INTERNAL BALLISTICS OF COMPOSITE CHARGES TAKING INTO ACCOUNT ANY POSSIBLE DIFFERENCE IN THE RATIOS OF SPECIFIC HEATS OF THE COMPONENT CHARGES

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ABSTRACT

In the present paper we have discussed the internal ballistics of composite charges when $\gamma_1 \neq \gamma_2$ (*i*) when the rate of burning is proportional to pressure, the equations for the first stage have been simplified, (*ii*) when the two component charges are tubular, it has been shown that in the first stage, the composite charge burns as a single charge and this enables us to write the fundamental differential equations together with initial conditions for both the power law and the Linear law, (*iii*) the continuity of the pressure-time curve at the end of the first stage and at all-burnt has been discussed, (*iv*) the conditions for maximum pressure to occur in any stage given in I have been improved to take into account the possible discontinuity in dp/dt, and an alternative method of discussing the problem has been indicated.

Introduction

In two recent papers, Aggarwal [1,2] has given a method for solving the equations of Internal Ballistics when γ_1, γ_2 for the two component charges are not necessarily equal. In paper I, he considered the solution for the linear law of burning when the component charges have the standard quadratic form-fuction and in paper II, he gave the solution for the power law of burning for the particular case, when both the component charges are tubular.

One object of the present paper is to suggest certain simplifications in the solution, for both the laws, for the first stage of burning i.e., the stage when both the component charges are burning. However, when both the component charges burn out simultaneously, there is only one stage of burning and the simplifications suggested here apply to the entire period of burning.

In the particular case, when both the component charges are tubular, we have shown here that for the first stage, for both the laws of burning, the composite charge behaves as a single charge with suitable values for the parameters defining its composition. For this case, a method for solving the equations of Internal Ballistics for a third law, viz, when the rate of burning is a linear function of the pressure has also been given.

We have also discussed here the continuity of the pressure-time derivative at the end of the first stage and at all-burnt and shown that, in general, even when $\gamma_1 \neq \gamma_2$, (*i*) dp/dt will be continuous at the end of the first stage if and only if the charge to burn out first is in cord form.

(*ii*) dp/dt will be continuous at all-burnt if and only if the charge to burn out second is in cord form.

When $\gamma_1=\gamma_2$, these results were first established by Kapur[5,6].

The stage in which the maximum pressure can occur has been discussed in 1 by the author and three possibilities are mentioned viz, that the maximum pressure can occur when

(*a*) both the charges are burning i.e., in the first stage.

(*b*) charge C_1 has been burnt out and charge C_2 is burning i.e., in the second stage.

(*c*) both the charges are burnt out i.e. at all-burnt.

The possible discontinuity in dp/dt at the end of the first stage enables us to discuss a fourth possibility viz., that the maximum pressure can occur in

(*d*) crossing from the first stage to the second.

We have obtained the conditions for case (*d*). Actually conditions given for case (*b*) apply to case (*d*) and the conditions given there for case (*c*) apply only when both the component charges burn out simultaneously. The conditions in other cases have also been improved by noting that for a pressure-maximum to occur at any instant, it is not necessary that dp/dt should vanish at that time, and that, in general, the change in sign of dp/dt from positive to negative at that instant would be sufficient to ensure a pressure-maximum here.

Basic equations and the conditions for simultaneous and non-simultaneous burning of the component charges

The four basic equations of Internal Ballistics, for the present problem are:

$$\frac{F_1 C_1 Z_1}{\gamma_1-1}+\frac{F_2 C_2 Z_2}{\gamma_2-1}=Ap(x+l)\,\frac{\dfrac{n_1 C_1 Z_1}{\gamma_1-1}+\dfrac{n_2 C_2 Z_2}{\gamma_2-1}}{n_1 C_1 Z_1+n_2 C_2 Z_2}+\tfrac{1}{2}\,w_1 v^2 \qquad (1)$$

or $F_1 C_1 Z_1 + k F_2 C_2 Z_2$

$$= Ap\,(l+x)\;\frac{n_1 C_1 Z_1 + k\, n_2 C_2 Z_2}{n_1 C_1 Z_1 + n_2 C_2 Z_2} + \tfrac{1}{2}(\gamma_1-1)\; w_1\; v^2 \qquad \ldots (1a)$$

where

$$w_1 = 1\cdot 5w + \tfrac{1}{3}(C_1 + C_2), \quad \ldots \quad (1b)$$

$$Al = Ko - \frac{C_1}{\delta_1} - \frac{C_2}{\delta_2}, \quad \ldots \quad (1c)$$

$$k = \frac{\gamma_1 - 1}{\gamma_2 - 1} \quad \ldots \quad (1d)$$

$$w_1 \frac{dv}{dt} = w_1 v \frac{dv}{dx} = Ap \quad \ldots \quad (2)$$

$$Z_i = (1 - f_i)(1 + \theta_i f_i) \quad [i=1, 2] \quad \ldots \quad (3)$$

$$D_i \frac{df_i}{dt} = -\beta_i p^\alpha \quad [i=1, 2] \quad \ldots \quad (4)$$

From (4)

$$\frac{1}{\beta'_1}\frac{df_1}{dt} = \frac{1}{\beta'_2}\frac{df_2}{dt}, \quad \ldots \quad (5)$$

where

$$\beta'_i = \frac{\beta_i}{D_i} \quad [i=1, 2] \quad \ldots \quad (6)$$

Integrating (5), we get

$$\frac{1-f_1}{\beta'_1} = \frac{1-f_2}{\beta'_2} \quad \ldots \quad (7)$$

since when ignition start, $f_1=1$, $f_2=1$.

When shot starts, let $f_1 = f_{10}$, $f_2 = f_{20}$, then (7) gives

$$\beta'_1 f_{20} - \beta'_2 f_{10} = \beta'_1 - \beta'_2 \quad \ldots \quad (8)$$

The condition that charge C_1 burns out first is that when $f_1=0$, f_2 should be positive. (7) then gives

$$\beta'_1 > \beta'_2 \quad \ldots \quad (9)$$

Similarly the condition that charge C_2 burns out first is $\beta'_2 > \beta'_1$, and the condition that they burn out simultaneously is $\beta'_1 = \beta'_2$. From (8) it is easily seen that these conditions are simpler, though equivalent to, the respective conditions.

$$\beta'_1 f_{20} > \beta'_2 f_{10}, \quad \beta'_1 f_{20} < \beta'_2 f_{10}, \quad \beta'_1 f_{20} = \beta'_2 f_{10} \quad \ldots \quad (10)$$

used in paper I.

Integration for the first stage of burning for the usual linear law ($\alpha=1$)

For $\alpha=1$, we have from (2), (4) and (6), for the first stage,

$$v = \frac{A}{\beta'_1 w_1}(f_{10} - f_1) = \frac{A}{\beta'_2 w_1}(f_{20} - f_2) \quad \ldots \quad (11)$$

From (1*a*), (3) and (7)

$$F_1 C_1 (1-f_1)\ (1+\theta_1 f_1) + k F_2 C_2\ (1-f_2) (1+\theta_2 f_2) - \tfrac{1}{2} (\gamma_1-1)\ w_1\ v^2$$

$$= Ap\ (1+x)\ \frac{n_1 C_1 (1-f_1) (1+\theta_1 f_1) + kn_2 C_2\ (1-f_2)\ (1+\theta_2 f_2)}{n_1 C_1 (1-f_1) (1+\theta_1 f_1) + n_2 C_2\ (1-f_2)\ (1+\theta_2 f_2)} \quad .. \quad (12a)$$

$$= Ap\ (1 + x)\ \frac{n_1 C_1 \beta'_1 (1 + \theta_1 f_1) + k\, n_2 C_2 \beta'_2 (1 + \theta_2 f_2)}{n_1 C_1 \beta'_1 (1 + \theta_1 f_1) + n_2 C_2 \beta'_2\ (1 + \theta_2 f_2)} \quad .. \quad (12b)$$

Substituting for f_1 and f_2 from (11) in (12*b*) and simplifying we get

$$\frac{d\xi}{\xi} = \frac{\bar{K}_1}{K\bar{K}_2}\ \frac{(\bar{a}_1 - v)\ v\ dv}{(a - v)\ (b + v)\ (\bar{a}_2 - v)} \quad .. \quad .. \quad .. \quad (13)$$

where

$$\xi = 1 + \frac{x}{l} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (14)$$

$$K = \frac{F_1 C_1\ \beta_1'^2\ w_1}{A^2}\ \theta_1 + \frac{kF_2C_2\beta_2'^2w_1}{A^2}\ \theta_2 + \tfrac{1}{2}\ (\beta_1 - 1) \quad .. \quad (15a)$$

$$K (a - b) = \frac{F_1 C_1 \beta'_1}{A}\ (1 - \theta_1 + 2\theta_1 f_{10}) + \frac{k\ F_2 C_2 \beta'_2}{A} (1 - \theta_2 + 2\theta_2 f_{20})$$

$$.. \quad .. \quad (15b)$$

$$Kab = \frac{F_1 C_1 z_{10} + k\ F_2 C_2 z_{20}}{w_1} \quad .. \quad .. \quad .. \quad .. \quad (15c)$$

$$\bar{K}_1 = \frac{n_1 C_1\ \theta_1\ \beta'^2_1\ w_1}{A} + \frac{k\ n_2\ C_2\ \beta'^2_2\ \theta_2\ w_1}{A} \quad .. \quad .. \quad (16a)$$

$$\bar{K}_1\ \bar{a}_1 = n_1 C_1 \beta'_1\ (1 + \theta_1 f_{10}) + k\ n_2 C_2 \beta'_2 (1 + \theta_2 f_{20}) \quad .. \quad (16b)$$

$$\bar{K}_2 = \frac{n_1 C_1\ \beta'^2_1\ \theta_1\ w_1}{A} + \frac{n_2 C_2\ \beta'^2_2\ \theta_2\ w_1}{A} \quad .. \quad .. \quad .. \quad (16c)$$

$$\bar{K}_2\ \bar{a}_1 = n_1 C_1\ \ '_1 (1 + \theta_1 f_{10}) + n_1 C_1\ \beta'_2 (1 + \theta_2 f_{20}) \quad .. \quad (16d)$$

Also

$$p = \frac{w_1}{A}\ v\ \frac{dv}{dx} = \frac{w_1}{Al}\ v\ \frac{dv}{d\xi}$$

$$\text{or } p = \frac{w_1}{Al}\ \frac{K\bar{K}_2}{\bar{K}_1}\ \frac{(a - v)\ (b + v)\ (\bar{a}_2 - v)}{(\bar{a}_1 - v)\ \xi} \quad .. \quad .. \quad (17)$$

Instead of (13) and (17), Paper I gives

$$\frac{d\xi}{\xi} = \frac{K_1}{KK_2}\ \frac{(a_1-v)\ (b_1+v)\ vdv}{(a-v)\ (b+v)\ (a_2-v)\ (b_2+v)} \quad .. \quad .. \quad .. \quad (13a)$$

$$p = \frac{KK_2}{K_1}\ \frac{w_1}{Al}\ \frac{(a-v)\ (b+v)\ (a_2-v)\ (b_2+v)}{(a_1-v)\ (b_1+v)\ \xi} \quad .. \quad .. \quad (17a)$$

where

$$\overline{K}_1 = K_1 \frac{w_1}{A}, \quad K_2 = \overline{K}_2 \frac{w_1}{A},$$

$$K_1(a_1 - b_1) = \frac{n_1 C_1 \beta'_1 w_1}{A}(1 - \theta_1 + 2\theta_1 f_{10}) + \frac{k n_2 C_2 \beta'_2 w_1}{A}(1 - \theta_2 + 2\theta_2 f_{20}) \quad (18a)$$

$$K_1 a_1 b_1 = n_1 C_1 Z_{10} + k_2 n_2 C_2 Z_{20} \quad (18b)$$

$$K_2(a_2 - b_2) = \frac{n_1 C_1 \beta'_1 w'_1}{A}(1 - \theta_1 + 2\theta_1 f_{10}) + \frac{n_2 C_2 \beta'_2 w_1}{A}(1 - \theta_2 + 2\theta_2 f_{20}) \quad (18c)$$

$$K_2 a_2 b_2 = n_1 C_1 Z_{10} + n_2 C_2 Z_{20} \quad (18d)$$

Comparison of (13) and (13*a*) shows that

$$\overline{a_1} = a_1, \ \overline{a_2} = a_2, \ b_1 = b_2 \quad (19)$$

It is obvious that (13) and (17) are respectively simpler than (13*a*) and (17*a*). The simplification arises due to the cancellation of a factor proportional to $1 - f_1$ or $1 - f_2$ from the numerator and denominator of (12*a*). This factor is zero at ignition, but is non-zero from the instant of shot-start to the instant when one of the component charges is burnt out and this is the period which corresponds to the first stage with which we are concerned.

(19) shows that (13) and (13*a*) or (17) and (17*a*) are essentially the same and for some purposes it may be convenient to still use the more complex forms (13*a*) and (17*a*) on account of their greater symmetry; remembering, however, that $b_1 = b_2$

Tubular Component Charges : Power Law of Burning

When $\theta_1 = \theta_2 = 0$, (12) gives for the energy equation

$$F_1 C_1 (1 - f_1) + k F_2 C_2 (1 - f_2) - \tfrac{1}{2}(\gamma_1 - 1) w_1 v^2 = Ap(l + x)\frac{n_1 C_1 \beta'_1 + k n_2 C_2 \beta'_2}{n_1 C_1 \beta'_1 + n_2 C_2 \beta'_2} \quad (20)$$

Using (7), we get

$$\lambda\left[F_1 C_1 + k \frac{\beta'_2}{\beta'_1} F_2 C_2\right]\left[1 - f_1\right] = Ap(l + x) + \tfrac{1}{2}(\gamma_1 - 1)\lambda w_1 v^2 \quad (21)$$

where

$$\lambda = \frac{n_1 C_1 \beta'_1 + n_2 C_2 \beta'_2}{n_1 C_1 \beta'_1 + k n_2 C_2 \beta'_2} \quad (22)$$

From (21), (1*b*), (1*c*), (2) and (4), we see that the equations of Internal Ballistics for the composite charge for the first stage are the same as those for a single charge with

mass $C = C_1 + C_2$, (23*a*)

force constant $F = \dfrac{\lambda\left[F_1 C_1 + k \frac{\beta'_2}{\beta'_1} F_2 C_2\right]}{C_1 + C_2}$ (23*b*)

form-factor $\theta = 0$ (23*c*)

propellant density $\delta = \dfrac{C}{C_1/\delta_1 + C_2/\delta_2}$ (23*d*) ratio of specific heats $\bar{\gamma} = 1 + \lambda (\gamma_1 - 1)$ .. .. .. (23e)

and ballistic size $\frac{D}{\beta} = \frac{D_1}{\beta_1}$ .. .. .. (23f)

Accordingly from Clemmow[3], the fundamental differential equation, for the power law, is

$$\frac{2XZ}{1+n}\frac{d^2X}{dZ^2} - \frac{2n(\bar{\gamma}-1)Z}{(1+\ )(\bar{\gamma}-n)}\left(\frac{dX}{dZ}\right)^2 + X\frac{dX}{dZ} = 1, \quad .. \quad (24)$$

where

$$\left.\begin{array}{l} X = \xi^{\frac{\gamma - n}{2n}} \qquad Z = \frac{n(\bar{\gamma}-n)}{1+n} M\left(\zeta\,\xi^{\gamma}\right)^{\frac{1}{n}} \\ M = \frac{A^2D^2}{FC\,\beta^2 w_1}\left(\frac{FC}{Al}\right)^{2-2\alpha}, \ \zeta = p\frac{Al}{FC}, \quad n = \frac{1}{3-2\alpha} \end{array}\right\} \quad .. \quad (55)$$

This is the main result of paper II.

When the charges burn out simultaneously, (21), (22), (23b) are slightly simplified and in this case discussion of the first stage of burning completes the discussion till the all-burnt position.

Tubular Component Charges : General Linear Law of Burning

Let the rate of burring be a linear function of the pressure, so that instead of (4) we have

$$D_i\frac{df_i}{dt} = -\beta_i\,(p + p_1) \quad [i = 1, 2] \quad .. \quad .. \quad .. \quad (26)$$

the solution of the equation of internal ballistics can be reduced to the solution of (Kapur[4]) either of the following equations:

$$\left(\frac{Y}{\xi}\right)^{\frac{\bar{\gamma}-1}{\bar{\gamma}}}\left[-y\,\zeta\zeta''(\zeta+\zeta_1) + \gamma\zeta'^2\,(\zeta + 2\,\zeta_1) - \zeta\zeta'(\zeta + 2\,\zeta_1)\right] = \frac{M'\,\bar{\gamma}\,\zeta^4}{\zeta+\zeta_1} \quad .. \quad (27)$$

$$\text{or} \quad \frac{Y'''}{Y''} - \frac{2Y''}{Y'} + \frac{Y'}{Y} = (\bar{\gamma}-1)\frac{M'Y}{Y''}\frac{Y'^{-\frac{2}{\bar{\gamma}-1}}}{\left(YY'^{-\frac{\bar{\gamma}}{\bar{\gamma}-1}} + \zeta_1\right)^2} + \frac{\xi_1\left(\frac{Y'}{Y} - \frac{\bar{\gamma}}{\bar{\gamma}-1}\frac{Y''}{Y'}\right)}{YY'^{-\frac{\bar{\gamma}}{\bar{\gamma}-1}} + \zeta_1} \quad .. \quad .. \quad (28)$$

or $$\xi^{\bar{\gamma}-1} \frac{d}{dY}\left[\left(Y+\zeta_1\xi^{\bar{\gamma}}\right)\xi^{-1}\frac{d\xi}{dY}\right] = \frac{M'Y}{Y+\zeta_1\zeta^{\bar{\gamma}}} \quad \text{.. (29)}$$

where

$$\xi = 1 + \frac{x}{l}, \quad \text{.. (30)}$$

$$\zeta = \frac{Al}{FC}p, \quad \zeta_1 = \frac{Al}{FC}p_1, \quad \text{.. (31)}$$

$$M' = \frac{A^2 D^2}{FC\ \beta^2 w_1}, \quad \text{.. (32)}$$

$$Y = \zeta\,\xi^{\bar{\gamma}} \quad \text{.. (33)}$$

and in (28) the independent variable is $(1 - f_1)$. Since this is not present explicitly in (28), it can be reduced to a second order differential equation.

The initial conditions for the integration of these equations are

$$\xi = 1,\ \zeta = \zeta_\circ = \frac{Al}{FC}p_\circ,\ Y = \zeta_\circ\ \zeta' = 1,\ 1-f_1 = \zeta_\circ\ Y' = 1,$$

$$Y'' = 1, \frac{d\xi}{dy} = 0, \quad \text{.. (34)}$$

where p_0 is the shot-start pressure.

Second Stage of Burning

For the second stage, the forumla corresponding to (13*a*) and (17*a*) are (Paper I).

$$\frac{d\xi}{\xi} = \frac{K_1'}{K'K'_2}\,\frac{(a'_1 - v)\ (b'_1 + v)\ vdv}{(a' - v)\ (b' + v)\ (a'_2 - v)\ (b'_2 + v)} \quad \text{.. (13}b\text{)}$$

and

$$p = \frac{K'_2\,K'}{K'_1}\,\frac{w_1}{Al}\,\frac{(a' - v)\,(b' + v)\,)\,(a'_2 - v)\,(b'_2 + v)}{(a'_1 - v)\ (b'_1 + v)\ \xi} \quad \text{.. (17}b\text{)}$$

where

$$K' = \frac{\gamma_1 - 1}{2} + \frac{k\,F_2\,C_2\,\rho'^2_2\,w_1}{A^2}\,\theta_2, \quad \text{.. (35}a\text{)}$$

$$K^1\,(a^1 - b^1) = \frac{k\,F_2\,C_2\,\beta'_2}{A}\,(1 - \theta_2 + 2\theta_2 f_{20}) \quad \text{.. (35}b\text{)}$$

$$K'\ a'\ b' = \frac{F_1\,C_1}{w_1} + \frac{k\,F_2\,C_2\,Z_{20}}{w_1} \quad \text{.. (35}c\text{)}$$

$$K'_1 = \frac{k\,n_2\,C_2\,\rho'^2_2\,w^2_1}{A^2}\,\theta_2, \quad \text{.. (36}a\text{)}$$

$$K_1'\,(a' - b'_1 = \frac{k\,n_2\,C_2\ '_2\,w_1}{A}(1 - \theta_2 + 2\theta_2\,f_{20}) \quad \text{.. (36}b\text{)}$$

$$K'_1\, a'_1\, b'_1 = n_1 C_1 + k\, n_2 C_2 Z_{20} \qquad \ldots \quad (36c)$$

$$K'_2 = \frac{n_2 C_2 \, f'^2_2 \, w^2_1}{A^2}\, \theta_2 \qquad \ldots \quad (37a)$$

$$K'_2\,(a'_2 - b'_2) = \frac{n_2 C_2 \, f'_2 \, w_1}{A}\,(1 - \theta_2 + 2\theta_2 f_{20}) \qquad \ldots \quad (37b)$$

$$K'_2\, a'_2\, b'_2 = n_1 C_1 + n_2 C_2 Z_{20} \qquad \ldots \quad (37c)$$

The formulae cannot, in general, be simplified except for the special case of tubular component charges. In this case (12) becomes

$$F_1 C_1 + k F_2 C_2 (1 - f_2) - \tfrac{1}{2}(\gamma_1 - 1)\, w_1 v^2$$
$$= Ap\,(l + x) \times \frac{n_1 C_1 + k\, n_2 C_2 (1 - f_2)}{n_1 C_1 + n_2 C_2 (1 - f_2)}$$

Substituting for f_2 from (11), we get a simplified expression for p of type (17).

For the power law, for the second stage no exact integration is possible. Use of a mean value of a variable coefficient is suggested in II but no method of evaluating this mean value is indicated.

Alternatively we may use some sort of an average of γ_1 and γ_2 in the second stage. We may use (23e) or

$$\frac{1}{\gamma - 1} = \frac{\dfrac{n_1 C_1}{\gamma_1 - 1} + \dfrac{n_2 C_2}{\gamma_2 - 1}}{n_1 C_1 + n_2 C_2} \qquad \ldots \quad (38)$$

Continuity of $\frac{dp}{dt}$ at the end of the first Stage and at All-Burnt

From (17a) and (17b)

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a - v} + \frac{1}{b + v} - \frac{1}{a_2 - v} + \frac{1}{b_2 + v} + \frac{1}{a_1 - v} - \frac{1}{b_1 + v} - \frac{1}{\xi}\frac{d\xi}{dv}$$

in the first stage and

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a' - v} + \frac{1}{b' + v} - \frac{1}{a'_2 - v} + \frac{1}{b'_2 + v} + \frac{1}{a_1 - v} - \frac{1}{b_1 + v} - \frac{1}{\xi}\frac{d\xi}{dv}$$

in the second stage.

Now

$$\frac{1}{\xi}\frac{d\xi}{dv} = \frac{1}{l + x}\frac{dx}{dv} = \frac{v}{l + x}\,\frac{w_1}{Ap},$$

and since x, p, v are continuous in crossing from one stage to the other, so is

$$\frac{1}{\xi}\frac{d\xi}{dv}$$

Let suffix 1 denote the end of the first stage, suffix 2 denote the beginning of the second stage, suffix 3 denote the end of the second stage and suffix 4 denote the beginning of the third stage (corresponding to motion after all-burnt), then

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_1-\left[\frac{1}{p}\frac{dp}{dv}\right]_2=\frac{K(a-b-2v_1)}{K(a-v_1)(b+v_1)}-\frac{K'(a'-b'-2v_2)}{K'(a^1-v_2)(b^1+v_2)}$$

$$+\frac{K'_2(a_2-b_2-2v_1)}{K'_2(a_2-v_1)(b_2+v_1)}-\frac{K'_2(a_2'-b_2'-2v_2)}{K'_2(a_2'-v_2)(b_2'+v_2)}$$

$$-\frac{K_1(a_1-b_1-2v_1)}{K_1(a_1-v_1)(b_1+v_1)}+\frac{K'_1(a'_1-b'_1-2v_2)}{K'_1(a_1^1-v_2)(b_1^1+v_2)}$$

Remembering that $p_1=p_2=p_{12}$ (say) and $v_1=v_2$

$$=\frac{A}{\beta_1'w_1}f_{10}=v_{12}\text{ (say)}$$

and substituting from (15), (18), (19), (35), (36), (37) and simplifying we get

$$\frac{1}{p_{12}}\left[\left(\frac{dp}{dv}\right)_1-\left(\frac{dp}{dv}\right)_2\right]$$

$$=\frac{C_1\beta'_1w_1[1-\theta_1]}{A}\left[\frac{F_1}{F_1C_1+kF_2C_2Z_{2,1}-\frac{1}{2}(\gamma_1-1)w_1v^2_{2,1}}+\frac{n_1}{n_1c_1+n_2c_2Z_{2,1}}\right.$$

$$\left.-\frac{n_1}{n_1C_1+kn_2C_2Z_{2,1}}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

where $Z_{2,1}$ is value of Z_2 at the end of the first stage, ie.—

$$Z_{2,1}=(1-f_{2,1})(1+\theta_2f_{2,1})=\frac{\beta'_2}{\beta'_1}\left[1+\theta_2\left(1-\frac{\beta'_2}{\beta'_1}\right)\right] \quad \ldots \quad (40)$$

If $k\geqslant 1$, the expression within brackets on the R.H.S. of (39) is definitely positive. Even if $k<1$, in all practical cases, this will be positive.

(*i*) if $\theta_1=1$ i.e. if the first charge is in cord form, dp/dv is continuous in the change-over from one stage to the other, and since

$$\frac{dp}{dt}=\frac{dp}{dv}\frac{dv}{dt}=\frac{Ap}{w_1}\frac{dp}{dt}$$

and p is also continuous at this instant, dp/dt will also be continuous.

(*ii*) If $\theta_1<1$, assuming the expression within brackets on the R.H.S. of (39) to be positive, dp/dv can change sign from positive to negative and not *vice-versa*. Consequently, in general, dp/dt is discontinuous in crossing from first stage to second and a pressure maximum can occur at the end of the first stage.

At the end of the second stage,

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_3 = \frac{K'(a'-b'-2v_3)}{K'(a'-v_3)(b'+v_3)} + \frac{K'_2(a'_2-b'_2-2v_3)}{K'_2(a_2'-v_3)(b'_2+v_3)}$$
$$-\frac{K'_1(a'_1-b'_1-2v_3)}{K'_1(a_1'-v_3)(b'_1+v_3)} - \left(\frac{1}{\xi}\frac{d\xi}{dv}\right)_3$$

Substituting from (35), (36), (37) and remembering that v_3 is the velocity at all-burnt so that

$$v_3 = \frac{A}{\beta'_2 w_1} f_{20},$$

we get

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_3 = \frac{-\frac{A}{\beta'_2}(\gamma_1-1) f_{20} + \frac{kF_2C_2\beta'_2w_1}{A}(1-\theta_2)}{F_1C_1 + k F_2C_2 - \frac{1}{2}(\gamma_1-1) w_1v_3^2}$$
$$+\frac{n_2C_2'\beta_2w_1(1-\theta_2)}{A(n_1C_1+n_2C_2)} - \frac{k n_2C_2\beta'_2w_1(1-\theta_2)}{A(n_1C'+k n_2C_2)}$$

After all-burnt from 1(*a*), since $z_1=1$, $z_2=1$

$$F_1C_1 + k F_2C_2 - \tfrac{1}{2}(\gamma_1-1) w_1v^2 = Ap(x+1)\frac{n_1c_1+k n_2c_2}{n_1c_1+n_2c_2}$$

so that

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_4 = \frac{-\frac{A}{\beta'_2}(\gamma_1-1) f_{20}}{F_1C_1 + k F_2C_2 - \frac{1}{2}(\gamma_1-1) w_1v_4^2} - \left[\frac{1}{\xi}\frac{d\xi}{dv}\right]_4 \quad \ldots (41)$$

Since

$$\left[\frac{1}{\xi}\frac{d\xi}{dv}\right]_3 = \left[\frac{1}{\xi}\frac{d\xi}{dv}\right]_4 \text{ and } v_3 = v_4 = \frac{A}{\beta'_2w_1} f_{20}$$

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_3 - \left[\frac{1}{p}\frac{dp}{dv}\right]_4$$
$$= (1-\theta_2)\frac{\beta'_2w_1}{A}C_2\left[\frac{kF_2}{F_1C_1+kF_2C_2-\frac{1}{2}(\gamma_1-1)w_1v_3^2} + \frac{n_2}{n_1C_1+n_2C_2}\right.$$
$$\left.-\frac{kn_2}{n_1C_1+kn_2C_2}\right] \quad \ldots (42)$$

(*i*) If $\theta_2 = 1$ i.e. if the second charge is in cord form, dp/dt is continuous at all-burnt position.

(*ii*) If $\theta_2 < 1$, assuming the expression within brackets on the R.H.S. of (42) to be positive, we find that in general dp/dt is discontinuous at all-burnt and since dp/dt can change from positive to negative at all-burnt a pressure maximum can arise here.

Maximum Pressure

Case a—Maximum Pressure in the first stage of burning:

The conditions obtained in I are—

$$\frac{f_{10}}{\beta'_1} > \frac{w_1}{A} v_{[2]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43a)$$

and
$$\frac{f_{20}}{\beta'_2} > \frac{w_1}{A} v_{[2]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43b)$$

where $v_{[\bar{2}]}$ is the velocity at which the expression for dp/dy in the first stage vanishes. We note that in virtue of (10), the two conditions are not independent. If charge C_1 burns out first, the second condition is implied by the first. Similarly if charge C_2 burns out first, the first condition is implied by the second. If the two charges burn out similtaneously, the two conditions become identical.

Case b—Maximum Pressure in the second stage of burning—

The conditions obtained in I are—

$$\frac{f_{10}}{\beta'_1} = \frac{w_1}{A} v_{[\bar{2}]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44a)$$

$$\frac{f_{20}}{\beta'_2} > \frac{w_1}{A} v_{[\bar{2}]}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44b)$$

where $v_{[\bar{2}]}$ is the velocity at which dp/dv in the second stage vanishes.

The first condition implies that at the instant dp/dv vanishes in the second stage, the first charge should be just completely burnt out, but the first charge is burnt out just at the beginning of the second stage. Thus conditions (44) are the conditions for the maximum pressure to occur at the beginning of the second stage. Out of these only (44*a*) is independent as 44(*b*) will be automatically satisfied if (44*a*) is satisfied. If the maximum pressure is to occur in the second stage proper

$$\frac{A}{w_1} \frac{f_{10}}{\beta'_1} < v_{[2]} < \frac{A}{w_1} \frac{f_{20}}{\beta'_2}$$

or

$$\frac{f_{10}}{\beta'_1} < \frac{w_1}{A} v_{[\bar{2}]} < \frac{f_{20}}{\beta'_2} \quad \ldots \quad \ldots \quad (45)$$

Case c—Maximum Pressure at all-burnt—

The conditions for the case of *non-simultaneous* burning out of component charges are obtained in I as

$$\frac{f_{10}}{\beta'_1} = \frac{w_1}{A} v_{[\bar{2}]} \quad \ldots \quad \ldots \quad \ldots \quad (46a)$$

$$\text{and} \quad \frac{f_{20}}{\beta^1_2} = \frac{w_1}{A} \, v_{[\bar{2}]} \quad \ldots \quad \ldots \quad \ldots \quad (46b)$$

but these two conditions imply

$$\frac{f_{10}}{\beta'_1} = \frac{f_{20}}{\beta'_2}$$

which is precisely the condition for the two charges to burn out *simultaneously*.

The fallacy appears to arise from the fact that at the instant when dp/dv vanishes in the second stage, both the first and the second charges are required to burn out and this can only happen when the charges burn out simultaneously.

Thus conditions (46) have to be replaced by

$$\frac{f_{10}}{\beta_1^1} < \frac{w_1}{A} \, v_{[\bar{2}]} = \frac{f_{20}}{\beta_2^1} \quad \ldots \quad \ldots \quad (47)$$

But even these are not strictly correct; for the maximum pressure will be at all-burnt if $\frac{dp}{dt} \geqslant o$ at all-burnt, as immediately after, we see from (41), $\frac{dp}{dt}$ is negative. If $\frac{dp}{dt}$ does not vanish in the second stage, even then the maximum pressure occurs at all-burnt. Thus (47) is to be replaced by

$$\frac{f_{10}}{\beta'_1} < \frac{f_{20}}{\beta'_2} \leqslant \frac{w_1}{A} \, v_{[\bar{2}]} \quad \ldots \quad \ldots \quad \ldots \quad (48)$$

but the first part is assumed to be satisfied as we are taking C_1 to be the charge to burnt out first. Thus the conditions for the maximum pressure to occur at all-burnt are—

$$\frac{f_{20}}{\beta'_2} \leqslant \frac{w_1}{A} \, v_{[\bar{2}]} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (49)$$

Case d—Maximum Pressure at the end of the first stage—

This possibility has not been considered in I, but it arises nevertheless due to the possible discontinuity in dp/dt at this instant. This can occur in three possible cases

$$\left.\begin{array}{ll} (i) & v_{[\bar{2}]} = \dfrac{A\, f_{10}}{\beta'_1 \, w_1} \\ (ii) & v_{[\bar{2}]} = \dfrac{A\, f_{10}}{\beta'_1 \, w_1} \\ (iii) & v_{[\bar{2}]} > \dfrac{A\, f_{10}}{\beta'_1 \, w_1} \quad \text{and} \quad v_{[\bar{2}]} < \dfrac{A\, f_{10}}{\beta'_1 \, w_1} \end{array}\right\} \quad (50)$$

An alternative approach

The approach of I discussed above for finding the stage in which the maximum pressure needs improvement in the following respects—

(*i*) The conditions obtained are expressed in terms of f_{10}, f_{20} while in practice we are likely to be given p_0, the shot-start pressure. It is therefore, desirable to express f_{10}, f_{20} in terms of p_o. From (17) at shot-start [$v = o$, $p = p_o$, $\xi = 1$]

$$p_o = \frac{w_1}{Al}\frac{K\bar{K}_2}{K_1}\frac{ab\,\bar{a}_2}{\bar{a}_1} = \frac{w_1}{Al}\frac{Kal\,\bar{K}_2\bar{a}_2}{\bar{K}_1\,\bar{a}_1}$$

$$= \frac{w_1}{Al}\frac{F_1C_1Z_{10} + k\,F_2C_2Z_{20}}{w_1}\frac{n_1C_1\beta'_1(1+\theta_1f_{10})+n_2C_2\beta'_2(1+\theta_2f_2)}{n_1C_1\beta'_1(1+\theta_1f_{10})+kn_2C_2\beta'_2(1+\theta_2f_{20})}$$

but from (8)

$$\frac{1-f_{10}}{\beta'_1} = \frac{1-f_{20}}{\beta'_2} = \mu \qquad \text{[say]}$$

so that

$$Alp_o\ [n_1C_1\beta'_1(1+\theta_1-\theta_1\mu\beta'_1)+kn_2C_2\beta'_2(1+\theta_2-\theta_2\mu\beta'_2)]$$
$$= \mu\,[F_1C_1\beta'_1(1+\theta_1-\theta_1\mu\beta'_1)+kF_2C_2\beta'_2(1+\theta_2-\theta_2\mu\beta'_1)]$$
$$\times\ [n_1C_1\beta'_1(1+\theta_1-\theta_1\mu\beta'_1)+n_1C_2\beta'_2(1+\theta_2-\theta_2\mu\beta_2)]$$

This is a cubic to determine μ and hence f_{10}, f_{20} in terms of p_o. When $k=1$, this becomes a quadratic in μ.

(*ii*) The conditions obtained are in terms of and $\bar{v}_{[2]}$, which are to be obtained as *approximate* algebraic solutions of the equations of the fifth degree. The use of approximate solutions in equations like (44*a*) can lead to erroneous results.

(*iii*) In general, we have to solve two equations of the fifth degree in order to determine even the stage in which the maximum pressure occurs.

It may be noted that when $\gamma_1=\gamma_2$, the two equations are of the first degree in v and exact solutions are easily obtained in that case.

In the present case the direct approach in terms of the discussion of the sign of $\frac{dp}{dv}$ is simpler.

In the first stage

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a-v} + \frac{1}{b+v} - \frac{1}{a_2-v} + \frac{1}{a_1-v} - \frac{vK_1(a_1-v)}{KK_2(a-v)(b+v)(a_2-v)} \qquad \text{(52)}$$

In the second stage

$$\frac{1}{p}\frac{dp}{dv} = -\frac{1}{a'-v} + \frac{1}{b'+v} - \frac{1}{a_2'-v} + \frac{1}{b_2'+v} + \frac{1}{a'_1-v} - \frac{1}{b'_1+v}$$
$$- \frac{v\,K'_1(a'_1-v)(b'_1+v)}{K'(a'-v)(b'+v)K'_2(a'_2-v)(b'_2+v)} \qquad \text{(53)}$$

and after all-burnt

$$\frac{1}{p}\frac{dp}{dv} = \frac{-v\,w_1\left[(\gamma_1 - 1) + \dfrac{n_1 C_1 + k n_2 C_2}{n_1 C_1 + n_2 C_2}\right]}{F_1 C_1 + k F_2 C_2 - \frac{1}{2}(\gamma_1 - 1) w_1 v^2} \quad \ldots \quad (54)$$

At the end of the first stage, and beginning of the second $v = \dfrac{A}{\beta'_1 w_1} f_{10}$ and at the end of the second stage and beginning of motion after all-burnt $v = \dfrac{A}{\beta'_2 w_1} f_{20}$

$\therefore$ from (52), (53), (54) we can easily find

$$\left[\frac{1}{p}\frac{dp}{dv}\right]_1, \quad \left[\frac{1}{p}\frac{dp}{dv}\right]_2, \quad \left[\frac{1}{p}\frac{dp}{dv}\right]_3, \quad \left[\frac{1}{p}\frac{dp}{dv}\right]_4$$

Let these values be ν_1, ν_2, ν_3, ν_4 respectively, then the following cases arise (assuming uniqueness of maximum pressure)

(*i*) maximum pressure occurs in the first stage if

$$\nu_1 < 0 \quad \ldots \quad (55)$$

(*ii*) the maximum pressure occurs at the end of the first stage if

$$\text{(a) } \nu_1 = 0 \text{ or (b) } \nu_1 > 0,\ \nu_2 = 0 \text{ or (c) } \nu_1 > 0,\ \nu_2 < 0 \quad (56)$$

(*iii*) the maximum pressure occurs in the second stage if

$$\nu_1 > 0,\ \nu_2 > 0,\ \nu_3 < 0 \quad \ldots \quad (57)$$

(*iv*) the maximum pressure occurs at all-burnt if

$$\nu_1 > 0,\ \nu_2 > 0,\ \nu_3 \geqslant 0 \quad \ldots \quad (58)$$

In case (*ii*) and (*iv*) even to find the value of the maximum pressure, we have not to solve any algebraic equation, but in case (*i*) and (*iii*) we have to solve an algebraic equation of the 5th degree in *v*.

When $_1 = \gamma_2$ uniqueness of maximum pressure has already been established Kapur[6]. Even when $\gamma_1 \neq \gamma_2$ we expect the maximum pressure to be unique, though it has not yet been rigorously demonstrated that it will always be so. Actually from the above, the condition that there is a pressure maximum in each stage is

$$\text{(v) } \nu_1 < 0,\ \nu_2 > 0,\ \nu_3 < 0 \quad \ldots \quad (59)$$

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