ON THE INTERNAL BALLISTICS OF COMPOSITE CHARGES WITH CUBIC FORM FUNCTION

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ABSTRACT

The equations of internal ballistics of composite charges are solved with cubic form function and including the co-volume correction term following Hunt-Hind system. Results are obtained in terms of algebraic functions or in terms of tabulated functions (Incomplete Beta Function) by making suitable approximation in evaluating the integral arising out of the inclusion of the co-volume correction term.

Introduction

The problem of composite charges has been tackled by Corner[1] and H.M.S.O.[2] by reducing the problem to one of an equivalent single charge. A direct treatment of the same problem has been given by Venkatesan and Patni[3] under the simplifying assumption that the co-volume of the gases equals their specific volumes. In this paper an attempt is made to treat the problem under a less restrictive condition viz. that the co-volume is not necessarily equal to the specific volume. Further this treatment is an extension to the case of cubic form function. The treatment is based on the assumption that the law of burning is linear and that the ratio of the specific heats for the two propellants is the same.

Basic Equations

Non-dimensionalisation of the equations leads to

$$z_1 + Lz_2 = \zeta_1 (\xi - B_1 z_1 - B_2 z_2) + \frac{\gamma - 1}{2M_1} \eta^2{}_1 \quad \ldots \quad (1)$$

$$\eta_1 \frac{df_1}{d\xi} = - \zeta_1 \quad \ldots \quad (2a)$$

$$\eta_2 \frac{df_2}{d\xi} = - \zeta_2 \quad \ldots \quad (2b)$$

$$\eta_1 \frac{d\eta_1}{d\xi} = M_1 \zeta_1 \quad \ldots \quad (3a)$$

$$\eta_2 \frac{d\eta_2}{d\xi} = M_2 \zeta_2 \quad \ldots \quad (3b)$$

$$z_1 = (1 - f)(1 + \theta_1 f_1 + \theta'_1 f^2{}_1) \quad \ldots \quad (4a)$$

$$z_2 = (1 - f_2)(1 + \theta_2 f_2 + \theta'_2 f^2{}_2) \quad \ldots \quad (4b)$$

where

$$L = \frac{F_2 C_2}{F_1 C_1} \quad \ldots \quad (5a)$$

$$\xi = 1 + \frac{x}{l} \qquad \text{(5b)}$$

$$\eta_1 = \frac{AD_1}{F_1\beta_1C_1}v \qquad \text{(5c)}$$

$$\eta_2 = \frac{AD_2}{F_2\beta_2C_2}v \qquad \text{(5d)}$$

$$\zeta_1 = \frac{Al}{F_1C_1}p \qquad \text{(5e)}$$

$$\zeta_2 = \frac{Al}{F_2C_2}p \qquad \text{(5f)}$$

$$M_1 = \frac{A^2D^2_1}{F_1\beta_1{}^2C_1W_1} \qquad \text{(5g)}$$

$$M_2 = \frac{A^2D^2_2}{F_2\beta^2_2C_2W_1} \qquad \text{(5h)}$$

$$\lambda = \frac{F_1C_1\beta'_1}{F_2C_2\beta'_2} \qquad \text{(5k)}$$

$$\beta'_1 = \beta_1/D_1 \qquad \text{(5}l\text{)}$$

$$\beta'_2 = \beta_2/D_2 \qquad \text{(5m)}$$

$$B_1 = \frac{C_1}{Al}\left(b_1 - \frac{1}{\delta_1}\right) \qquad \text{(5n)}$$

$$B_2 = \frac{C_2}{Al}\left(b_2 - \frac{1}{\delta_2}\right) \qquad \text{(5p)}$$

Hence $\eta_2 = \lambda\eta_1$ .. (6)

Initial conditions

$$z_1 = z_{10};\ z_2 = z_{20};\ \zeta_1 = \zeta_{10};\ \zeta_2 = \zeta_{20};\ \xi = 1;\ \eta_1 = \eta_2 = 0$$

Hence

$$z_{10} + Lz_{20} = \zeta_{10}(1 - B_1z_{10} - B_2z_{20}) \qquad \text{(7)}$$

Now from (2*a*) and (2*b*) on integration

$$\frac{(1 - f_1)}{\beta'_1} = \frac{(1 - f_2)}{\beta'_2}$$

$$\text{i.e., } \beta'_2 - \beta'_1 = \beta'_2 f_1 - \beta'_1 f_2 = \beta'_2 f_{10} - \beta'_1 f_{20} \qquad \text{(8)}$$

$$\text{i.e., } f_{20} = \beta'_1 - \beta'_2 (1 - f_{10}) \qquad \text{(9)}$$

Substituting (9) in (7) gives us a cubic equation in f_{10}. However since this is a physical case there will be only one admissible value of f_{10} (that lying between 0 and 1).

Solution of the equation

From (2*a*) and (3*a*) on integration we get

$$\eta_1 = M_1 (f_{10} - f_1) \qquad \text{(10a)}$$

and similarly from (2*b*) and (3*b*) we have

$$\eta_2 = M_2 \; (f_{20} - f_2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{10b}$$

Substituting (10*a*), (10*b*), (6) and (3*a*) in (1) we get

$$M^3{}_1 \, \eta_1 \frac{d\eta_1}{d\xi} (\xi - B_1 z_1 - B_2 z_2) = (a - \eta_1)(b + \eta_1)(c - k\eta_1) \quad \ldots \tag{11}$$

where

$$abc = M^3{}_1 \, (z_{10} + L z_{20}) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{12a}$$

$$- bc + ac - kab = M^2{}_1 \Big[(1 - \theta_1) + 2 (\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2{}_{10} + \frac{\lambda L M_1}{M_2} \Big\{ 1 - \theta_2 + 2 (\theta_2 - \theta'_2) f_{20} + 3\theta'_2 f^2{}_{20} \Big\} \Big] \quad \ldots \tag{12b}$$

$$- c - ka + kb = M_1 \Big[\theta_1 - \theta'_1 + 3 \theta'_1 f_{10} + \frac{\lambda^2 M^2{}_1 L}{M^2{}_2} \times (\theta_2 - \theta'_2 + 3 \theta'_2 f_{20}) + \frac{\gamma - 1}{2} M_1 \Big] \quad \ldots \quad \ldots \tag{12c}$$

and
$$k = \theta'_1 + \frac{\lambda^3 M^3{}_1 L}{M^3{}_2} \theta'_2 \quad \ldots \quad \ldots \quad \ldots \tag{12d}$$

Equation (11) can be written as

$$\frac{d\xi}{d\eta_1} - \frac{M_1{}^2 \, \xi \, \eta_1}{(a - \eta_1)(b + \eta_1)(c - k\eta_1)} = \frac{-M_1{}^2 \, \eta_1 \, (B_1 z_1 + B_2 z_2)}{(a - \eta_1)(b + \eta_1)(c - k\eta_1)} \tag{13}$$

Hence on integrating and substituting initial conditions we get

$$\xi \, (a - \eta_1)^{\frac{M^2{}_1 a}{(a + \;\;)(c - ka).}} (b + \eta_1)^{\frac{M_1{}^2 b}{(a + b)(c + kb).}} (c - k\eta_1)^{\frac{M^2{}_1 c}{(ka - c)(kb + c)}}$$
$$- a^{\frac{M^2{}_1 a}{(a + b)(c - ka)}} \; b^{\frac{M^2{}_1 b}{(a + b)(c + kb)}} \; c^{\frac{M^2{}_1 c}{(ka - c)(kb + c)}}$$
$$= - \frac{I}{M_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \tag{14}$$

where

$$I = \int_0^{\eta_1} \Big[M^3{}_1 \, (B_1 z_{10} + B_2 z_{20}) + \eta_1 M^2{}_1 \Big\{ B_1 (1 - \theta_1 + 2 \overline{\theta_1 - \theta'_1} f_{10} + 3\theta'_1 f^2{}_{10}) \Big\}$$
$$+ \frac{B_2 M_1 \lambda}{M_2} (1 - \theta_2 + 2 \, \overline{\theta_2 - \theta'_2} \, f_{20} + 3 \, \theta'_2 f^2{}_{20})$$
$$- M \eta_1{}^2{}_1 \Big\{ B_1 (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) + \frac{\lambda^2 B_2 M_1{}^2}{M^2{}_2} (\theta_2 - \theta'_2 + 3\theta'_2 f_{20}) \Big\}$$
$$+ \eta^3{}_1 \Big(B_1 \theta'_1 + \frac{\lambda^3 M^3{}_1 B_2}{M^3{}_2} \theta'_2 \Big) \Big] \times \eta_1 (a - \eta_1)^{\frac{M^2{}_1 a}{(a + b)(c - ka)} - 1}$$
$$\times (b + \eta_1)^{\frac{M^2{}_1 b}{(a + \;\;)(c + kb)} - 1} \times (c - k\eta_1)^{\frac{M^2{}_1 c}{(ka - c)(kb + c)} - 1} \times d\eta_1 \tag{15}$$

Now from equation (3*a*)

$$\frac{1}{\zeta} = \frac{M_1}{\eta_1} \frac{d\xi}{d\eta_1}$$

$$= \frac{M^3_1 \xi - M^3_1 (B_1 z_1 + B_2 z_2)}{(a - \eta_1)(b + \eta_1)(c - k\eta_1)} \quad \ldots \quad (16)$$

Equations (15) and (16) give ξ and ζ_1 as functions of η_1 and with (10*a*), ξ and ζ are known as functions of f_1. These equations are valid so long as both the propellants are burning.

From (8) we see that if

(1) $\beta'_2 > \beta'_1$ then f_1 cannot become zero before f_2. Hence charge C_2 is burnt earlier.

(2) $\beta'_2 < \beta'_1$ then f_2 cannot become zero before f_1.

Hence charge C_1 is burnt earlier.

(3) $\beta'_2 = \beta'_1$ then both the charges are burnt out together.

Hence two different cases arise viz.,

(A) Both the propellants burn out together.

(B) The propellants burn out at different times.

Let us call that propellant which will burn out first C_2.

Case A

This has to be dealt within two parts.

(*a*) When the propellants are burning

(*b*) When the propellants are burnt out.

Of these (*a*) has already been dealt with and if we denote the values at 'all-burnt' by the suffix B we have

$$\eta_{1B} = M_1 f_{10} \quad \ldots \quad (17)$$

and ξ_B is given by

$$\xi_B (a - \eta_{1B})^{\frac{M^2_1 a}{(a+b)(c-ka)}} (b+\eta_{1B})^{\frac{M^2_1 b}{(a+b)(c+kb)}} (c-k\eta_{1B})^{\frac{M^2_1 c}{(ka-c)(kb+c)}}$$

$$- a^{\frac{M^2_1 a}{(a+b)(c-ka)}} b^{\frac{M^2_1 b}{(a+b)(c+kb)}} c^{\frac{M^2_1 c}{(ka-c)(kb+c)}}$$

$$= -\frac{I_B}{M_1} \quad \ldots \quad (18)$$

and

$$\frac{1}{\zeta_{1B}} = \frac{M^3_1(\xi_B - \overline{B_1 + B_2})}{(a - \eta_{1B})(b + \eta_{1B})(c - k\eta_{1B})} \quad \ldots \quad (19)$$

After 'all-burnt'

Equation (1) becomes

$$1 + L = \zeta_1 (\xi - B_1 - B_2) + \frac{\gamma - 1}{2M_1} \eta^2_1 \quad \ldots \quad (20)$$

On integration this gives

$$\left(\xi - B_1 - B_2\right)^{\gamma - 1}\left(1 + L - \frac{\gamma - 1}{2M_1}\eta^2{}_1\right) = \text{const} = \Phi \text{ (say)} \qquad (21)$$

and

$$\Phi = \left(\xi_B - B_1 - B_2\right)^{\gamma - 1}\left(1 + L - \frac{\gamma - 1}{2M_1}\eta^2{}_{1B}\right) \qquad .. \qquad (22)$$

Now from equation (20)

$$\left(1 + L - \frac{\gamma - 1}{2M_1}\eta^2{}_1\right) = \zeta_1 (\xi - B_1 - B_2)$$

Hence

$$\zeta_1 (\xi - B_1 - B_2)^{\gamma - 1} = \Phi \qquad .. \qquad .. \qquad .. \qquad .. \qquad (23)$$

i. e.

$$\eta^2{}_1 = \left\{1 + L - \frac{\Phi}{(\xi - B_1 - B_2)^{\gamma - 1}}\right\}\frac{2M_1}{\gamma - 1} \qquad .. \qquad .. \qquad (24)$$

Hence

$$\eta^2{}_{1E} = \frac{2M_1}{\gamma - 1}\left\{1 + L - \frac{\Phi}{(\xi_E - B_1 - B_2)^{\gamma - 1}}\right\} \qquad .. \qquad .. \qquad (25)$$

Case B

(*a*) When both charges are burning

(*b*) When C_2 is burnt out

(*c*) When C_1 is also burnt out.

Case B(a) has already been dealt with.

If we denote the values at the instant when charge is burnt out by the suffix B_2 we have

$$\eta_{2,B_2} = M_2 f_{20} \qquad .. \qquad .. \qquad .. \qquad .. \qquad .. \qquad .. \qquad (26a)$$

and $\eta_2 = \lambda\, \eta_1$

Hence $\eta_{1,B_2} = \dfrac{M_2}{\lambda} f_{20}$ $\qquad .. \qquad .. \qquad .. \qquad .. \qquad .. \qquad$ (26b)

and ξ_{B_2} is given by the relation

$$\xi_{B_2}(a - \eta_{1,B_2})^{\frac{M^2{}_1 a}{(a + b)(c - ka)}}(b + \eta_{1,B_2})^{\frac{M^2{}_1 b}{(a + b)(c + kb)}}(c - k\eta_{1,B_2})^{\frac{M_1{}^2 c}{(ka - c)(kb + c)}} - a^{\frac{M_1{}^2 a}{(a + b)(c - ka)}}\, b^{\frac{M_1{}^2 b}{(a + b)(c + kb)}}\, c^{\frac{M_1{}^2 c}{(ka - c)(kb + c)}}$$

$$= -\frac{I_{B_2}}{M_1} \qquad .. \qquad .. \qquad .. \qquad .. \qquad .. \qquad .. \qquad (27)$$

and

$$\frac{1}{\zeta_{1,B_2}} = \frac{M^3_1 \left(\xi_{B_2} - B_1 z_{1,B_2} - B_2 \right)}{(a-\eta_{1,B_2})\ (b+\eta_{1,B_2})\ (c-k\eta_{1,B_2})} \quad \text{(28)}$$

Equation (1) becomes

$$z_1 + L = \zeta_1 (\xi - B_1 z_1 - B_2) + \frac{\gamma - 1}{2M_1} \eta^2_1 \quad \text{(29)}$$

Substituting we get

$$M_1^3 \eta_1 \frac{d\eta_1}{d\xi} (\xi - B_1 z_1 - B_2) = (a_1 - \eta_1)(b_1 + \eta_1)(c_1 - k_1\eta_1) \quad \text{(30)}$$

where

$$a_1 b_1 c_1 = M^3_1 (z_{10} + L) \quad \text{(31a)}$$

$$- b_1 c_1 + a_1 c_1 - k_1 a_1 b_1 = M^2_1 \left\{1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2_{10}\right\} \quad \text{(31b)}$$

$$- c_1 - k_1 a_1 + k_1 b_1 = - M_1 \left(\theta_1 - \theta'_1 + 3\theta'_1 f_{10} + \frac{\gamma - 1}{2} M_1\right) \quad \text{(31c)}$$

$$k_1 = \theta'_1 \quad \text{(31d)}$$

Hence

$$\frac{d\xi}{d\eta_1} - \frac{M^2_1 \xi \eta_1}{(a_1-\eta_1)(b_1+\eta_1)(c-k_1\eta_1)} = \frac{-M^2_1\eta_1 (B_1 z_1 + B_2)}{(a_1-\eta_1)(b_1+\eta_1)(c_1-k_1\eta_1)} \quad \text{(32)}$$

On integration and substitution of initial conditions we have

$$\xi\, (a_1-\eta_1)^{\frac{M^2_1 a_1}{(a_1+b_1)(c_1-k_1 a_1)}} \, (b_1+\eta_1)^{\frac{M^2_1 b_1}{(a_1+b_1)(c_1+k_1 b_1)}} \times (c_1-k_1\eta_1)^{\frac{M^2_1 c_1}{(k_1 a_1-c_1)(k_1 b_1+c_1)}}$$

$$-\xi_{B_2}\, (a_1-\eta_{1,B_2})^{\frac{M^2_1 a_1}{(a_1+b_1)(c_1-k_1 a_1)}} \, (b_1+\eta_{1,B_2})^{\frac{M^2_1 b_1}{(a_1+b_1)(c_1+k_1 b_1)}} \times (c_1-\eta_{1,B_2})^{\frac{M^2_1 c_1}{(k_1 a_1-c_1)(k_1 b_1+c_1)}}$$

$$= - I^*/M1 \quad \text{(33)}$$

where

$$I^* = \int_{\eta_{1,B_2}}^{\eta} \Big[M^3_1 (B_1 z_{10} + B_2) + M^2_1 \eta_1 B_1 \left\{ 1 - \theta_1 + 2\overline{\theta_1 - \theta'_1} f_{10} + 3\theta'_1 f^2_{10} \right\}$$

$$- M\eta^2_1 B_1 (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) + B_1 \theta'_1 \eta^3_1 \Big] \times$$

$$\eta_1 (a_1 - \eta_1)^{\frac{M^2_1 a_1}{(a_1+b_1)(c_1-k_1 a_1)} - 1} \times (b_1 + \eta_1)^{\frac{M^2_1 b_1}{(a_1+b_1)(c_1+k_1 b_1)} - 1}$$

$$\times (c_1 - k_1 \eta_1)^{\frac{M^2_1 c_1}{(k_1 a_1 - c_1)(k_1 b_1 + c_1)} - 1} \times d\eta_1 \quad \text{(34)}$$

and

$$\frac{1}{\zeta} = \frac{M^3{}_1\xi - M_1{}^3(B_1z_1 + B_2)}{(a_1 - \eta_1)(b_1 + \eta_1)(c_1 - k_1\eta_1)} \quad \ldots \quad \ldots \quad \ldots (35)$$

'All-burnt'

$$\eta_{1B} = M_1f_{10}$$

and ξ_B is given by

$$\xi_B\,(a_1 - \eta_{1B})^{\frac{M^2{}_1a_1}{(a_1+b_1)(c_1-k_1a_1)}}\,(b_1 + \eta_{1B})^{\frac{M^2{}_1b_1}{(a_1+b_1)(c_1+k_1b_1)}}$$

$$(c_1 - \eta_{1B})^{\frac{M^2{}_1c_1}{(k_1a_1-c_1)(k_1b_1+c_1)}}$$

$$-\,\xi_{B,_2}\,(a_1 - \eta_{1,B_2})^{\frac{M^2{}_1a}{(a_1+b_1)(c_1-k_1a_1)}}\,(b_1 + \eta_{1,B_2})^{\frac{M^2{}_1b_1}{(a_1+b_1)(c_1+k_1b_1)}}$$

$$(c_1 - k_1\eta_{1,B_2})^{\frac{M^2{}_1c_1}{(k_1a_1-c_1)(k_1b_1+c_1)}}$$

$$= -\frac{I^*_B}{M_1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (36)$$

and

$$\frac{1}{\zeta_{1B}} = \frac{M^3{}_1(\xi_B - B_1 - B_2)}{(a_1 - \eta_{1B})(b_1 + \eta_{1B})(c_1 - k_1\eta_{1B})} \quad \ldots \quad \ldots \quad \ldots (37)$$

After 'all-burnt'

The form of the equations are the same as in (20) to (25) and hence

$$(\xi - B_1 - B_2)^{\gamma-1}\left(1 + L - \frac{\gamma - 1}{2M_1}\eta_1{}^2\right) = \Phi^* \quad \ldots \quad \ldots (38)$$

where

$$\Phi^* = (\xi_B - B_1 - B_2)\left(1 + L - \frac{\gamma - 1}{2}M_1f^2{}_{10}\right) \quad \ldots \quad \ldots (39)$$

and ξ_B is given by equation (36);

ζ_1

$$\zeta_1(\xi - B_1 - B_2)^{\gamma} = \Phi^* \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots (40)$$

Maximum Pressure

The following cases may arise viz., the maximum pressure occurs when

(*a*) Both the charges are burning

(*b*) C_1 only is burning

(*c*) At the position of All-burnt.

Both charges are burning

In this case the pressure is given by equation (16)

$$\frac{1}{\zeta_1} = \frac{M^3{}_1\xi - M^3{}_1(B_1z_1 + B_2z_2)}{(a - {}_1)(b + {}_1)(c - k_1\eta_1)}$$

At the positions of maximum pressure $d\zeta_1=0$. Hence differentiating (16) and putting $d\zeta_1=0$ we get if we denote values at maximum pressure by the suffix '*m*'

$$M^3_1\left(\frac{d\xi}{d\eta_{1m}}\right)_m = \frac{M^3_1\,\xi_m\{3k\eta^2_{1m}-2\eta_{1m}(ka+c-kb)+ac-bc-kab\}}{(a-\eta_{1m})\;(b+\eta_{1m})\;(c-k\eta_{1m})}$$

$$+\Big[M_1^2\Big\{B_1(1-\theta_1+2\,\overline{\theta_1-\theta'_1}\,f_{10}+3\,\theta'_1 f'_{10})+\frac{B_2\lambda M_2}{M_1}(1-\theta_2)$$

$$+2\,\overline{\theta_2-\theta'_2}\,f_{20}+3\,\theta'_2 f'_{20})\Big\}$$

$$-2M_1\eta_{1m}\Big\{B_1(\theta_1-\theta'_1+3\,\theta'_1 f_{10})+\frac{B_2\lambda M^2_2}{M^2_1}(\theta_2-\theta'_2+3\,\theta'_2 f_{20})\Big\}$$

$$+3\,\eta^2_{1m}\,(B_1\theta'_1+\frac{\lambda^3 M^3_1}{M^3_2}\,B_2\,\theta'_2)\Big]\times$$

$$\Big[M^3_1(B_1 z_{10}+B_2 z_{20})+M^2_1\;{}_{1m}\Big\{B_1(1-\theta_1+\overline{2\theta_1-\theta'_1}\,f'_{10}\,3\theta'_1 f^2_{10})$$

$$+\frac{B_2\lambda M_2}{M_1}(1-\theta_2+\overline{2\,\theta_2-\theta'_2}\,f_{20}+3\,\theta'_2 f'_{20})\Big\}$$

$$-M\eta^2_1\Big\{B_1(\theta_1-\theta'_1+3\,\theta'_1 f_{10})+\frac{B_2\lambda^2 M^2_2}{M^2_1}(\theta_2-\theta'_2+3\,\theta'_2\,f_{20})\Big\}$$

$$+\eta^3_{1m}(B_1\theta'_1+B_2\frac{\lambda^3 M^3_2}{M^3_1}\theta'_2)\Big]$$

$$-\frac{\{3k\eta^2_{1m}-2\eta_{1m}(ka+c-kb)+ac-bc-kab\}}{(a-\eta_{1m})\;(b+\eta_{1m})\;(c-k\eta_{1m})} \qquad \ldots \qquad (41)$$

But from equation (13)

$$\left(\frac{d\xi}{d\eta_1}\right)_m = \frac{M^2_1\,\xi_m\,\eta_{1m}-M^2_1\,\eta_{1m}\,(B_1 z_{1m}+B_2 z_{2m})}{(a-\eta_{1m})\;(b+\eta_{1m})\;(c-k\eta_{1m})} \qquad \ldots \qquad (42)$$

Substituting (42) in (41) and simplifying we get

$$M_1\,\xi_m\Big[3\Big(\theta'_1+\frac{\lambda^3 M^3_1}{M^3_2}\,L\theta'_2\Big)\eta^2_{1m}-2M_1\eta_{1m}\Big\{\theta_1-\theta'_1+3\,\theta'_1 f_{10}$$

$$+\frac{\lambda^2 M^2_1 L}{M^2_2}(\theta_2-\theta'_2+3\,\theta'_2\,f_{20})+\frac{\gamma M_1}{2}\Big\}$$

$$+M_1^2\Big\{1-\theta_1+2\,(\theta_1-\theta'_1)\,f_{10}+3\,\theta'_1\,f^2_{10}+\frac{\lambda L M_1}{M_2}(1-\theta_2+$$

$$2\,\overline{\theta_2-\theta'_2}\,f_{20}+3\,\theta'_2\,f'_{20})\Big\}\Big]$$

$$= \Big[\frac{M^4_1 \lambda}{M_2} z_{10} \{1 - \theta_2 + 2(\theta_2 - \theta'_2) f_{20} + 3\theta'_2 f'_{20}\} (B_1 L - B_2)$$

$$+ M^3_1 z_{20} \{1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f'^2_{10}\} (B_1 L - B_2) \Big]$$

$$+ M^3_1 \eta_{1m} \Big[z_{10} \Big\{ \frac{2\lambda^2 M_1}{M^2_2} (\theta_2 - \theta'_2 + 3\theta'_2 f_{20}) (B_2 - B_1 L) - \gamma \Big\}$$

$$+ z_{20} \Big\{ \frac{2}{M_1} (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) (B_1 L - B_2) - \gamma \Big\} \Big]$$

$$+ M_1 \eta^2_{1m} \Big[\frac{3\lambda^3 M^3_1 \theta'_2}{M^3_2} z_{10} (B_1 L - B_2) + 3\theta'_1 z_{20} (B_2 - B_1 L)$$

$$+ \frac{\lambda}{M_2} (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) \Big\{1 - \theta_2 + 2(\theta_2 - \theta'_2) f_{20} + 3\theta'_2 f^2_{20}\Big\} (B_1 L - B_2)$$

$$+ \frac{\lambda^2 M^2_1}{M^2_2} (\theta_2 - \theta'_2 + 3\theta'_2 f_{20}) \Big\{1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2_{10}\Big\} (B_2 - B_1 L)$$

$$- B_1 \Big\{1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2_{10}\Big\} \frac{\gamma + 1}{2}$$

$$- B_2 \frac{\lambda M_1}{M_2} \Big\{1 - \theta_2 + 2(\theta_2 - \theta'_2) f_{20} + 3\theta'_2 f'_{20}\Big\} \frac{\gamma + 1}{2} \Big]$$

$$+ M_1 \eta^3_{1m} \Big[\frac{2\lambda}{M_2} \theta'_1 \Big\{1 - \theta_2 + 2(\theta_2 - \theta'_2) f_{20} + 3\theta'_2 f^2_{20}\Big\} (B_1 L - B_2)$$

$$+ \frac{2\lambda^3}{M^3_2} M^2_1 \theta'_2 \Big\{1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f'_{10}\Big\} (B_1 L - B_2)$$

$$+ B_1 (\theta_1 - \theta'_1 + 3\theta'_1 f_{10})$$

$$+ \frac{B_2 \lambda^2 M^2_1}{M^2_2} (\theta_2 - \theta'_2 + 3\theta'_2 f_{20}) \Big]$$

$$+ \eta^4_{1m} \Big[\frac{\lambda^2 M_1}{M^2_2} \theta'_1 (\theta_2 - \theta'_2 + 3\theta'_2 f_{20}) (B_1 L - B_2)$$

$$+ \frac{\lambda^3 M^2_1}{M^3_2} \theta'_2 (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) (B_2 - B_1 L)$$

$$+ \frac{\gamma - 3}{2} \Big(B_1 \theta'_1 + \frac{B_2 \lambda^3 M^3_1}{M^3_2} \theta'_2 \Big) \Big] \quad \ldots \quad (43)$$

Equations (14) with $\eta_1 = \eta_{1m}$ and (43) give ξ_m and η_{1m} and from (16) we can get ζ_{1m}

C_1 only is burning

In this case the pressure is given by equation (35)

$$\frac{1}{\zeta_1} = \frac{M^3{}_1 \xi - M^3_1 (B_1 z_1 + B_2)}{(a_1 - \eta_1)(b_1 + \eta_1)(c_1 - k_1 \eta_1)}$$

Hence differentiating (35) putting $d\zeta_1 = o$ and proceeding as in the case when both charges are burning we get after simplification

$$
\begin{aligned}
M\xi_m \Big[3\theta'_1 \eta^2_{1m} - 2M_1 \eta_{1m} \left(\theta_1 - \theta'_1 + 3\theta'_1 f_{10} + \frac{\gamma M_1}{2} \right) \\
+ M^2_1 \left\{ 1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2_{10} \right\} \Big] \\
= M^3_1 \left\{ 1 - \theta_1 + 2(\theta_1 - \theta'_1) f_{10} + 3\theta'_1 f^2_{10} \right\} (B_2 - B_1 L) \\
+ M^2_1 \eta_{1m} \left\{ 2(\theta_1 - \theta'_1 + 3\theta'_1 f_{10})(B_1 L - B_2) - \gamma M_1 B_1 z_{10} \right\} \\
+ M_1 \eta^2_{1m} \Big[3\theta'_1 (B_2 - B_1 L) - B_1 \Big\{ 1 - \theta_1 + 2\overline{\theta_1 - \theta'_1} f_{10} \\
+ 3\theta'_1 f^2_{10} \Big\} \times \frac{\gamma + 1}{2} M_1 \Big\} \Big] \\
+ \eta^3_{1m} B_1 (\theta_1 - \theta'_1 + 3\theta'_1 f_{10}) M_1 \\
+ \eta^4_{1m} B_1 \theta'_1 \frac{\gamma - 3}{2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)
\end{aligned}
$$

Equation (33) with $\eta_1 = \eta_{1B}$ and equation (44) give ξ_m and η_{1m} and from (35) we get ζ_{1m}.

'At-Burnt'

The maximum pressure is given by equation (19) and (37) in the two cases.

Approximate solution when B_1 *and* B_2 *are small*

In a number of systems of internal ballistics the co-volume b of the gases is taken equal to the specific volume, so that $B = \frac{C}{Al}\left(b - \frac{1}{\delta}\right)$ becomes equal to zero. In this section B_1 and B_2 are not taken to be zero but are treated as small quantities and as correction terms. Hence second and higher powers of B_1 and B_2 are neglected. The treatment here also has, one of necessity, to be divided into two parts.

(*i*) $\beta'_2 = \beta'_1$

(*ii*) $\beta'_2 > \beta'_1$

Upto 'All-Burnt' $\beta'_2 = \beta'_1$

ξ is given in terms of η_1 by equations (14) and (15). In order to evaluate the integral either in terms of algebraic or tabulated functions we make some simplyfying assumptions. The approximation is not too drastic since they are made only in I every term of which is multiplied by one or the other of the small quantities B_1 and B_2.

If we are interested in evaluating the integral I in terms of tabulated functions only it is enough if we made the assumption that the shot-start pressure is zero or $\theta'_1=\theta'_2=0$. Then it is possible to express I in terms of Incomplete Beta Function. However for the sake of simplicity and for the sake of bringing this treatment in line with the one for single charge[4] we shall make use of both of the assumptions in evaluating I. But in the case where both the propellants burn at different times we shall make only one assumption viz. that $\theta'_1 = 0$.

Before 'Burnt'

In evaluating I we put

$$\left.\begin{aligned} p_0 &= 0 \\ \theta'_1 &= \theta'_2 = 0 \end{aligned}\right\} \qquad (45)$$

If we denote by primed letters the values when equation (45) holds good we have

$$a' = \frac{M_1\left\{1 + \theta_1 + \dfrac{\lambda L M_1}{M_2}(1+\theta_2)\right\}}{\theta_1 + \dfrac{\lambda^2 L M^2_1}{M^2_2}\theta_2 + \dfrac{\gamma - 1}{2}M_1} \qquad (46a)$$

$$b' = 0 \qquad (46b)$$

$$c' = M_1\left(\theta_1 + \frac{\lambda^2 L M^2_1}{M^2_2}\theta_2 + \frac{\gamma-1}{2}M_1\right) \qquad (46c)$$

and

$$I' = \left(\frac{1}{C'}\right)^{\frac{M^2_1}{C'}+1} \int_0^{\eta_1}\Big[B_1\left\{M_1^2\eta_1(1+\theta_1) - M_1\eta_1^2\theta_1\right\} + B_2\left\{\frac{\lambda M_1^3}{M_2}(1+\theta_2)\eta_1 - \frac{\lambda M_1^3}{M_2^3}\theta_2\eta^2_1\right\}\Big] \times (a'-\eta_1)^{M_1^2/c_1 - 1}\, d\eta_1 \qquad (46d)$$

$$= B_1 I_1 + B_2 I_2 \quad \text{(say)} \qquad (47)$$

where

$$I_1 = \left(\frac{1}{C'}\right)^{\frac{M^2_1}{C'_1}+1} \int_0^{\eta_1}\left\{M^2{}_1\eta_1(-1+\theta_1) - M_1\eta^2{}_1\theta_1\right\}(a'-\eta)^{\frac{M^2_1}{c'}-1}\, d\eta_1 \qquad (48a)$$

and

$$I_2 = \left(\frac{1}{C'}\right)^{\frac{M^2_1}{C'}+1} \int_0^{1}\left\{\frac{\lambda M^3_1}{M_2}(1+\theta_2)\eta_1 - \frac{\lambda^2 M^3_1}{M^2_2}\theta_2\eta^2_1\right\}(a'-\eta_1)^{\frac{M^2_1}{c'}-1}\, d\eta_1 \qquad (48b)$$

Hence

$$I_1 = M_1\, a'^{\frac{M^2_1}{c'}}\; c'^{-\frac{M^2_1}{c'}}\; K_1 \quad \ldots \quad (49a)$$

and

$$I_2 = M_1\, a'^{\frac{M^2_1}{c'}}\; c'^{-\frac{M^2_1}{c'}}\; K_2 \quad \ldots \quad (49b)$$

where

$$K_1 = \left[\frac{1}{M^3_1}\left\{\frac{2M_1\theta_1 c'^2(a'-\eta_1)^2}{(M^2_1+c')(M^2_1+2c')} - \frac{(M^2_1\overline{1+\theta_1} - 2M_1\theta_1\eta^1)(a'-\eta_1)c'}{(M^2_1+c')} - (M^2_1\,\overline{1+\theta_1}\eta_1 - M_1\theta_1\eta^2_1)\right\}\left(1-\frac{\eta_1}{a'}\right)^{\frac{M^2_1}{c'}} + \frac{a'c'}{M^3_1(M^2_1+c')}\left\{M_1(1+\theta_1) - \frac{2M_1\theta_1 a'c'}{M^2_1+2c'}\right\}\right] \quad \ldots \quad (50a)$$

and

$$K_2 = \left[\left\{\frac{2\lambda^2c'^2(a'-\eta_1)^2}{M^2_2(M^2_1+c')(M^2_1+2c')} - \frac{c'\lambda\left(1+\theta_2-\frac{2\lambda}{M_2}\theta_2\eta_1\right)(a'-\eta_1)}{M_2(M^2_1+c')} - \frac{\lambda}{M_2}\left(\overline{1+\theta_2}\eta_1 - \frac{\lambda}{M_2}\theta_2\eta_1^2\right)\right\}\left(1-\frac{\eta_1}{a'}\right)^{\frac{M^2_1}{c'}} + \frac{a'c'}{(M^2_1+c')}\frac{\lambda}{M_2}\left\{1+\theta_2 - \frac{2\lambda\theta_2 a'c'}{M_2(M^2_1+2c')}\right\}\right] \quad \ldots \quad (50b)$$

Hence

$$\xi = \left(1-\frac{\eta_1}{a}\right)^{-\frac{M^2_1a}{(a+b)(c-ka)}}\left(1+\frac{\eta_1}{b}\right)^{-\frac{M^2_1b}{(a+b)(c+kb)}} \times \left(1-\frac{k\eta_1}{c}\right)^{-\frac{M^2_1c}{(ka-c)(kb+c)}} \times\left[1-\frac{I'}{M_1}\, a^{-\frac{M^2_1a}{(a+b)(c-ka)}}\, b^{-\frac{M^2_1b}{(a+b)(c+kb)}}\, c^{-\frac{M^2_1c}{(ka-c)(kb+c)}}\right] \quad \ldots \quad (51)$$

Since B_1 and B_2 are small we put $p_o = o$; $\theta'_1 = \theta'_2 = o$ in terms multiplying I′

Hence

$$\xi = \left(1-\frac{\eta_1}{a}\right)^{-\frac{M^2_1a}{(a+b)(c-ka)}}\left(1+\frac{\eta_1}{b}\right)^{-\frac{M^2_1b}{(a+b)(c+kb)}}\left(1-\frac{k\eta_1}{c}\right)^{-\frac{M_1^2c}{(ka-c)(kb+c)}}(1-B_1K_1-B_2K_2) \quad \ldots \quad (52)$$

If $\xi^{(1)}$, $\eta_1^{(1)}$, $\zeta^{(1)}$ etc. represent the values of ξ, η_1 and ζ_1 when B_1 and B_2 are zero then

$$\xi = \xi^{(1)} (1 - B_1K_1 - B_2K_2) \quad .. \quad .. \quad .. \quad (53)$$

Maximum Pressure

Equation (1) gives

$$\zeta_1 (\xi - B_1 z_1 - B_2 z_2) = \zeta_1^{(1)} \xi^{(1)} \quad .. \quad .. \quad .. \quad (54)$$

i.e.

$$\zeta_{1m} = \zeta_{1m}^{(1)} \left\{ 1 + B_1\left(K_{1m} + \frac{z_{1m}}{\xi_m^{(1)}}\right) + B_2\left(K_{2m} + \frac{z_{2m}}{\xi_m^{(1)}}\right)\right\} \quad .. \quad (55)$$

neglecting second and higher powers of B_1 and B_2.

Here also we can put $p_o = o$; $\theta'_1 = \theta'_2 = o$ in evaluating K_{1m} and K_{2m} since they are multiplied by B_1 and B_2 respectively.

'All-Burnt'

$$\xi_B = \xi_B^{(1)} (1 - B_1 K_{1B} - B_2 K_{2B}) \quad .. \quad .. \quad .. \quad (56)$$

In evaluating K_{1B} and K_{2B} we make the usual assumptions.

After "All-burnt'

ξ, η_1 and ξ_1 are connected by the relation

$$\zeta_1 (\xi - B_1 - B_2)^{\gamma} = (\xi - B_1 - B_2)^{\gamma-1}\left(1 + L - \frac{\gamma - 1}{2 M_1} \eta^2_1\right) = \Phi$$

where

$$\Phi = (\xi_B - B_1 - B_2)^{\gamma-1}\left(1 + L - \frac{\gamma - 1}{2 M_1} \eta^2_{1B}\right)$$

$$= (\xi'_B)^{\gamma-1}\left(1+L-\frac{\gamma-1}{2 M_1} \eta^2_{1B}\right)[1-B_1(\gamma-1)c'_1-B_2(\gamma-1)c'_2] \quad (57)$$

where

$$c'_1 = K_{1B} + \frac{1}{\xi'_B} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (58a)$$

$$c'_2 = K_{2B} + \frac{1}{\xi'_B} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (58b)$$

Hence

$$\Phi = \Phi^{(1)} [1 - B_1 (\gamma - 1) c'_1 - B_2 (\gamma - 1) c'_2] \quad .. \quad .. \quad (59)$$

where

$$\Phi^{(1)} = \xi_B^{(1)\,\gamma-1}\left(1 + L - \frac{\gamma - 1}{2 M_1}\eta^2_{1B}\right) \quad .. \quad .. \quad .. \quad (60)$$

The pressure and velocity after all-burnt are therefore given, Φ being known by equation (59) by

$$\zeta_1 = \Phi\,(\xi - B_1 - B_2\,)^{-\gamma} \quad \ldots \quad (61)$$

and

$$\eta^2{}_1 = \frac{2\,M_1}{\gamma-1}\left[\,1 - \Phi\;(\,\xi - B_1 - B_2\,)^{1-\gamma}\,\right] \quad \ldots \quad (62)$$

The muzzle velocity η_{1E} is given by

$$\eta^2{}_{1E} = \frac{2\,M_1}{\gamma-1}\left[\,1 - \Phi\;(\xi_E - B_1 - B_2)^{1-\gamma}\right] \quad \ldots \quad (63)$$

$\underline{\beta'_2 > \beta'_1}$

In evaluating I* we make the following approximation viz.

$$\theta'_1 = o \quad \ldots \quad (64)$$

If primed letters denote the corresponding things when equation (64) holds

$$k'_1 = o \quad \ldots \quad (65a)$$

$$c'_1 = M_1\,(\theta_1 + \frac{\gamma - 1}{2}\,M_1) \quad \ldots \quad (65b)$$

$$a'_1 - b'_1 = M_1\,(1 - \theta_1 + 2\theta_1 f_{10}) \quad \ldots \quad (65c)$$

$$a'_1 b' = \frac{M_1{}^2\,(\,z_{10} + L\,)}{\theta_1 + \frac{\gamma - 1}{2}\,M_1} \quad \ldots \quad (65d)$$

Hence

$$I'^* = \int_{\eta_{1,B2}}^{\eta}\Bigg[\Big\{M^3{}_1\,(B_1 z_{10} + B^2) + M^2{}_1\eta_1 B_1\,(1 - \theta_1 + 2\theta_1 f_{10}) - M_1\eta_1{}^2 B_1\theta_1\Big\}$$

$$\eta_1\,(a'_1 - \eta_1)^{\frac{M^2{}_1 a'_1}{(a'_1 + b'_1)\,c'} - 1}\,(b'_1 + \eta_1)^{\frac{M^2{}_1 b'_1}{(a'_1 + b'_1)\,c'} - 1}\,c'_1{}^{-\frac{M^2{}_1}{c'} - 1}\Bigg]\,d\eta_1$$

$$= c'_1{}^{-\frac{M^2{}_1}{c'_1} - 1}\int_{\eta_{1,B2}}^{\eta}\psi\;(\eta'_1)\;(a'_1 - \eta)^{\frac{M^2{}_1\,a'_1}{(a'_1 + b'_1)\,c'_1} - 1}$$

$$\times\,(b'_1 + \eta'_1)^{\frac{M^2{}_1 b'_1}{(a'_1 + b'_1)\,c'_1} - 1}\quad d\eta_1 \quad \ldots \quad (67)$$

where

$$\psi(\eta_1) = M^3_1 (B_1 z_{10} + B_2)\, \eta_1 + M^2_1 B_1 (1-\theta_1+2\theta_1 f_{10})\eta^2_1 - M_1 B_1 \theta_1 \eta^3_1 \quad .. (68)$$

$$I'^* = c'_1{}^{-\frac{M^2_1}{c'_1}-1} \int_{a'_1}^{\chi} {}^{\frac{M^2_1}{c'_1}-1} \Big[\psi(a'_1) - \left(\frac{d\psi}{d\eta_1}\right)_{a'_1} a'_1 (1-\chi)$$

$$\frac{b'_1}{a'_1}\left(1 + \frac{\eta_{1,B_2}}{b'_1}\right)$$

$$+ \frac{1}{2!}\left(\frac{d^2\psi}{d\eta^2_1}\right)_{a'_1} a'^2_1 (1-\chi)^2 - \left(\frac{d^3\psi}{d\eta^3_1}\right)_{a'_1} \frac{1}{3!} a'^3_1 (1-\chi)^3 \Big] \times$$

$$\chi^{\frac{M^2_1\, b'_1}{(a'_1+b'_1)c'}-1} (1-\chi)^{\frac{M^2_1\, a'_1}{(a'_1+b'_1)c'_1}-1} d\chi \quad .. (69)$$

$$\text{where } \chi = \frac{b'_1}{a'_1}\left(1 + \frac{\eta'_1}{b'_1}\right) \quad .. \quad .. \quad .. (70)$$

$$i.e.\ I'^* = a'_1{}^{\frac{M^2_1}{c'_1}-1}\, c'_1{}^{-\frac{M^2_1}{c'_1}-1} \Big[\psi(a'_1)\left(B_{m,n;\,\chi} - B_{m,n;\,\chi_{B_2}} \right)$$

$$-\left(\frac{d\psi}{d\eta_1}\right)_{a'_1}\left(B_{m,n+1;\,\chi} - B_{m,n+1;\chi_{B_2}} \right) a'_1$$

$$+\left(\frac{d^2\psi}{d\eta^2_1}\right)_{a'_1}\left(B_{m,n+2;\chi} - B_{m,n+2;\,\chi_{B_2}} \right) \frac{a'^2_1}{2!}$$

$$-\left(\frac{d^3\psi}{d\eta^3_1}\right)_{a'_1}\left(B_{m,n+3;\ \chi} - B_{m,n+3;\ \chi_{B_2}} \right) \frac{a'^3_1}{3!} \Big] \quad .. (71)$$

where $B_{m,n;x}$ is the Incomplete Beta Function.

$$B_{m,n;x} = \int_0^x x^{m-1}(1-x)^{n-1}\,dx \quad .. \quad .. \quad .. (72a)$$

and

$$m = \frac{M^2_1\, b'_1}{(a'_1+b'_1)c'_1} \quad .. \quad .. \quad .. \quad .. \quad .. (72b)$$

$$n = \frac{M^2_1\, a'_1}{(a'_1+b'_1)c'_1} \quad .. \quad .. \quad .. \quad .. (72c)$$

Hence ξ, η relation is obtained in terms of tabulated functions and 'All-burnt' values are got by putting

$\eta_1 = \eta_{1B}$ *ie* $\chi = \chi_B$.

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References

1. Corner, J. Theory of Interior Ballistics of Guns.—John Wiley and Sons. (1950)

2. H. M. Stationery Office—Internal Ballistics. (1951)

3. Venkatesan, N. S. & Patni, G. C.—Influence of Composite Charges on maximum pressure and muzzle velocity—Def. Sc. Jour. Vol. 3, 1953.

4. Venkatesan, N. S.—On the solutions of the equations of Internal Ballistics with a cubic form function—Proc. N. I. S. I. (In press).