DIMENSIONAL ANALYSIS AND THEORY OF MODELS—I*

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Introduction

Purpose of Models—Whenever it is desired to obtain the solution of a physical problem the first attempt usually is to obtain an analytical solution. In the analytical method the problem is given a mathematical formulation leading usually to a differential equation. This differential equation is then solved subject to the boundary conditions of the problem. Such a solution, whenever it is possible, is a complete solution of the problem since it determines all the constants involved in the problem and provides further an insight into the mechanism of the phenomenon under consideration. In most of the engineering problems, the number of variables or the complexity of the situation makes the application of the usual analytical procedures tedious and may lead to a mathematically cumbersome solution. In still other problems the general laws governing the behaviour of the system are unknown and analytical procedures have not been developed.

For such problems dimensional analysis supplemented by experimental evidence may lead to the formulation of a general law governing the phenomenon. However, if a large number of variables is involved, the collection of data and its subsequent reduction to a general formula may be too time consuming to be feasible. In addition much supplementary data may be needed to make the range of the resultant equation sufficiently broad to obviate extrapolation in the majority of applications. In many such instances a general formula is not necessary; all that the engineer requires for the design is an indication of the relationship of the variables for a specific design or within a narrow range of variation of the significant variables. Under these circumstances a model gives the desired result quickly and cheaply.

Definition of a Model.—A model is a device which is so related to a physical system that observations on the model may be used to predict the performance of the physical system in the desired respect. The physical system for which the predictions are to be made is called the prototype. The laws of similitude make it possible to determine the performance of the prototype from tests made on the model. In order that a model will reproduce the behaviour of the prototype it must satisfy certain requirements again based on the principle of similitude. Such laws will be developed later.

A model is not necessarily smaller than the prototype. Actually it may even be larger. Thus the flow in a carburetor might be studied in a very large model. In fluid flow problems it is also not necessary to use the same fluid for the model as for the prototype. The laws of similitude make it possible to carry out experiments with a convenient fluid such as water, and then apply the results to a fluid which is less convenient to work with such as

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air, gas, steam or oil. The flow of water at the entrance to a small centrifugal pump runner might be investigated by the flow of air at the entrance to a large model of the runner. In Hydraulics and Aeronautics, however, experiments are performed with small scale models. In fact the chief difficulty in Hydraulic models is to make them large enough.

Types of models.—Models may be broadly classified into the following four types:

(1) True models.

(2) Adequate models.

(3) Distorted models.

(4) Dissimilar models.

True models are models in which all significant characteristics of the prototype are faithfully reproduced to scale. In addition to being geometrically similar to the prototype, the model satisfies all other restrictions introduced by the design conditions.

Adequate models are models from which accurate predictions of one characteristic of the prototype may be made, but which will not necessarily yield accurate predictions of other characteristics.

A distorted model is a model in which some design condition is violated sufficiently to require correction of the prediction equation. Under certain conditions, particularly where flow of fluids is involved it is impracticable, if not impossible, to satisfy all the design conditions with a length scale other than unity. Lack of availability of suitable materials or specified dimension of members may lead to the adoption of two or more different scales. When the model is distorted with respect to some characteristic, this distortion will affect the prediction equation and corrections must be made to obtain reliable results. Distortion is usually required in models of river channels, floodways, harbours and estuaries for which the horizontal dimensions are large in proportion to the vertical ones. In such cases the horizontal scales are limited by space and cost restrictions. When these scales result in model depths and slopes which are too small to yield significant results, a vertical exaggeration or a distorted vertical scale is required.

Dissimilar models are models which bear no apparent resemblance to the prototype but which through suitable analogies give accurate predictions of the behaviour of the prototype. Thus information concerning the torsional stresses in a shaft may be obtained from measurements taken on a soap film and the characteristics of a vibrating mechanical system may be duplicated in an electrical circuit.

In the subsequent Section the first three types of models are considered. Since the main aim is to develop the theory of models as applicable to ship resistance problems, emphasis will be laid on fluid flow models. The primary objective is to deduce the conditions for the validity of experiments on models and the laws of comparison of models with their prototypes. Such laws can be established if either the differential equation for the flow under consideration or the expressions representing the forces acting on the fluid are known. For most of the problems encountered in actual practice such information is hardly available. In such cases dimensional analysis can be used to deduce the laws governing the relationship of model to prototype. With the help of dimensional analysis a relationship between the physical variables of a problem can be reduced to a relationship between dimensionless groups which by their numerical value characterise the type of flow under consideration. For equal values of these dimensionless parameters the flow patterns are similar in the model and the prototype.

DIMENSIONAL ANALYSIS

The Method of dimensions

The method of dimensions had its origin in the Principle of Similitude referred to by Newton[1]. Newton applied this principle with reference to the equation of motion which he had deduced by applying the laws of motion enunciated by him. Thus in Newtonian dynamics the final velocity 'v' of a body of mass 'm' starting from rest and travelling in a straight line for a distance 's' under the action of a constant force 'F' is given by

$$v^2 = 2\frac{F}{m}s$$

With reference to this equation the principle of similitude states that if we consider a number of different masses such that the ratio $\frac{F}{m}$ is the same for all, then the ratio $\frac{v^2}{s}$ is the same for all. Or, again, if the final velocities attained by different masses in equal displacements are equal, then the applied forces are directly proportional to the masses. Newton made frequent use of the principle of similitude and this appears to have been the first application of what is known as the method of dimensions.

More than a hundred years after Newton's work the subject was examined by Fourier [2] who introduced two important concepts in the theory of dimensional analysis. The first is the concept of what today is termed the 'dimensional formula' and the second is the recognition of the dimensional homogeneity of physical equations. These ideas are so fundamental to the theory of dimensions that it is worthwhile to explain their significance.

Dimensional formula

Every physical observation has two characteristics associated with it, the qualitative and the quantitative. The qualitative aspect of an observation serves to describe accurately the nature of this observation so that it can be identified sufficiently from other observations. This description of the nature of a physical quantity may be given in terms of the fundamental entities : mass, length and time. The concepts of mass, length and time were regarded as fundamental and independent by Newton. They are still regarded as such and metaphysical speculation has so far not succeeded in showing that any one of them is dependent on either of the other two. The sciences of Eelectricity and Magnetism and Heat require the use of two other primary concepts such as charge and temperature but it is still a matter of discussion whether they should be regarded as primary or secondary. In terms of the fundamental concepts velocity, for example, is given as $\frac{\text{Length}}{\text{Time}}$,

or

$$V = LT^{-1}$$

Thus the physical quantity velocity has a dimension of 1 in length and —1 in time and its dimensional formula is LT^{-1}. The dimensional forumula for acceleration is LT^{-2}. The dimensional formula of a physical quantity may thus be looked upon as a short hand statement of the definition of that physical quantity and as revealing its physical nature. It is derived from the definition of the physical quantity.

The quantitative description of a physical observation is necessary to indicate the extent or the degree of occurrence and to assist in distinguishing it from qualitatively similar occurrences of different magnitude. This quantitative description involves both a number and a standard of comparison. The standard of comparison which is arbitrarily established is called a unit. The number indicates the extent to which the unit quantity is duplicated in the measured quantity.

The dimensional formulae of various physical quantities are given in the following table in terms of Force, Length and Time chosen as fundamental dimensions. The M.L.T. system also could be used but it is simpler and more convenient in engineering practice to use the F.L.T. system.

TABLE 1

	F	L	T
Mass	1	—1	2
Velocity	0	1	—1
Acceleration	0	1	—2
Force	1	0	0
Pressure	1	—2	0
Pressure gradient..	1	—3	0
Mass density	1	—4	2
Viscosity	1	—2	1
Modulus of Elasticity	1	—2	0
Surface tension	1	—1	0

Dimensional homogeneity

Fourier's second contribution to the theory of dimensional analysis was the principle of dimensional homogeneity. A physical equation normally consists of an algebric sum of the two or more terms. The equation is said to be dimensionally homogeneous if and only if all the terms have the same dimensions. This principle applies to differential and integral equations as well as to algebric equations. Empirical equations, however, are not necessarily dimensionally homogeneous unless they contain all the variables that would appear in an analytical derivation of the equation. For example, in the problem of the drag on a spherical body in an air stream, it might be argued that density and viscosity may be disregarded, since they are constants for standard air. The equation of the drag force R would than be of the form $R = f(V, D)$ where V is the velocity of the stream and D is the diameter of the body. However, it is obviously impossible to construct a dimensionally homogeneous equation of this form, since the variables V and D do not contain the dimensions of force or mass[3].

Value of dimensional analysis

The most important use of dimensional analysis to the engineer is in establishing principles of model design, operation and interpretation. It helps the experimenter in the selection of experiments capable of yielding significant information and in the avoidance of redundant experiments. An application of dimensional analysis to a physical problem reduces substantially the number of functionally related quantities below the number of relevant physical quantities. The whole analysis is summed up in the Buckingham's [4] π — theorem which states that "if n variables are connected by an unknown dimensionally homogeneous equation, then this equation can be expressed in the form of a relationship between (n—r) independent dimensionless product combinations of the physical variables involved in the problems". In most cases, 'r' is equal to the number of fundamental dimensions in the problem. However, this is not an infallible rule, since the number of fundamental dimensions in a problem may depend on the system of fundamental dimensions used. For example problems of stress analysis usually involve only two dimensions F and L. However, since $F = MLT^{-2}$, there are three dimensions if the mass system is used.

The calculation of such non-dimensional parameters of a physical problem can be made by the Rayleigh's[5] method with the help of the following example. A liquid of density ρ and viscosity μ is flowing with a velocity V in a smooth straight pipe of circular cross section of diameter D. It is required to find an expression for the pressure gradient (pressure drop per unit length) G in the pipe.

The pressure gradient G would obviously depend on V, D, ρ and μ and we can, therefore, write—

$$G = f(V, D, \rho, \mu) \qquad \ldots \quad (2\cdot1)$$

The Rayleigh method consists in writing the function f as the product of powers of the variables V, D, ρ, μ. We thus write

$$G = C\, V^{k_1}\, D^{k_2}\, \rho^{k_3}\, \mu^{k_4} \qquad \ldots \quad (2\cdot2)$$

where C is a dimensionless constant and k_1, k_2, k_3, k_4 are numbers whose values are to be determined. The justification of writing (2·1) in the form (2·2) is shown in the Appendix. Writing the variables in (2·2) in terms of their dimensional formulae we have

$$FL^{-3} = (LT^{-1})^{k_1}\, L^{k_2}\, (FL^{-4}T^2)^{k_3}\, (FL^{-2}T)^{k_4}$$

$$\text{or, } FL^{-3} = F^{k_3+k_4}\, L^{k_1+k_2-4k_3-2k_4}\, T^{-k_1+2k_3+k_4} \qquad \ldots \quad (2\cdot3)$$

Since equation (2·2) must be dimensionally homogeneous the exponents of the fundamental dimensions F, L, T must be the same on either side in (2·3). This gives

$$\left.\begin{array}{l} k_3+k_4=1 \\ k_1+k_2-4/k_3-2\,k_4=-3 \\ -k_1+2\,k_3+k_4=0 \end{array}\right\} \qquad (2\cdot4)$$

There are three equations and four unknowns. Therefore, three of the unknowns can be determined in terms of the fourth.

In terms of k_1 we have

$k_1=k_1\,;\,k_2=-3+k_1\,;\,K_3=-1+k_1\,;\,k_4=2-k_1$

so that

$$G=C\,\frac{\mu^2}{D^3\rho}\left(\frac{VD\rho}{\mu}\right)^{k_1}$$

or, $$\phi^{(i)}\left(\frac{D^3\rho G}{\mu^2},\ \frac{VD\rho}{\mu}\right)=0 \qquad (2\cdot5)$$

where $\phi^{(i)}$ is an unknown function of the parameters $\frac{D^3\rho G}{\mu^2}$ and $\frac{VD\rho}{\mu}$.

It is easy to verify that these parameters are dimensionless.

In terms of k_2, we have

$k_1=3+k_2\,;\,k_2=k_2\,;\,k_3=2+k_2\,;\,k_4=-1-k_2$

These values give

$$G=C\,V^{3+k_2}\,D^{k_2}\,\rho^{2+k_2}\,\mu^{-1-k_2}$$

or,

$$G=C\,\frac{V^3\rho^2}{\mu}\left(\frac{VD\rho}{\mu}\right)^{K_2}$$

which can be further written as

$$\phi^{(ii)}\left(\frac{\mu G}{V^3\rho^2},\ \frac{VD\rho}{\mu}\right)=0 \qquad (2\cdot6)$$

It can be verified that $\frac{\mu G}{V^3\rho^2}$ is a non-dimensional parameter.

In terms of k_3 we have

$k_1=1+k_3\,;\,k_2=-2+k_3\,;\,k_3=k_3\,;\,k_4=1-k_3$

These values give, as before,

$$\phi^{(iii)}\left(\frac{D^2G}{V\mu},\ \frac{VD\rho}{\mu}\right)=0 \qquad (2\cdot7)$$

$\frac{D^2G}{V\mu}$ is again a non-dimensional parameter.

In terms of k_4, the solution of equation (2·4) is

$$k_1 = 2 - k_4 ; k_2 = -1 - k_4 ; k_3 = 1 - k_4 ; k_4 = k_4$$

We then have

$$\phi^{(iv)}\left(\frac{DG}{V^2\rho}, \frac{VD\rho}{\mu}\right) = 0 \quad \dots \quad (2 \cdot 8)$$

In this case also $\frac{DG}{V^2\rho}$ is a non-dimensional parameter.

It is thus seen that in each case the relationship between the five variables G, V, D, ρ and μ involved in the problem reduces to a relationship between two non-dimensional independent parameters. The relationship can thus be put in the form

$$\phi\,(\pi_1, \pi_2) = 0 \quad \dots \quad (2 \cdot 9)$$

where π_1, and π_2, are the non-dimensional parameters. That there are only two in this problem is a direct consequence of the π—theorem, since there are five variables and these have been expressed in terms of three fundamental dimensions.

A rapid method of calculating the π —terms.

In this section a more rapid method of evaluating the complete set of π—terms of a physical problem has been developed. For simplicity the problem of flow of a fluid in a pipe solved in the preceding section has again been taken up. The dimensions of the variables involved in the problem are rewritten in the following table for convenience.

	k_1	k_2	k_3	k_4	k_5	
	G	V	D	ρ	μ	
F	1	0	0	1	1	
L	—3	1	1	—4	—2	(2·10)
T	0	—1	0	2	1	

A typical dimensionless product of the variables will be of the form

$$\pi = G^{k_1}\, V^{k_2}\, D^{k_3}\, \rho^{k_4}\, \mu^{k_5} \quad \dots \quad (2 \cdot 11)$$

where the k's are numbers to be determined.

Writing both sides of (2·11) in terms of the fundamental dimensions F, L, T we have

$$F^{\circ}L^{\circ}T^{\circ} = (FL^{-3})^{k_1}\, (LT^{-1})^{k_2}\, L^{k_3}\, (FL^{-4}\, T^2)^{k_4}\, (FL^{-2}\, T)^{k_5}$$

Comparing the exponents of F, L, T on both sides we have

$$\left.\begin{array}{r} k_1 + k_4 + k_5 = 0 \\ -3k_1 + k_2 + k_3 - 4k_4 - 2\,k_5 = 0 \\ -k_2 + 2k_4 + k_5 = 0 \end{array}\right\} \quad \dots \quad (2 \cdot 12)$$

It should be noted that the coefficients of the k's in each equation are the row of numbers shown in (2·10). Such a rectangular array of numbers as (2·10) displaying the dimensions of the variables is known as a dimensional matrix. The equations for the exponents of a dimensionless product can be written down directly by inspection of the dimensional matrix. The writing down of such a matrix is the first step in the evaluation of the π—terms.

Any solution of the eqn. (2·12) is a set of exponents for forming the π—terms with the help of (2·11). The system of eqn. (2·12) is undetermined, since there are three equations in five unknowns, and possesses an infinite number of solutions. In the present case any arbitrary values may be assigned to two of the unknowns and the remaining three may then be determined in terms of these two provided that the determinant of the coefficients of the remaining three unknowns is not equal to zero. Let us assume, for example, that k_1 and k_2 are given arbitrary values. Then a rearrangement of eqn. (2·12) gives

$$\left.\begin{aligned} k_4+k_5 &= -k_1 \\ k_3-4k_4-2k_5 &= 3k_1-k_2 \\ 2k_4+k_5 &= k_2 \end{aligned}\right\} \quad \ldots \quad (2\cdot13)$$

The determinant of the coefficients of the K's on the left of eqn. (2·13) is

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -4 & -2 \\ 0 & 2 & 1 \end{vmatrix} \neq 0 \qquad (2\cdot14)$$

Therefore, the eqn. (2·13) are independent and the solution is

$$\left.\begin{aligned} k_3 &= 3k_1+k_2 \\ k_4 &= k_1+k_2 \\ k_5 &= -2k_1-k_2 \end{aligned}\right\} \quad \ldots \quad (2\cdot15)$$

It is seen that the determinant (2·14) is the third order determinant formed by the elements of the last three columns of the dimensional matrix. Since this determinant is not zero, the rank of the dimensional matrix is 3. This is the significance of r used in the statement of the π—theorem in section 2·4. Since $r=3$ in the present case and $n=5$ the number of dimensionless products in any solution of the problem will be $5-3=2$. There is no theoretical reason for picking the determinant on the right side of the dimensional matrix; if the matrix contains any third order non-zero determinant its rank is 3 and the three k's corresponding to the columns of the non-zero determinant can be evaluated in terms of the remaining ones. It is obvious that the number of rows in the dimensional matrix is 3 so that the rank of the matrix can not exceed three which is the number of fundamental dimensions employed in the problem. It may happen in some cases that the rank of the matrix is less than the number of fundamental dimensions. In such cases the value of r is less than the number of fundamental dimensions and is equal to the rank of the highest order non-zero determinant in the dimensional matrix. Once the value of r is known the number of π—terms constituting a complete set of dimensionless products of the problem is known as $n-r$. For convenience it is desirable to rearrange the dimensional matrix in such a manner that a non-zero determinant of order r occurs in the right hand r columns of the matrix. In the present case this rearrangement is not necessary but in cases where it is necessary it will be assumed that this has been done.

Let arbitrary values $k_1=1$ and $k_2=0$ be assigned for the first solution. Then the solution of eqn. (2·13) is

$$k_3=3,\ k_4=1,\ k_5=-2$$

Similarly if $k_1=0$, $k_2=1$ for the second solution eqn. (2·13) yield

$$k_3=1,\ k_4=1,\ k_5=-1$$

The solutions may be neatly arranged in the matrix form shown below

	k_1	k_2	k_3	k_4	k_5
	G	V	D	ρ	μ
π_1	1	0	3	1	—2
π_2	0	1	1	1	—1

(2·16)

It should be observed that the third, fourth and fifth columns in the matrix of solution (2·16) are merely the coefficients in the solutions (2·15) for k_3, k_4, k_5. The first two columns of the matrix of solutions consist of zeros, except for the ones on the principal diagonal. Consequently, the matrix of solutions can be written down immediately by inspection of eqn. (2·15). Each row in (2·16) is a set of exponents in a dimensionless product. Accordingly in the present case the following complete set of π—terms is obtained.

$$\pi_1 = \frac{GD^3\rho}{\mu^2}, \quad \pi_2 = \frac{VD\rho}{\mu}$$

These are the π—terms corresponding to the solution (2·5). It is to be noted that the first variable G occurs only in π_1 and the second variable V occurs only in π_2. This is an important characteristic of the method. It verifies that the products are independent of each other.

The procedure outlined above gives a complete set of π—terms corresponding to the solution (2·5). The set of π—terms corresponding to the solution (2·6), (2·7) and (2·8) may be found by giving arbitrary values to other k's instead of k_1 and k_2. As a matter of fact infinitely many different complete sets of π—terms can be formed from a given set of variables. In so far as Buckingham's π—theorem is concerned any complete set of dimensionless products is admissible. What set to use depends mainly on experimental convenience. Buckingham has pointed out that we obtain the maximum amount of experimental control over the dimensionless variables if the original variables that can be regulated each occur in only one dimensionless product. For example, if a velocity V is easily varied experimentally, then V should occur in only one π—term. That π—term can then be regulated by varying V. Likewise if a pressure P can be easily varied without affecting V, then P should occur in only one of the π—terms, but not in the same one as V.

The dependent variable of the problem must also be considered. Usually it is desired to know how this variable depends on the other variables. The dependent variable, consequently should not occur in more than one π—term.

Since the first (n—r) variables in the dimensional matrix each occur in only one dimensionless product, the preceding conditions will be realised, as nearly as possible, if the following rule is observed. In the dimensional matrix, let the first variable be the dependent variable. Let the second variable be that which is easiest to regulate experimentally. Let the third variable be that which is next easiest to regulate, and so on. In exceptional cases, this arrangement may lead to an impasse, because the dimensional matrix does not contain a non-zero determinant of order r in the right hand r columns. The variables should then be rearranged without altering the recommended arrangement more than necessary.

When a transformation of a complete set of π—terms is performed to obtain a different set to achieve greater experimental control over the variables, it is necessary to ascertain that there are as many π—terms in the new set as in the old and that the new π—terms are independent of each other. Otherwise the new set of π—terms will not be a complete set.

An extension of the method of dimensions. Directed lengths and dual role of mass

Bridgman[6] emphasised the fact that there is nothing sacrosanct about MLT or FLT as fundamental dimensions and that dimensional analysis is merely a man-made tool that may be manipulated at will. The only justification of dimensional analysis is its utility and, therefore, in judging the value of any new method in dimensional analysis the sole criterion must be the pragmatic one. This is the touch-stone by which any claim to advancement of knowledge or improvement of method in this subject must be tested. It will be shown in this article that two of the entities, so long treated as basic, namely length and mass, are each capable of analysis into simpler and more fundamental constituents and that the use of appropriate constituents results in the clearing away of certain ambiguities and in an increased power of the analysis.

In dimensional analysis certain concepts such as those of Mass, Length and Time are chosen as primary and all other physical concepts are expressed in terms of these primary concepts in the form of their dimensional formulae. The number of primary concepts as well as their nature may be varied at will. The dimensional formulae will, of course, change with the selection of the primary concepts. In order to avoid ambiguity the dimensional formulae ascribed to two different physical concepts must be different i.e. there must be a 1 : 1 correspondence, between concepts and their dimensional formulae. In the MLT system this requirement is violated in a number of cases. In the dimensional formulae for energy or work and torque, both of them are measured as force$\times$distance and are, therefore, represented by the same dimensional formula ML^2T^{-2}. Despite this identity in dimensions, physicists have always regarded energy and torque as distinct concepts. Another example is the confusion between normal stress and shearing stress. These concepts are often considered as identical from the mistaken notion that 'dimensions' are associated with units instead of concepts and that identical units imply identical concepts. From an operational standpoint, however, the above concepts are as distinct as energy and torque. Such ambiguities tend to introduce into dimensional analysis a vagueness that is anything but helpful. In the example quoted above ambiguity arises because no attention has been paid in the dimensional formulae to the direction in which the length is measured. In the concept of energy length is measured in the direction of the force, while with torque length is measured in a direction perpendicular to the force. This is the characteristic that distinguishes the two concepts and this very characteristic has been ignored in the MLT system. Thus in the dimensional formula for torque ML^2T^{-2} the length dimension occurs twice but although these lengths are measured in mutually perpendicular directions no distinction between them has been made on this account. If this is done the dimensional formula for torque could be represented as $ML_x L_y T^{-2}$ or by the cyclic variations

$ML_y L_z T^{-2}$ or $ML_z L_x T^{-2}$ Energy or work, on the other hand, is the product of two collinear vectors and is accordingly represented by $ML^2_x T^{-2}$ or $ML^2_y T^{-2}$ or $ML^2_z T^{-2}$. Thus by resolving the length dimension into three mutually perpendicular directions it is possible to distinguish in a significant way between energy and torque.

The resolution of the length dimension L into three independent length dimensions L_x, L_y, L_z is useful in many problems. Since such a resolution leads to an increase in the number of fundamental dimensions the number of π—terms in the solution will be less and, therefore, the solution will be more complete. In some cases the use of vector lengths enables a complete solution to be determined which by the traditional method would not have been possible. In other cases, as will be shown later the employment of these vector lengths makes results more significant than they otherwise appear to be.

It is possible to carry out the analysis of the length dimension one stage further by attributing positive and negative signs to the components. By distinguishing between L_x and L_{-x} etc. We may resolve L into six components. Such a procedure has not so far proved to be of any practical value, but the remarkable increase in power of the method accruing from the employment of components of L justifies the expectation that the use of even more fundamental entities would on appropriate occasions be valuable.

Like length mass also can be resolved into two components. In Physics mass is commonly regarded from two quite different points of view: (1) as quantity of matter and (2) as inertia. While it is true that there is strict proportionality between the two, that does not make them identical. They are in fact quite different in nature and should, therefore, be differentiated by distinct dimensional symbols. We can make this distinction by writing.

M_μ = Quantity of matter

M_i = Inertia.

Although examples commonly illustrated are concerned with mass as inertia, in certain cases such as in problems involving heat and temperature it is mass as quantity of matter that must be taken into account. The resolution of the mass dimension into two components brings further increase in clarity and power of the method of dimensional analysis.

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APPENDIX

In the Rayleigh's method the secondary quantity is written as the product of powers of the primary quantities on which it is assumed to depend. Thus if a measurable physical quantity α depends on the primary quantities a_1, a_2, a_3, ..., a_n we write

$$\alpha = C_\alpha \, a_1^{c_1} a_2^{c_2} \ldots\ldots\ldots a_n^{c_\alpha} \qquad \ldots\ldots(A\cdot 1)$$

where C_α is a dimensionless constant and c_1, c_2......c_n are numbers whose values are determined by a procedure outlined in page 8. The validity of the formula (A·1) is to be established. The proof is based on the following two axioms which are inherent in our methods of measurement and evaluation of quantities.

Axiom I—Absolute numerical equality of quantities may exist only when the quantities are similar qualitatively.

That is, a general relationship may be established between quantities only when they have the same dimensions. For example, a quantity that is measured in terms of force can be equal only to a quantity that is evaluated in terms of force, and cannot be equal to a quantity having dimensions different from that of force.

Axiom II—The ratio of the magnitudes of two like quantities is independent of the units used in their measurement, provided that the same units are used for evaluating each.

For example, the ratio of the length of a table to its width is the same, regardless of whether the dimensions are measured in inches, feet or metres. This axiom is the direct result of our standard linear system of measurement.

The general relationship between a secondary quantity α and the primary quantities a_1, a_2, ..., a_n may be written as

$$\alpha = f\,(a_1, a_2, \ldots, a_n) \qquad \ldots\ldots(A\cdot 2)$$

The secondary quantity might, for example, be the horse power of an engine and the primary quantities items such as bore, stroke, rpm, fuel consumption etc.

The form of the function f is to be determined. In this equation α is the number denoting the magnitude of the secondary quantity and a_1, a_2, a_3, ..., a_n are numbers denoting the magnitudes of the primary quantities involved.

Let β be another magnitude of the same secondary quantity (power of some other engine) which is to be evaluated in terms of the same primary quantities. Then, in general,

$$\beta = f\,(b_1, b_2, \ldots, b_n) \qquad \ldots\ldots(A\cdot 3)$$

in which b_1, b_2, ..., b_n are quantities identical in nature to a_1, a_2, ..., a_n but different in magnitude. The nature of the functions in (A·2) and (A·3) is identical; only the numerical values are different.

If the sizes of the units in which the primary quantities are evaluated are changed, the number representing the magnitude of the first secondary quantity will change from α to another number α', and β will change to β'. That is,

$$\alpha' = f(x_1 a_1, x_2 a_2, \ldots, x_n a_n) \quad \text{(A·4)}$$

and

$$\beta' = f(x_1 b_1, x_2 b_2, \ldots x_n b_n) \quad \text{(A·5)}$$

in which $x_1, x_2, \ldots, x_n$ represent the ratios of the size of the first set of units to the size of the second set of units. For example, if a_2 is measured in feet and $x_2 a_2$ in inches, x_2 equals 12.

From axiom 2 we have

$$\frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}$$

or,

$$\alpha' = \frac{\alpha}{\beta} \cdot \beta'$$

or,

$$f(x_1 a_1, x_2 a_2, \ldots, x_n a_n) = \frac{f(a_1, a_2, \ldots, a_n)}{f(b_1, b_2, \ldots, b_n)} f(x_1 b_1, x_2 b_2, \ldots x_n b_n) \quad \text{(A·6)}$$

If both sides of eqn. (A·6) are differentiated partially with respect to each x_i, there will result a series of equations of the form

$$a_i \frac{\partial f(x_1 a_1, x_2 a_2, \ldots, x_n a_n)}{\partial (a_i x_i)} = \frac{f(a_1, a_2, \ldots, a_n)}{f(b_1, b_2, \ldots, b_n)} b_i \frac{\partial f(x_1 b_1, x_2 b_2, \ldots, x_n b_n)}{\partial (b_i x_i)}$$

Let all the x's become unity. Then

$$a_i \frac{\partial f(a_1, a_2, \ldots, a_n)}{\partial a_i} = \frac{f(a_1, a_2, \ldots, a_n)}{f(b_1, b_2, \ldots, b_n)} b_i \frac{\partial f(b_1, b_2, \ldots, b_n)}{\partial b_i}$$

or,

$$a_i \frac{1}{f(a_1, a_2, \ldots, a_n)} \frac{\partial f(a_1, a_2, \ldots, a_n)}{\partial a_i} = b_i \frac{1}{f(b_1, b_2, \ldots, b_n)} \frac{\partial f(b_1, b_2, \ldots, b_n)}{b_i} \quad \text{(A·7)}$$

The left hand side of the above equation relates to the variables of the first system and the right hand side to the variables of the second system. The two sides can, therefore, be equal only when each one of them is equal to a constant, so that

$$a_i \frac{\frac{\partial f(a_1, a_2, \ldots, a_n)}{\partial a_i}}{f(a_1, a_2, \ldots, a_n)} = C_i$$

where C_i is a constant.

$$\frac{\frac{\partial J(a_1, a_2, \ldots, a_n)}{\partial a_i}}{f(a_1, a_2, \ldots, a_n)} = \frac{C_i}{a_i} \quad \text{A·8)}$$

An equation of the type (A·8) will exist for each value of a_i and b_i, each equation being a differential equation of the general relationship between $f(a_1, a_2, \ldots, a_n)$ and the particular a_i involved. Since $a_1, a_2, \ldots, a_n$ are independent quantities, eqn (A·8) may be written as

$$\frac{d f(a_1, a_2, \ldots, a_n)}{f(a_1, a_2, \ldots, a_n)} = C_i \frac{d a_i}{a_i}$$

or, $\log f(a_1, a_2, \ldots, a_n) = C_i \log a_i + \text{constant.}$

If the same procedure is carried out for each value of a_i there results the general solution

$$\log f(a_1, a_2, \ldots, a_n) = C_1 \log a_1 + C_2 \log a_2 + \ldots + C_n \log a_n + \log C_\alpha$$

$$= \log (C_\alpha a_1^{c_1} a_2^{c_2} \ldots\ldots a_n^{c_n})$$

or, $f(a_1, a_2, \ldots a_n) = C_\alpha a_1^{c_1} a_2^{c_2} \ldots a_n^{c_n}$

so that

$$\alpha = C_\alpha a_1^{c_1} a_2^{c_2} \ldots a_n^{c_n}$$