PROJECTILES IN FLIGHT

by

B. S. Madhava Rao

Institute of Armament Studies, Kirkee

ABSTRACT

General properties of trajectories are indicated. Recent methods for trajectory calculations, and for the setting up of range tables are presented briefly.

Introduction

Taken in the widest sense, the whole Universe is a system of projectiles in flight. The spinning earth moving in its orbit round the sun, along with the planets, their satellites, the comets, meteors, the zodiacal light and the cosmic rays are all projectiles. The sun itself, along with the components of the solar system, is a projectile moving in our galaxy, and one step further leads to galaxies, and to extra-galactic nebulae as systems of projectiles.

Coming from the macroscopic to the microscopic scale, the atom itself is a projectile system with all kinds of elementary particles taking part in the flight subject to strange types of motion governed by the quantum and relativistic laws.

It is not intended to deal in this article with the inter-planetary types of projectiles moving in the exosphere, or the earth satellites whose region of motion is mainly the ionosphere, although in a way, the study of these satellite trajectories is simpler than that of normal trajectories in the tropo-and stratospheres, consequent on the assumption of molecular flow in the ionosphere. Also, in this region, the composition of oxygen and nitrogen is considered to be in the same ratio as at sea level viz 30 per cent and 70 per cent respectively, but due to absorption of ultra-violet solar radiation, molecular oxygen is dissociated and exists as atomic particles, while nitrogen remains in the molecular state. Thus the mechanism of flight in the ionosphere is described by the equations of the free molecular flow theory, or the so-called super-aerodynamics and detailed analysis of the lift and drag co-efficients occurring for various body configurations in high speed flight can be carried out on the basis of this theory. It can also be shown that the realm of super aero-dynamics can be defined by the ratio of the body size of the vehicle to the mean free path length of the particles in the gas flow. Such hypersonic research is also bound to be of significance in the case of ordinary trajectories. In fact, recent rockets, having sundered the sound barrier are now running into the aerodynamic heating problem relating to the terrific, heat encountered on re-entry, and the success achieved in dealing with these problems by using wind tunnel experiments for finding pressure, temperature and density ratios for given Mach numbers, has emphasised the utility of such experiments in ordinary trajectory calculations too.

Laws of resistance

We now come to the subject of exterior ballistics related to projectiles in the tropo-and strato-spheres of which the principal problem is the calculation of trajectories based on the laws of air resistance, and the conditions of the atmosphere. Considering first the resistance of the air, the resistance function obviously is a function of (*i*) the form and dimensions of the projectile (*ii*) the velocity v of the projectile relative to the air (*iii*) the obliquity δ or the angle between v and the axis of rotation of the projectile and (*iv*) the physical and thermodynamical characteristics of the atmosphere. As is well known, it can be shown on this basis that the retardation function R can be written as

$$R = \frac{ia^2\Delta}{w} v^2 \phi\left(\frac{v}{c^*}, \frac{vd}{\nu}\right) \quad .. \quad .. \quad (1)$$

where i depends on the form of the projectile, a is its diameter, w its weight, $\Delta = \rho g$ is the specific weight of air, c^* the velocity of sound so that v/c^* is the Mach number M, ν the coefficient of viscosity, and d a linear dimension of the projectile so that vd/ν is the Reynold's number. In the old form of the resistance function, one had

$$R = i\frac{a^2\Delta}{w} F(v) = C\frac{\Delta}{\Delta_0} F(v) \quad .. \quad .. \quad (2)$$

where $c = i\dfrac{a^2\Delta}{w}$ is the reciprocal of the normal ballistic coefficient, and the last factor is a function of v only. The difference between (1) and (2), even neglecting the effect of viscosity, is the appearance of c^* the effect of which cannot be neglected, and this might be considered an essential improvement of modern ballistics.

Considering next the atmospheric conditions, and denoting by the suffix y the conditions at altitude y above sea level, and by the suffix 0 the conditions at sea level, the main problem is to find the variations of the atmospheric pressure H and of the temperature with y. Introducing the absolute temperature T′ corrected for humidity viz. $T' = T\left(1 + \dfrac{3f}{8H}\right)$ where f is the tension of water vapour, it can be shown that

$$\frac{dH}{H} = -k\frac{dy}{T'}, \ k \text{ being a constant.}$$

and assuming a law $\rho p^{-\lambda} = \text{const}$, or $H\Delta^{-\lambda} = \text{const}$, with λ as given by experimental ballistics, one has

$$\frac{dT'}{T'} = \frac{\lambda-1}{\lambda}\frac{dH}{H}, \text{ giving } T'_y = T'_0 - \frac{k(\lambda-1)}{\lambda} y.$$

Finally, for the variation of c^*, one has

$$\left(\frac{c^*_y}{c^*_0}\right)^2 = \left(\frac{H_y}{\Delta_y}\right) : \left(\frac{H_0}{\Delta_0}\right) = \frac{T'_y}{T'_0} \quad .. \quad .. \quad (3)$$

These enable the putting of the retardation function (1), after neglecting viscosity, in the form

$$R = \frac{i\, a^2\, \Delta_o}{w} \cdot \frac{\Delta}{\Delta_o}\, v^2\, \phi\left(\frac{v}{c^*}, j\right) \quad .. \quad .. \quad (4)$$

where j denotes the influence of the dimensions of the projectile.

Now (4) is equivalent to

$$R = c\, \frac{H_y}{H_o}\, \frac{T'_o}{T'_y}\, v^2\, \varphi\left(\frac{v}{c^*}, j\right), \text{ using (3)}$$

Since $v^2 \frac{T'_o}{T'_y} = v^2 \left(\frac{c_o^*}{c_y^*}\right)^2$, with c_o^* a constant, this expression is proportional to the square of the Mach number M, and can be taken inside the function φ, and one has the important result

$$R = c\, \frac{H_y}{H_o}\, \psi\,(M, j) \quad .. \quad .. \quad .. \quad (5)$$

Experimental results show that $\psi\,(M, j)$ can be written in the form

$$\psi\,(M, j) = F_1\,(M) - j\, F_2\,(M) \equiv F\,(M),$$

so that finally

$$R = c\, \zeta\, F\,(M) \quad .. \quad .. \quad .. \quad (6)$$

with $\zeta = H_y / H_o$. It is to be noted that Δ has completely vanished from the formulae.

Equation (6) forms the basis of modern ballistics, specially of the French School, and it deserves wide adoption. What is actually done is to choose judiciously three values of j depending on the general exterior profile of the projectile, and then develop a table called Table B, for numerical values of F(M) for the three values, the methods of experimental ballistics being adopted for setting up the table. Similarly, normal atmospheric conditions as functions of altitude, for example, values of ζ are not obtained by means of formulae but by experimental determinations using explosive soundings and embodied in another table, called Table A. The old relations

$$R = c\, \frac{\Delta_y}{\Delta_o}\, F\,(v), \text{ and } \Delta_y = \Delta_o\, e^{-hy} \left(h = 10^{-4}\right)$$

are notoriously insufficient, and it is therefore an advantage, also leading to greater generality, to replace them by tables A & B.

The next step is the actual calculation of the trajectories and the setting up of the range tables. But before doing this, we might consider some general questions relating to the integration of the equations of motion, some general properties of trajectories, and methods of calculation of arcs of trajectories.

Some general questions

Treating the problems related to perturbations of variations of the initial or atmospheric conditions at time $t = 0$, and to the motion of the projectile round its centre of gravity as secondary ones, the principal problem is the calculation of the trajectory in the vertical plane of the initial velocity, knowing the retardation function. A general solution of the principal problem should constitute a procedure for the integration of the fundamental system of equations, freeing itself from every hypothesis relating either to the standard atmosphere, or to a law of resistance. Nevertheless, solutions obtained by making such hypotheses as $R = c \frac{\triangle_y}{\triangle_o} F(v)$, and $\triangle_y = \triangle_o e^{-hy}$, and other simpifying assumptions would be of value as approximations to the general theory. To this category belong the theorems of Saint-Robert based on the assumption $\triangle_y = \triangle_o$, other general properties of trajectories are the principle of rigidity of the trajectory, the principle of utilisation of oblique axes which avoids the initial angle α, the formula valid for range under large values of the site of the target not involving either the initial velocity or the ballistic co-efficient, and finally the problems of inverse ballistics. Generalities on the integration of the equations of motion are illustrated by the Siacci method, and the method of development in convergent series in powers of the ballistic coefficient.

Among one of the most important general methods is that of successive calculation by arcs, of which the fundamental principle is that, given the ballistic co-ordinates x_o, y_o, t_o, θ_o, v_o at the origin of the arc, and θ, at its extremity, one should be able to calculate the other co-ordinates at the extremity. This method originally applied using the quadratic law of retardation, and the exponential law for $\triangle$, leads to two types of errors of a ballistic and geometrical nature. Recent methods of trajectory calculation amount to an elimination of these errors by refining the method of arcs.

The method of G-H-M

We will now describe a method developed and recently perfected by the French Ballistic School called the G-H-M after the work of Garnier, Haags and Marcus. This is of the general type mentioned at the beginning of the previous article as being independent of any special laws relating either to R or the atmospheric conditions, but only using Table A for the latter, and Table B for the empirical functions of air resistance. Thus, the method permits, keeping the method of calculation invariant, the derival of a profit due to every improvement in these two laws. In addition to A & B, two other tables C and D are used devoted respectively to analytic functions for calculating trajectories, and to analytic functions for the calculations of variations or alterations. This method is based essentially on calculations of arcs, but generalises the earlier inelegant methods by perfecting the numerical calculus, so that besides fixing the closed rules of calculation of the amplitude of the arcs for a given precision, it also satisfies the conditions necessary for a practically feasible method for successive calculations by arcs. Further, it gives, with the same ease and precision, the methods of calculation for the perturbations and the differential corrections.

The method uses the modern law for R given by (6) with c* the velocity of sound at altitude y being a function of T′. The following quantities k and l whose values are given in Table A are used in the calculations.

$$\left.\begin{aligned} k &= -\frac{d}{dy}(\log H) \\ l &= -\tfrac{1}{2}\frac{d}{dy}(\log T') \end{aligned}\right\} \qquad \ldots \quad (7)$$

while Table B gives

$$n = \frac{M\,F'(M)}{F(M)} \quad \ldots \quad (8)$$

i.e. the degree of resistance of the air. Table B gives, besides, three laws of resistance with the common form

$$F(M) = F_1(M) - j\,F_2(M),$$

where j characterises the external profile of the projectile, and three round figures, j = 0·04, 0·14 and 0·24 are taken, the first one corresponding to a modern, and the third to an old type of projectile. The table also contains a sub-table J (M) which corresponds to variation of j by 0·01. In view of the discontinuity of F′ (M) and hence also of n at the point M = 1, Table B is suitably modified by giving in blue coloured pages the values of F (M) for $M<1$, calculated for M slightly greater than 1, and in red coloured pages the values of f(M) for $M>1$, calculated for M slightly less than 1.

Using $u = v\cos\theta$ and $\tan\theta$ as the essential variables, and introducing the pure numbers $\rho = R/g$, and

$$\xi = \int_0^\theta \frac{d\theta}{\cos\theta} = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right),$$

the fundamental equation of the hodograph becomes

$$\frac{du}{u} = p\,d\xi \quad \ldots \quad (9)$$

Using the indices 0 and 1 for the several ballistic variables at the beginning and end of the arc, and D for total variations, for example, $Dx = x_1 - x_o$ etc., except that $D\xi = \xi_o - \xi_1$ and also the index "ia" to characterise the values which are only a first approximation to the end values, the integration of (9) leads to

$$\log u_o - \log u_1 = \int_{\xi_1}^{\xi_o} \rho N\,d\xi = \beta, \quad \ldots \quad (10)$$

when $N = \log_{10} e$. If we know β, we find $Dx = u_o\,u_1 \cdot \frac{1}{g}(\tan\theta_o - \tan\theta_1)$ and thus we are led to the calculation of x_1. It is essential to find the first derivative of p (ξ) as a simple expression, so that calculation of higher derivatives may be possible, at least upto $\rho''(\xi)$. This is achieved by using. $R = g\rho = c\xi F(M)$ and getting

$$\frac{d\rho}{\rho} = n\,\frac{dv}{v} - h\,dy \quad \ldots \quad (11)$$

and $$\frac{d\rho}{\rho} = (np + m\sin\theta)\,d\xi \quad \ldots \quad (12)$$

where $m = n + hv^2/g,\ h = k - nl$, and

$$\rho'_o = \left(\frac{d\rho}{d\xi}\right)_o = p_o\,(n_o\,\rho_o + m_o\sin\theta_o) \quad \ldots \quad (12')$$

Writing (10) as

$$\log u_o - \log u_1 = \beta = \rho_i \; N \, d \, \xi \quad .. \quad .. \quad (10')$$

with ρ_i as the mean value over the arc, the next step is the development in a series in the neighbourhood of ξ_o in the form:

$$\rho_1 = \rho_o - \rho_o' D\xi + \frac{1}{2!} \rho_o'' D\xi^2 - \frac{1}{3!} \rho_o''' D\xi^3 + \quad .. \quad (13)$$

$$\int_{\xi_1}^{\xi_o} \rho d\xi = \rho_o D\xi - \frac{1}{2!} \rho_o' D\xi^2 + \frac{1}{3!} \rho_o'' D\xi^3 - \quad .. \quad (14)$$

$$ie. \quad \rho_i = \rho_o - \frac{1}{2!} \rho_o' D\xi + \frac{1}{3!} \rho_o'' D\xi^2 - \quad .. \quad (15)$$

The second approximation is based on employing

$$\rho_{ia} = \rho_o - \tfrac{1}{2} \rho_o' D\xi$$

and similarly for higher approximations. The essential results are as follows :— Starting from ρ_{ia} exact to the second order, one gets ρ_{ia} correct to the third order which permits calculation of u_1 correct to the fourth order, after having calculated ρ_o'' and so on. All the G-H-M methods run on these lines. The trouble taken in calculating ρ_o' is largely compensated by the utilisation made possible of large values of $D\xi$, from where results a considerable diminution in the number of arcs used.

The scheme for calculation of an arc, the plan of calculation, and the details of calculation can all be easily systematised. The fixation of the amplitudes of the arcs, the errors committed at the successive stages for the several D's and the limitations on these errors can also be derived. One could further define *Coefficients of security* for the several ballistics elements; for example considering θ, this coefficient f_β is defined as $\frac{D}{D\theta}\theta_\beta$, and these coefficients too can be calculated, and serve as practical rules of limitation on the amplitudes of the arcs used in the calculation.

It is worthy of note that the labour of the G-H-M method which is purely numerical in character has recently been shown to be capable of much simplification if we use the Runge-Kutta metohd of numerical integration.

Two particular problems amenable to the G-H-M method are the calculation of zenithal trajectories, and the calculation of trajectories of the Keplerian Orbit type, in particular elliptic types, when the projectile attains a high altitude in the atmosphere, where the resistance of the air is negligible. These latter types of orbits are related to the inter-continental ballistic missiles.

Conclusion

We have indicated above the barest outlines of the principal problem of theoretical exterior ballistics. It is necessary to supplement this by considering the secondary problems mentioned above. Specially in view of the fact that modern methods do not rely on particular laws of retardation, the importance of experimental ballistics, specially of wind tunnel experiments for the calculation of tables of type B becomes evident. The same is true for construction of tables of type A also.

Lastly, it is necessary to deal with problems of applied ballistics, specially the fundamental one of the construction of range tables based on the previously established theoretical and experimental results.

Reference

Besse, L. "Cours de Ballistiques exterieure" Mem. le L' Artillerie Francaise, t. 31,1957, 2e —fasc. pp 391—446; 3e fasec pp. 593—626 and 4e fasc pp. 939—1016.

Discussion

At the outset Mr. B.N. Sen (Kirkee) wanted to know whether the G.H.M. method discussed was better than the other method either in being quicker or in giving results agreeing more closely with those of Range Tables.

Prof. Rao replied that he had confined himself to the theoretical calculation of the trajectory and the G.H.M. method was only one of such methods, its chief merit being its elegance. Comparison of theoretically calculated results with the range tables involved a number of factors and was an aspect of Applied Ballistics which he had not touched upon.

Dr. Thiruvenkatachar pointed out that the computation of trajectories was a prerequisite for the compilation of Range Tables and hence the question of any method for computation of the trajectories giving results agreeing with the range tables did not arise.

Dr. Varma and Dr. Thiruvenkatachar drew attention to the fact that in practice the calculated trajectories and the fired results had to be matched by suitably adjusting the ballistic coefficient.

The use of two parameters in the Dupont's resistance law was the subject of considerable discussion. It was explained by Mr. N.S. Venkatesan (New Delhi) that the introduction of two parameters enabled one to get a fairly constant ballistic coefficient over the range of elevations. He mentioned that since the value of the ballistic coefficient was obtained by interpretation, from the three or four prints obtained from actual firings, it was desirable to have a C_0 which varied with elevation. Generally that value of j, which gave the least variation for C_0 was chosen.

The discussion then centered on the necessity for the use of electronic machines for computing the trajectories and from this point of view Dr. S.N. Mitra (Calcutta) wanted to know whether this method had any iterative process in it so as to eliminate the chance of any error being propagated during successive stages of computation. Prof. Rao replied that the method was iterative and was to be carried out in a large number of steps.

Major H.S. Subba Rao drew attention to the problem of the stability of a projectile. He mentioned that the minimal conditions for stability usually accepted is $S>1$. This naturally led one to enquire as to how high a value should S have and whether a value for $S = 9$ or 10 is acceptable. This condition had been later modified by Nielson, Synge, McShane and others and they had introduced three conditions for stability. He felt that this problem was a very important one and needed careful study.

The chairman agreed with the above remarks and he felt that due consideration had not been given to this problem so far. He stressed the need for further investigation.