the points z_i at an angle ψ_i with *a priori* probability p_i. When the target is known to have arrived at $z(s)$ of (5) at an angle $\psi(s)$ with a probability $p(s)$, the field is

$$E(z, \psi) = \int_{o}^{l} p(s)\, \Delta\left\{z - z(s)\right\} \hat{\delta}\left\{\psi - \psi(s)\right\} ds \qquad (33)$$

Taking the curve to be the circular arc

$$z(s) = \zeta + ae^{i\lambda}, \; |\lambda - \varphi| \leqslant \alpha \qquad (34)$$

where ζ, φ are constant complex and real numbers respectively, and taking

$$\left.\begin{aligned} s &= a(\lambda - \varphi) \\ \psi(s) &= \lambda \\ \text{and} \quad p(s) &= 1/2\,\alpha a \end{aligned}\right\} \qquad (35)$$

(33) Leads to the result that, if the target has reached ζ at an angle φ the probability field at the next instant is

$$\left.\begin{aligned} &p(\zeta, \varphi\,; z, \psi) = o \text{ or } \Delta(z - \zeta - ae^{i\psi})/2\alpha \\ &\text{according as } |\psi - \varphi| < \text{ or } \geqslant \alpha \end{aligned}\right\} \qquad (36)$$

Hence, on the same principle as under lies (1).

$$Q(z, t, \psi) = \iint d\zeta \int_{K}^{2\pi + K} d\varphi \, Q(\zeta, t-1, \varphi) \, p(\zeta, \varphi; z, \psi) \quad (37)$$

$$= \iint d\zeta \int_{\psi-\alpha}^{\psi+\alpha} Q(\zeta, t-1, \varphi) \frac{\Delta(z - \zeta - ae^{i\psi})}{2\alpha} d\varphi \quad (38)$$

$$= \frac{1}{2\alpha} \int_{\psi-\alpha}^{\psi+\varkappa} Q(z - ae^{i\psi}, \; t-1, \varphi) \, d\varphi \quad .. \quad (39)$$

Suppose the target is known to have arrived at an angle between θ_1 and θ_2 (with equal likelihood) at the initial position, viz. the origin. One may then take the initial probability field as

$$Q(z, o, \psi) = \{\Delta(z)/\theta_2 - \theta_1\} \text{ or } o \quad .. \quad .. \quad .. \quad (40)$$

according as ψ lies between θ_1 and θ_2 or not. Then equation (39) leads to

$$\left. \begin{aligned} Q(z, t, \psi_o) &= (2\alpha)^{-t} \int_{\psi_o - \alpha}^{\psi_o + \alpha} d\psi_1 \int_{\psi_1 - \alpha}^{\psi_1 + \alpha} d\psi_2 \, .. \int_{\psi_{t-2} - \alpha}^{\psi_{t-2} + \alpha} d\psi_{t-1} \; I \\ \text{where } I &= \int \Delta(z - a \sum_{r=o}^{t-1} e^{i\psi_r}) \, d\psi_t \end{aligned} \right\} \quad .. \quad (41)$$

where the range of integration w.r.t. ψ_t is the interval common to $(\psi_{t-1} - \alpha, \psi_{t-1} + \alpha)$ and (θ_1, θ_2). Particularly, when any course was equally likely at the instant of known contact,

$$Q(z, t, \psi_o) = (2\alpha)^{-t} \int_{\psi_o - \alpha}^{\psi_o + \alpha} d\psi_1 \ldots \int_{\psi_{t-1} - \alpha}^{\psi_{t-1} + \alpha} d\psi_t \, \frac{\Delta(z - a \sum_{o}^{t-1} e^{i\psi_r})}{2\pi}$$

$$= (2\alpha)^{1-t} (2\pi)^{-1} \int_{\psi_o - \alpha}^{\psi_o + \alpha} d\psi_1 \ldots \int_{\psi_{t-2} - \alpha}^{\psi_{t-2} + \alpha} d\psi_{t-1} \; \Delta\left(z - a \sum_{o}^{t-1} e^{i\psi_r}\right) \quad \ldots\ldots\ldots\ldots (42)$$

The consistency of the methodology is verified by noting how the equations above reduce to those of the previous case (of random turn) when $\alpha = \pi$. The integrand in (42) being periodic in ψ_r $(1 \leq r \leq t-1)$,

$\alpha = \pi$ implies that

$$Q(z, t, \psi_o) = (2\pi)^{-t} \int_{o}^{2\pi} d\psi_1 \ldots \int_{o}^{2\pi} d\psi_{t-1} \, \Delta(z - a \sum_{r=o}^{t-1} e^{i\psi_r}) \quad (43)$$

This combined with (28) leads to (19) as it should. For explicit evaluation of $Q(z, t, \psi)$ and $P(Z, t)$ in the case of restricted turn, define $q(Z_1, t_1, \psi_1; Z_2, t_2, \psi_2)$ as the probability that the target which arrives at Z_1 at time t_1 by an angle ψ_1 reaches Z_2 by angle ψ_2 at time t_2. It is clear that

$$q(o, o, \psi_1; z, t, \psi) = Q(z, t, \psi) \qquad \ldots \ldots \ldots \ldots (44)$$

when $\theta_1 = \theta_2 = \psi$ and $1/(\theta_2 - \theta_1)$ is replaced by $\hat{\delta}(\psi - \psi_1)$ in (40).
By induction it can be proved that

$$q(z_1 + z, t_1 + t, \psi_1; z_2 + z, t_2 + t, \psi_2) \text{ is independent of } z \text{ and } t. \qquad (45)$$

with the help of the relation

$$q(z_1, t_1, \psi_1; z, t, \psi) = \frac{1}{2\alpha} \int_{\psi-\alpha}^{\psi+\alpha} q(z_1, t_1, \psi_1; z - ae^{i\psi}, t-1, \lambda) \qquad (46)$$

which is derived on the same lines as (39). Induction proves also that

$$q(z_1, t_1, \psi_1; z_2, t_2, \psi_2) = \iint dz \int_0^{2\pi} d\psi \, q(z_1, t_1, \psi_1; z, t, \psi) \, q(z, t, \psi; z_2, t_2, \psi_2)$$

for all t such that $t_1 \leqslant t \leqslant t_2$ $\qquad \ldots \ldots \ldots \ldots \ldots (47)$

Set now that

$$F(\zeta; \psi_1, \psi_2; t) = \iint dz \, q(o, o, \psi_1; z, t, \psi_2) e^{i\zeta \wedge z} \qquad \ldots \ldots \ldots (48)$$

Using (45) and (47)

$$\int_0^{2\pi} d\psi F(\zeta; \psi_1, \psi; t_1) F(\zeta; \psi, \psi_2; t_2)$$

$$= \int_0^{2\pi} d\psi \iint dz e^{iz\wedge\zeta} \iint dw \, q(o, o, \psi_1; w, t_1, \psi) q(o, o, \psi; z - w, t_2, \psi_2)$$

$$\left[\text{using}(45)\right] = \int_0^{2\pi} d\psi \iint dz e^{iz\wedge\zeta} \iint dw \, q(o, o, \psi_1; w, t_1, \psi) \, q(w, t_1, \psi; z, t_1 + t_2, \psi_2)$$

$$[\text{using }(47)] = \iint dz \, e^{i\zeta\wedge z} \, q(o, o, \psi_1; z, t_1 + t_2, \psi,)$$

$$= F(\zeta; \psi_1, \psi_2; t_1 + t_2) \qquad \ldots \ldots \ldots \ldots (49)$$

(39), (44) and (48) imply

$$F(\zeta; \psi_1, \psi_2; o) = \hat{\delta}(\psi_2 - \psi_1) \qquad \ldots \ldots \ldots (50)$$

and $F(\zeta; \psi_1 \psi_2; 1) = \dfrac{1}{2\alpha} exp\{ia\rho Cos(\psi_2 - \varphi)\}$ or 0 $\qquad \ldots \ldots (51)$

according as $|\psi_2 - \psi_1| \leqslant$ or $> \alpha$, (where $\zeta = \rho e^{i\varphi}$)

(49) and (51) combine to give

$$F\left(\zeta;\psi_0;t\right)=\int_{\psi_0-\alpha}^{\psi_0+\alpha} d\psi_1\ F\left(\zeta;\psi,\psi_1;t-1\right)\frac{1}{2\alpha}\ exp.\left\{ia\rho\ Cos(\psi_0-\varphi\right\}$$

$$=\int_{\psi_0-\alpha}^{\psi_0+\alpha} d\psi_1 \int_{\psi_1-\alpha}^{\psi_1+\alpha} d\psi_2\, F\left(\zeta;\psi,\psi_2;t-2\right)\left(\frac{1}{2\alpha}\right)\times$$

$$exp\left\{ia\rho\sum_{j=0}^{1}\cos\left(\psi_j-\varphi\right)\right\}$$

$$=\left(2\alpha\right)^{-t}\int_{\psi_0-\alpha}^{\psi_0+\alpha} d\psi_1\ldots\int_{\psi_{t-1}-\alpha}^{\psi_{t-1}+\alpha} d\psi_t\ \hat{\delta}\left(\psi_t-\psi\right)exp.\left\{ia\rho\sum_{0}^{t-1}\mathrm{Cos}\left(\psi_j-\varphi\right)\right\} \quad (52)$$

The set of equations (52), (48), (28) and the following one which can be proved by induction *viz.*

$$Q(z,t,\psi,)=\frac{1}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} q\left(o,o,\lambda;z,t,\psi\right)d\lambda \quad \ldots \quad (53)$$

combine to give, for the case of restricted turn,

$$\iint P(z,t,\ )e^{i\zeta\wedge z}dz=\frac{(2\alpha)^{1-t}}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2} d\lambda_1\int_{\lambda_1-\alpha}^{\lambda_1+\alpha} d\lambda_2\ldots\int_{\lambda_{t-1}-\alpha}^{\lambda_{t-1}+\alpha} d\lambda_t\times\ exp\left\{ia\rho\sum_{j=1}^{t}Cos(\lambda_j-\varphi)\right\}$$

$$\ldots \quad (54)$$

When the initial course (just before contact) is random,

$$\iint P(z,t)e^{i\zeta\wedge z},\ dz=\frac{(2\alpha)^{1-t}}{2\pi}\int_{0}^{2\pi} d\lambda_1\int_{\lambda_1-\alpha}^{\lambda_1+\alpha} d\lambda_2\ldots\int_{\lambda_{t-1}-\alpha}^{\lambda_{t-1}+\alpha} d\lambda_t\times exp\left\{ia\rho\sum_{1}^{t}Cos(\lambda_j-\varphi)\right\}$$

$$\ldots \quad (55)$$

Hence $P\left(Z,t\right)=\left(2\alpha\right)^{1-t}\left(8\pi\right)^{-3}\int_{0}^{2\pi}d\varphi\int_{0}^{\infty}d\rho.\ \rho I$

$$\text{Where } I=\int_{0}^{2\pi} d\lambda_1\int_{\lambda_1-\alpha}^{\lambda_1+\varkappa} d\lambda_2\ldots\int_{\lambda_{t-1}-\alpha}^{\lambda_{t-1}+\alpha} d\lambda_1\, exp\left\{ia\rho\sum_{1}^{t}Cos\left(\lambda_i-\varphi\right)-|z|\,Cos\varphi\right\}$$

$$\ldots \quad (56)$$

Acknowledgment

Grateful acknowledgments are due to Dr. J.N. Nanda, P. Sc. O. (Navy) for the encouragement given in pursuing this problem.