MOTION OF A PROJECTILE IN A ROTATING EARTH

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ABSTRACT

Semi-theoretical expressions for the corrections to be included in the Range Tables for rotation of the earth have been deduced and numerical values for 25 pdr., streamlined projectile fired with super charge have been calculated. The expressions are in good agreement with similar attempts by other workers.

Introduction

When a particle of mass "m" is projected from the surface of the earth, the forces acting on it are

(*a*) $m\vec{g}$, the force due to gravity.

(*b*) $m\vec{V}F(v)$, the air resistance corresponding to velocity $\vec{V}$. In the standard notation the air resistance is expressed as $V^2 P(v/a) \times 10^{-4}/C$, where the constant C is the ballistic coefficient of the projectile, and P (v/a) is tabulated in 1940 tables.

(*c*) $2m[\vec{v} \times \vec{\omega}]$, the Coriolis force due to the rotation of the earth where $\vec{\omega}$ is the angular velocity vector of rotation, pointed towards the North Pole and its magnitude is of the order of $7 \cdot 3 \times 10^{-5}$ radians per sec.

(*d*) $m[\vec{r}_0 \times \vec{\omega}] \times \vec{\omega}$, the centrifugal force, where $\vec{r}_0$ is the radius vector of the projectile from the centre of the earth.

(*e*) Stray forces due to wind etc.

Owing to the smallness of the magnitude of ω, effects proportional to ω^2 such as those due to centrifugal force can be neglected in comparison with other forces mentioned above. Hence to a first approximation, neglecting the effects of wind, centrifugal force and curvature of the earth, the equation of motion of the particle can be written as

$$\frac{d\vec{v}}{dt} = -\vec{g} - \vec{v}F(v) + 2[\vec{\omega} \times \vec{v}] \quad \ldots \quad (1)$$

and the prescribed initial conditions are given as V_0, the velocity of projection; α, the angle of projection; λ, the latitude of the place of firing and **β**, the azimuth of the line of fire. For $|\omega|=0$ the equation (1) reduces to

$$\frac{d\vec{v}}{dt} = -\vec{g} - \vec{v}F(v) \quad \ldots \quad (2)$$

Solutions of (2) for given values of V_0, α and C have been obtained by various numercial methods and numerical values of R, the horizontal range; V_s, & θ_s the striking velocity and angle of descent; T, the time of fall etc. have been computed for different values of V_0, α and C. These values are available from the Range Tables for particular guns.

If analytic solutions of (2) were available, solutions of (1) could be expressed in terms of those of (2) treating the term containing $\vec{\omega}$ as a small perturbation. These being not possible, attempts will be made in this paper to express the solutions of (2) by approximate analytic functions, agreeing with the numerical solutions within reasonable degree of accuracy and then to express the solutions of (1) in terms of these approximate but analytic solutions.

Approximate solutions

Formal solutions of (1) and (2) can respectively be expressed as

$$\vec{v}(t) = \left\{ \left[\vec{v}_o - \vec{g} \int_o^t \frac{dt}{\varphi} \right] + 2 \int_o^t \left[\vec{\omega} \times \vec{v} \right] \frac{dt}{\varphi} \right\} \phi(t) \quad .. \quad (3)$$

and

$$\vec{v}(t) = \left[\vec{v}_o - \vec{g} \int_o^t \frac{dt}{\phi} \right] \phi(t) \quad .. \quad .. \quad .. \quad .. \quad (4)$$

with

$$\phi(t) = \frac{\text{Horizontal component } V_x \text{ at any time } t}{\text{Horizontal component } V_{ox} \text{ at } t = 0}$$

From equation (2) it can be shown that $\phi(t)$ continually decreases with time "t" but at a decreasing rate. The best analytic representation for such a function is of the form

$$\text{Exp.} \left[-\sum_{n=1}^{\infty} (A_n t^n) \right] \quad , \quad A_n \geqslant 0, n = 1, 2 \quad .. \quad .. \quad \infty.$$

For larger values of "t" the behaviour is more akin to exponential function and as a compromise between accuracy and simplicity, ϕ (t) can be replaced by ae^{-bt} when t is large and the constants a, b depend on V_0, α and C. Validity of such an assumption can be tested by evaluating values of R, T V & θ_s, the horizontal range, time of fall, speed of striking and angle of descent from this approximate method and comparing these values with those given in the range tables. Thus from (4).

$$(v_s)_x = v_{ox} \phi(T),$$

$$v_{ox} (\tan \alpha + \tan \theta s) / g = \int_o^T \frac{dt}{\phi(t)} \quad .. \quad .. \quad .. \quad (5)$$

$$R/v = \int_o^T \varphi(t)\, dt,$$

and $$\frac{R \tan \alpha}{g} = \int_o^T \phi(t) \left(\int_o^t \frac{dt}{\phi} \right) dt.$$

If $\varphi(t)$ is of the form ae^{-bt},

$$(v)_x = a \cdot e^{-bT} v_{ox},$$

$$v_{ox} (\tan \alpha + \tan \theta_s)/g = (e^{bT} - 1)/ab, \quad .. \quad .. \quad .. \quad (6)$$

$$R/v_{ox} = \frac{a}{b}\left[1 - e^{-bT}\right],$$

$$R \tan \alpha/g = \frac{1}{b}\left[T - \frac{1-e^{-bT}}{b}\right]$$

Eliminating 'a' and 'b' from the first three relations in (6) it can be shown that

$$\chi = \frac{v_{ox}(\tan \theta_s + \tan \alpha)(gT - v_{ox} \tan \theta_s)}{gR \tan \alpha} = 1$$

Values of λ when V_{ox}, V_x, θ_s, T & R are replaced by their corresponding range table values can be evaluated (see column IV of Table 1). Validity of replacing $\varphi(t)$ by the exponential function can be justified if χ differs little from unity. Alternatively, if values of a and b are chosen suitably such that calculated values of R, T, V_s and θ are found to be in close agreement with the range table values, the method can be justified for adoptation in calculating the effect of perturbation due to the rotation of the earth.

Values of a, b and χ for 25 pdr. projectiles fired with super charge and at higher register have been calculated with the help of relations (6) and range table values of R, T, V_s & θ_s. Departure from actual values are expected to be maximum for this values of V_o and angles of projection. From column IV of Table I, it is found that χ is near about unity throughout the whole range of angles of firing. For lower values of the velocity of projection, the agreement is still better.

Effect of rotation on the trajectory

The solutions of the equation (1) with the perturbing term can now be expressed easily in terms of the constants a, b evaluated from the solutions of the unperturbed equation. Since ω is small, the function $\vec{V}$ occurring in side the integral of the formal solution (3) can be replaced by $\vec{v}(t)$, the solution of equation (2). Let $\vec{\rho}$ and $T + \triangle T$ be the radius vector of the particle and the time of flight for the horizontal range when $\omega \neq o$. Then

$$\vec{\rho}(T+\triangle T) = \int_o^{T+\triangle T} \left\{\left[\vec{v}_o - \vec{g}\int_o^t \frac{dt}{\phi}\right] + \left[2\int_o^t \vec{\omega} \times \left(\vec{v}_o - g\int_o^t \frac{dt}{\phi}\right) dt\right\}\right] \phi(t)dt$$

$$\simeq \vec{r}(T) + \triangle T \vec{v}_s + 2[\omega \times \vec{v}_o] f_1(T) - 2[\vec{\omega} \times \vec{g}] f_2(T) \quad .. \quad (7)$$

where $\vec{r}(T)$ and $\vec{V}_s(T)$ are the functions for unperturbed equation representing the radius vector and the velocity of striking for the horizontal range and

$$f_1(T) = \int_0^T t\,.(t)\,dt = \frac{a}{b^2}\left[1 - (1 + bT)\,e^{--bT}\right] = \frac{R - v_x\,T}{b\,v_{ox}},$$

$$f_2(T) = \int_0^T \phi\left\{\int_0^t dt\left(\int_0^t \frac{dt}{\phi}\right)\right\} dt = \frac{(av_{ox} + v_x\,)\,T - 2R}{ab\,v_{ox}},$$

$$a = gR/v_{ox}\,[\tan\theta_s + \tan\alpha)\,v_x$$

$$b = [a - v_x/v_{ox}]/[(R/v_{ox})] = \frac{gR - v^2{}_x\,[\tan\alpha + \tan\,\theta_s]}{Rv_x\,[\tan\alpha + \tan\,\theta_s]}$$

Values of f_1 (T) and f_2 (T) with $V_0 = 1700$ f. s., and

$\alpha = 46°26', 55°9', 62°10', 65°42'$ & $69°56'$ for 25 pdr. stream lined shells have been calculated and entered in columns V & VI of Table I.

As α, λ and β represent respectively the angle of projection, latitude of the place of firing and the azimuth of the line of fire,

$$\omega_x = \omega\cos\lambda\cos\beta,\ \omega_y = \omega\sin\lambda,\ \omega_z = \omega\cos\lambda\sin\beta,$$

$$v_{ox} = v_o\cos\alpha,\ v_{oy} = v_o\sin\alpha,\ v_{oz} = 0,$$

$$g_x = 0\quad,\quad g_y = g\qquad g_z = 0,$$

$$(v_s)_x = v_s\cos\theta s,\ (v_s)_y = v_s\sin\theta_s,\ (v_z)_z = o.$$

Breaking the solution (7) into components and simplifying, it can be shown that

$$\triangle T = -\frac{2\omega\,v_o}{v_s}\cdot f_1(T)\ .\ \cos\alpha\ .\ \cos\lambda.\ \sin\beta;$$

$$\triangle R = 2\,\omega\cos\lambda\sin\beta\left[gf_2(T) - \frac{v_o\cos(\alpha - \theta_s)}{\sin\theta_s}\cdot f_1(T)\right],$$

and $D = 2\omega\cos\lambda\cos\beta\,[v_o\sin\alpha\,f_1(T) - gf_2] - 2\omega v_o f_1(T)\,.\,\sin\lambda.\,\cos\alpha.\,..$ (8)
where $\triangle T$, $\triangle R$ and D represent respectively the increments in time of fall, horizontal range and the lateral displacement. D is to the right in the N. H. and vice-versa.

TABLE I

Q.F. 25 pdr. Streamlined shell

Angle of firing	a	b	λ	$f_1(T)$	$f_2(T)$
46°26	·806	·0105	·996	880	20482
55°9	·826	·0103	·988	1093	30027
62°10′	·835	·0097	·986	1255	41215
65°42′	·837	·0094	·984	1333	44099
69°56′	·844	·0093	·977	1411	45723

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