SOME NEW METHODS FOR TESTING RANDOMNESS OF A BINOMIAL SEQUENCE AND ITS APPLICATIONS IN TWO SAMPLE PROBLEMS

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ABSTRACT

The t-test commonly used for testing two samples is based on the assumption that the samples are random and belong to the same normal population. These assumptions may or may not be valid for different types of experimental data. In cases where these assumptions do not hold good, it would be preferable to use tests which are independent of the nature of the distribution of the parent population. A number of such tests, some developed in the Defence Science Laboratory, is given in this paper.

The tests depend on a sequence of A's and B's obtained by pooling together the two samples $\{x_m\}$ and $\{y_n\}$ and arranging them in ascending or descending order and treating the observations belonging to $\{x_m\}$ and $\{y_n\}$ as A's and B's respectively. For this sequence the number of AB's or AB's and BA's are noted for the following cases:

(1) Between any two observations of the sequence separated by (k—1) observations or less;

(2) Between any two observations in blocks of (k+1) consecutive observations moving from one end to the other end.

It has been found that the standardized deviates of these statistics serve as more reliable tests than any of the other existing tests. Further work is in progress to confirm these findings.

Introduction

The main problem in dealing with two samples is to decide whether two given samples $\{x\ \}$ and $\{y_n\}$ belong to the same or different populations. This is usually done by evaluating the quantity

$$t = \frac{(\bar{x} - \bar{y})}{s} \sqrt{\frac{nm}{n+m}}$$

where $\bar{x}$ and $\bar{y}$ are the means of samples $\{x_m\}$ and $\{y_n\}$ and s^2 is the pooled variance within the two samples. The above 't' is compared with the theoretical 't' tabulated by Fisher for the desired level (5 percent or 2·5 percent or 1 percent) of significance. If the observed value of 't' is greater than the theoretical value for the desired level of significance, then we come to the conclusion that the two samples differ significantly from each other.

It will be noted that the theoretical values of 't' have been calculated on the assumptions that (*i*) the samples belong to one and the same normal population and (*ii*) the samples $\{x_m\}$ and $\{y_n\}$ are random ones. If the distributions of the parents are not normal, suitable changes should be made in the values of 't' used for purposes of comparison. Though for all practical purposes the assumptions regarding the form of the distributions may hold good, in actual practice the nature of the distributions is not known and therefore one is not sure of the theoretical value of 't' to be used for purposes of comparison. Besides this point, the second assumption that the two samples are random may or may not be true in which case our conclusions regarding the samples will be biased. The *t*-test does not take into account any information on this point. Therefore the conclusions arrived at by using the t-test are subject to these limitations. It may further be observed that the application of the t-test involves the calculation of the means and the variances and when there are large number of samples for examination the statistical analysis will take a good deal of time. In view of these circumstances, tests which are independent of the nature of the distributions and means and variances of the samples and are at the same time dependent on the random distribution of the observations will be preferable, provided the efficiency of these tests compares favourably with the 't' test. The purpose of this paper is to describe a number of such tests some of which are new and have arisen as an outcome of investigations undertaken in this laboratory. These tests can be used for testing (*i*) whether a given sequence of binomial observations is random or not and (*ii*) the significance of the difference between two samples without evaluating the means and the variances.

Description of some of the tests reported in literature

A number of such tests has been developed from time to time by various authors. The most important of them are the following: (1) Run test[1], (2) Median test[2], (3) Dixon's[3] C^2-statistic and (4) Wilcoxon's[4] T test. We shall now give a brief description of these tests.

(1) *Run Test*—This test was evolved by Wald and Wolfowitz (1940). The two samples $\{x_m\}$ and $\{y_n\}$ are pooled together and arranged in ascending or descending order. The observations of the ordered sequence are identified as A or B according as a particular observation belongs to sample $\{x_m\}$ or $\{y_n\}$ respectively*. This process leads to a sequence of two quantities A and B as the one indicated below:

AABBBAABBAAA

Defining a run to be a succession of A's or B's we note the number of runs in the above sequence. By comparing this observed number with the expected value on the basis of the variance for the total number of runs in the given sequence the significance of the difference between the two samples $\{x_m\}$ and $\{y_n\}$ can be decided. The expected number of runs and the variance for the distribution of the number of runs for m x's and n y's are given by

$$\frac{2mn}{m+n}+1 \quad \text{and} \quad \frac{2mn\,(2mn-m-n)}{(m+n)^2\,(m+n-1)}$$

respectively.

* The ordering is best done by plotting the observations of x and y as A and B on a line.

If X is the observed number of runs noted for given samples $\{x_m\}$ and $\{y_n\}$

$$Z = \frac{X - \frac{2mn}{m+n} - 1}{\sqrt{\frac{2mn.(2mn-m-n)}{(m+n)^2\ (m+n-1)}}}$$

enables us to decide whether the two samples are significantly different or not. If the value of Z $\leqslant -1 \cdot 64$ or $-1 \cdot 96$ or $-2 \cdot 33$ then we consider the two samples to be significantly different according to the level of probability required viz., 5 percent, 2·5 percent or 1 percent. It may be noted that the test is essentially a one tail test.

It may be remarked that this test is not sensitive when the difference between the populations to which the samples belong is very small. But when this difference is large this test is very sensitive.

2. *Median test*—The details of this test were discussed by Mood (1954). The test is based on the number of x's to the right or left of the median of the combined samples $\{x_m\}$ and $\{y_n\}$. As in (1) above, the pooled samples are arranged in ascending or descending order and the number of x's to the left or right of the combined median (i.e. the middle value of the m+n observations) is compared with the expected value on the basis of the variance for the distribution of this number. The expected value and the variance of this statistic when (m+n) is even or odd are as follows:

m=n; median value lies between nth & (n+1)th observations

$$\left.\begin{aligned} \mu_1(M) &= \frac{(m+n)}{2}.\frac{m}{(m+n)} = \frac{m}{2} \\ \mu_2(M) &= \frac{mn}{4\ (m+n-1)} = \frac{m^2}{4\ (2m-1)} \end{aligned}\right\}$$

m+n is odd; there are (m+n—1)/2 observations on either side of the median.

$$\left.\begin{aligned} \mu_1(M) &= \frac{m+n-1}{2}\ \frac{m}{m+n} \\ \mu_2(M) &= \frac{mn(m+n+1)}{4\ (m+n)^2} \end{aligned}\right\}$$

The test here consists in evaluating the quantity

$$Z = \frac{M - \mu_1(M)}{\text{S.D. of } M}$$

where M stands for the actual number of x's observed to the left of the median value. If $|Z| \geqslant 1 \cdot 96$, then the two samples can be considered to be significantly different at the 5 percent level.

The efficiency of this test as compared to the 't' test for a normal population is $2/\pi$, but is more than the run test when the difference between the means of the populations to which $\{x_m\}$ and $\{y_n\}$ belong is very small.

3. *Dixon's test*—The criterion proposed by Dixon (1940) for testing two samples $\{x_m\}$ and $\{y_n\}$ depends on the quantity

$$C^2 = \sum_{i=1}^{m+1} \left(\frac{1}{(n+1)} - \frac{m_i}{m} \right)^2$$

where m_i is the number of observations of $\{x_m\}$ between i and (i+1) the observations of $\{y_n\}$ when $\{x_m\}$ and $\{y_n\}$ are pooled and arranged in ascending or descending order. It has been shown by Dixon that nkC^2 follows a χ^2 distribution with ν degres of freedom, where

$$k = am\,(n+2)/n, \quad \nu = an\,(m+n+1)/(n+1)$$

$$\text{and} \quad a = \frac{m\,(n+3)\,(n+4)}{2\,(m-1)\,(m+n+2)\,(n+1)}$$

and therefore if k and nkC^2 are known, the question whether $\{x_m\}$ and $\{y_n\}$ are from the same population can be settled from the λ^2 Tables. When m and n are not very small we may approximately assume $mC^2\,(n+1)$ to be distributed as χ^2 with n degrees of freedom.

The actual efficiency of this test as compared to the other has not been worked out. This test includes more information than the test (2) and therefore will be more reliable than (1) and (2).

4. *Wilcoxon's test*—This test is based on the sum of the ranks of x's or y's when $\{x_m\}$ and $\{y_n\}$ together are arranged in ascending or descending order. Mann & Whitney[5] have shown that the number of times y's precede x's in this sequence and the ranks of x's are connected by the relation, sum of the ranks of

x's + number of times y's precede the x's $= mn + \frac{m\,(m+1)}{2}$ i.e.,

$$T + U = mn + m\,\frac{(m+1)}{2}, \quad \text{where}$$

T = sum of the ranks of x's and U = number of times that y's precede the x's in the combined sequence. The expectation and the variance of U for two samples belonging to the same population are as follows:

$$\left. \begin{aligned} E\,(U) &= \frac{mn}{2} \\ V\,(U) &= \frac{mn}{12}\,(m+n+1) \end{aligned} \right\}$$

A test of significance for the two samples can be worked out as follows. We note U, the number of times that y's precede x's or (x's precede y's), and calculate

$$Z = \frac{U - \frac{mn}{2}}{\sqrt{\frac{mn}{12}\,(m+n+1)}}$$

If now $|Z| \geqslant 1\cdot96$ we consider the two samples as belonging to two separate populations. Otherwise both the samples can be considered as coming from the same population. The efficiency of this test works out to be $3/\pi$ as compared to '*t*' for very small difference in the population means.

All the methods described above are valid only if m and n are not very small i.e. not less than 7 or 8. The actual distributions for small samples have been given for some of these tests and can be used to determine the probability of Z taking a value greater than or equal to the observed one.

Some new tests

We shall now describe a few similar tests recently developed in this laboratory. These tests are more efficient than the existing ones. They are based on the sequence of A's and B's obtained by pooling the two samples $\{x_m\}$ and $\{y_n\}$ together and ordering them. The tests consist in noting down the number of AB's or BA's, or AB and BA's in the sequence on the basis of the following scheme: Note the number of AB or BA's, or AB and BA's by considering the sequence from left to right for the following combinations.

(1) Adjacent observations like 1 and 2; 2 and 3; 3 and 4;......;N—1 and N.

(2) Alternate observations like 1 and 3; 2 and 4; 3 and 5;.....;N—2 and N.

(3) Observations 1 and 4; 2 and 5; 3 and 6; ....;N—3 and N. and so on.

Define

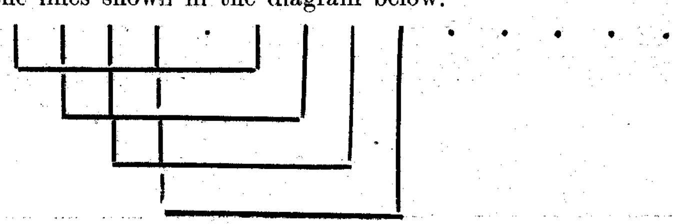
$$X_1=(12)+(23)+(34)\ldots.+(N-1, N)$$
$$X_2=(13)+(24)+(35)\ldots.+(N-2, N)$$
$$X_3=(14)+(25)+(36)\ldots.+(N-3, N)$$

Assume that the combination (rs) takes the value 1 or 0 respectively according as r is A and s is B or otherwise. In this case X_1, X_2, X_3....represent the number of AB combinations arising from adjacent, alternate and other observations separated by varying numbers. Now we take

$$T_k = X_1+X_2+X_3+\ldots\ldots\ldots+X_k$$

and note the number of AB's observed in the sequence. If AB and BA are taken, then the score given is 1 according as (rs) is (AB) or (BA). Let T_k' represent the number of AB's and BA's in $X_1+X_2+\ldots\ldots\ldots X_k$.

In addition we shall consider the number of AB, or AB and BA joins in the sequence by treating it as (N—k) moving blocks of (k+1) consecutive observations on the lines shown in the diagram below:



Each of the $(k+1)$ observation gives $k(k+1)/2$ differences. Take the total number of AB or AB and BA joins for the $(N-k)$ blocks Let W_k and W_k' represent respectively the number of AB's or AB and BA's.

Now T_k, T_k', W_k and W_k' can form the basis of a family of tests for testing the significance of the difference between two samples. The expectation and the variance of these statistics for various values of k are noted in the tables below:—

TABLE I

Values of T_k *for AB joins in spacings of varying sizes*

Values of (k+1)	Expected Value	Variance
(1)	(2)	(3)
2	$\frac{mn}{N}$	$\frac{m^2n^2}{N^2(N-1)}$
3	$\frac{(2N-3)mn}{N(N-1)}$	$\frac{(4N-7)mn}{N(N-1)} + \frac{(4N^2-26N+44)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{\frac{(2N-3)mn}{N(N-1)}\right\}^2$
4	$\frac{3(N-2)mn}{N(N-1)}$	$\frac{(9N-22)mn}{N(N-1)} + \frac{(9N^2-69N+146)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{\frac{3(N-2)mn}{N(N-1)}\right\}^2$
5	$\frac{2(2N-5)mn}{N(N-1)}$	$\frac{2(8N-25)mn}{N(N-1)} + \frac{(16N^2-140N+350)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{\frac{2(2N-5)mn}{N(N-1)}\right\}^2$

TABLE II

Values of W_k *for AB joins for blocks of varying sizes*

Values of (k+1)	Expected Value	Variance
(1)	(2)	(3)
2	$\frac{mn}{N}$	$\frac{m^2n^2}{N^2(N-1)}$
3	$3(N-2)\frac{mn}{N(N-1)}$	$\frac{(9N-22)mn}{N(N-1)} + \frac{(9N^2-67N+128)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \frac{9(N-2)^2m^2n^2}{N^2(N-1)^2}$
4	$6(N-3)\frac{mn}{N(N-1)}$	$\frac{4(9N-34)mn}{N(N-1)} + \frac{(36N^2-346N+884)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \frac{36(N-3)^2m^2n^2}{N^2(N-1)^2}$
5	$10(N-4)\frac{mn}{N(N-1)}$	$\frac{4(25N-127)mn}{N(N-1)} + \frac{(100N^2-1170N+3690)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \frac{100(N-4)^2m^2n^2}{N^2(N-1)^2}$

NOTE—$N=m+n$

TABLE III

AB and BA joins T'_k in spacing for various sizes

Values of (k+1)	Expected Value	Variance
(1)	(2)	(3)
3	$2(2N-3)\dfrac{mm}{N(N-1)}$	$2(8N-19)\dfrac{mn}{N(N-1)} + \dfrac{(16N^2-104N+176)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{2(2N-3)\dfrac{mn}{N(N-1)}\right\}^2$
4	$6(N-2)\dfrac{mn}{N(N-1)}$	$4(9N-29)\dfrac{mn}{N(N-1)} + \dfrac{(36N^2-276N+584)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{6(N-2)\dfrac{mm}{N(N-1)}\right\}$
5	$4(2N-5)\dfrac{mn}{N(N-1)}$	$(64N-260)\dfrac{mn}{N(N-1)} + \dfrac{(64N^2-560N+1400)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \left\{\dfrac{4(2N-5)mn}{N(N-1)}\right\}^2$

TABLE IV

AB and BA joins W'_k for blocks of varying sizes

Values of (k+1)	Expected Value	Variance
(1)	(2)	(3)
2*	$\frac{2mn}{m+n}$	$\frac{2mn\,(2mn-m-n)}{(m+n)^2\,(m+n-1)}$
3	$6(N-2)\frac{mn}{N(N-1)}$	$4(9N-26)\frac{mn}{N(N-1)} + \frac{(36N^2-268N+512)m(m-1)n(n-1)}{N(N-1)\,(N-2)\,(N-3)} - \left\{6(N-2)\frac{mn}{N(N-1)}\right\}^2$
4	$12(N-3)\frac{mn}{N(N-1)}$	$36(4N-7)\frac{mn}{N(N-1)} + \frac{(144N^2-1384N+3536)m(m-1)n(n-1)}{N(N-1)\,(N-2)\,(N-3)} - \left\{12(N-3)\frac{mn}{N(N-1)}\right\}^2$
5	$20(N-4)\frac{mn}{N(N-1)}$	$80(5N-28)\frac{mn}{N(N-1)} + \frac{(400N^2-4680N+14760)m(m-1)n(n-1)}{N(N-1)\,(N-2)\,(N-3)} - \left\{20(N-4)\frac{mn}{N(N-1)}\right\}^2$

NOTE—$N=m+n$

* This applies for Table III also.

Any of the tests described above can be used for testing the randomness of a binomial sequence of observations and also for testing the significance of the difference between two samples $\{x_m\}$ and $\{y_n\}$. When $(k+1)=2$, $W'_k=T'_k$ and reduces to Wald and Wolfowitz's test discussed earlier. Also when $(k+1)=m+n$ T'_k, W'_k reduce to Wilcoxon's test. When $(k+1)>2$ we obtain a number of new tests which are all more efficient than any of the similar existing tests. The tests are performed by finding the standardized deviate

$$Z=\frac{X-E(x)}{S.D. \text{ of } x}$$

where X stands for the observed W_k, W'_k, T_k or T'_k as the case may be. When k is about 1/5 or 1/6 of (m+n); W_k and T_k seem to be more efficient than for other values. If $|Z|\geqslant 1\cdot 96$ we consider the test to be significant at the 5 per cent level. (The efficiency of W'_k and T'_k is maximum when $k=m+n$).

It would be noted that the application of the test is very much facilitated if the expected values and the variances of W_k, W'_k, T_k and T'_k are available. With this end in view some of these values are given in the Appendices I & II, III & IV for some chosen values of m, n and k.

Applications

Randomness in a sequence—Suppose a sample of 40 bulbs has been collected as it emerges from a factory and it has been found that ten of them are defective. It is required to know whether the defective ones occur at random or not. This can be decided by noting the sequence of the order in which the bulbs have been produced in the factory. Let the sequence of the sample be as follows:

GGGDDGGGGGDGGGGDDDGGGGGGGDGGDGGGGGGGGDDGG

G denotes a good bulb

D denotes a defective bulb

The observed and the expected values of W and T *i.e.*, the number of times that G comes before D in blocks and at spacings of length 3, 4, 5 and 6, and other quantities associated with the distribution of the standardised deviate are given below in Tables V and VI respectively.

TABLE V

Number of times that G precedes D for varying block sizes

Size of block k+1	Observed	Expected	Variance	Standardised Deviate $\frac{(O-E)}{S.D.}$
3	21	21·92	7·34	—0·34
4	45	42·69	21·87	0·49
5	75	69·23	51·46	0·80
6	107	100·96	105·59	0·59

TABLE VI

Number of times that G precedes D for varying spaces

Length of spacing k+1	Observed	Expected	Variance	Standardised Deviate $\frac{(O-E)}{S.D.}$
3	15	14·81	3·07	0·11
4	24	21·92	5·12	0·92
5	33	28·85	7·83	1·48
6	40	35·58	11·41	1·31

On comparing the values of the standardised deviates with 1·96, the value of the normal deviate at the 5 per cent level, we find them all to be less than 1·96 indicating thereby that the validity of the null-hypothesis is not to be questioned. Hence the above tables show that the given sequence can be considered to be a random one. For blocks of five observations the standardised deviate is a maximum.

In actual practice it is not conveniently possible to fix in advance the size of the block for which the standardised deviate is a maximum. However it may be noted that the efficiency of the test increases with k, reaches a maximum for a particular k and then decreases as k increases further.

Two sample testing—Table VII gives the data for two random samples of twenty-five observations each, from a normal population with mean 25 and standard deviation 10.

TABLE VII

Data of two samples considered for their analysis

Serial No.	Sample A	Sample B
1	36·75	36·55
2	27·09	29·87
3	25·15	34·78
4	37·10	18·62
5	21·09	23·28
6	18·96	23·89
7	9·19	22·67
8	23·33	10·74
9	2·73	34·50
10	23·25	30·12
11	26·38	44·77
12	12·46	7·72
13	23·44	15·40
14	25·12	35·27
15	31·71	15·17
16	21·52	24·79
17	22·62	16·40
18	31·71	19·69
19	21·28	24·12
20	22·25	23·23
21	28·23	31·12
22	30·50	32·55
23	9·107	22·46
24	22·77	14·80
25	30·47	18·25

Pooling together the two sampies and arranging the fifty values in ascending order and identifying each of these values as A or B according as the observations belong to sample A or B we obtain the following sequence:

ABABABBBBBBABAAAAABABABBAABBBAAAAABBAABAAB BBBBAAAB

The results of examining the above sequence for A preceding B in blocks of various sizes are noted in Tables VIII and IX.

TABLE VIII

Number of times that A precedes B for blocks of varying sizes

Size of Block (k+1)	Observed	Expected	Variance	Standardised Deviate $\frac{(O-E)}{S.D.}$
3	33	36·73	15·37	—0·95
4	66	71·94	42·01	—0·92
5	107	117·35	88·82	—1·10
6	162	172·19	163·24	—0·80

TABLE IX

Number of times that A precedes B for varying sizes of the spacings

Length of spacing k+1	Observed	Expected	Variance	Standardised Deviate $\frac{(O-E)}{S.D.}$
3	23	24·74	6·44	—0·69
4	36	36·73	10·02	—0·23
5	47	48·47	14·22	—0·39
6	61	59·95	19·26	0·24

The values of the standardised deviates (being less than 1·96) are not significant and hence they reveal that there is no difference between the two samples. The value of 't' for the two samples works out to be 0·43. The value of Z for the two samples is greater than 0·43 when the size of the block or spacing, *i.e.* (k+1), is 3. This indicates that the present test may prove to be better than even the 't' test. Further work is needed to confirm this finding.

References

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APPENDIX I

Values of T_2 and T_3 for a few values of m and n

(a) Spacings 3: and less

$$\begin{cases} \mu'_1 = (2N-3)nm/N(N-1) \\ \mu_2 = (4N-7)\dfrac{nm}{N(N-1)} \end{cases}$$

$$+(4N^2-26N+44)\frac{n(n-1)m(m-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{(2N-3)nm/N(N-1)\}^2$$

(b) Spacings 4: and less

$$\mu'_1 = 3(N-2)nm/N(N-1)$$

$$\mu_2 = (9N-22)\frac{nm}{N(N-1)}$$

$$+(9N^2-69N+146)\frac{n(n-1)m(m-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{3(N-2)nm/N(N-1)\}^2$$

			(a)		(b)	
N	m	n	Expected Value μ_1	Variance μ_2	Expected Value μ_1	Variance μ_2
30	20	10	13·10	3·21	19·31	5·26
	15	15	14·74	3·96	21·72	6·30
	24	6	9·43	1·81	13·90	3·25
	25	5	8·19	1·43	12·07	2·68
40	20	20	19·74	5·22	29·23	8·21
	30	10	14·81	3·09	21·92	5·13
	32	8	12·64	2·35	18·71	4·00
	35	5	8·64	1·20	12·79	2·33
50	25	25	24·74	6·49	36·73	10·01
	40	10	15·84	2·84	23·51	4·77
	30	20	23·76	5·97	35·27	9·37
	45	5	8·91	1·02	13·22	1·95
60	30	30	29·75	7·74	44·24	11·92
	50	10	16·53	2·58	24·58	4·38
	40	20	26·44	6·16	39·32	9·84
	45	15	22·31	4·49	33·18	[illegible]

APPENDIX II

Values of W_2 and W_3 for a few values of m and n

(a) Blocks of 3:

$$\begin{cases} \mu'_1 = 3(N-2)nm/N(N-1) \\ \mu_2 = (9N-22)\dfrac{mn}{N(N-1)} \end{cases}$$

$$+(9N^2-67N+128)\frac{n(n-1)m(m-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{3(N-2)nm/N(N-1)\}^2$$

(b) Blocks of 4:

$$\begin{cases} \mu'_1 = 6(N-3)nm/N(N-1) \\ \mu_2 = (36N-136)nm/N(N-1) \end{cases}$$

$$+(36N^2-346N+884)\frac{n(n-1)m(m-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{6(N-3)nm/N(N-1)\}^2$$

			(a)		(b)	
N	m	n	Expected Value μ'_1	Variance μ_2	Expected Value μ'_1	Variance μ_2
30	20	10	19·31	7·44	37·44	21·08
	15	15	21·72	9·12	41·90	24·48
	24	6	13·90	4·30	26·81	13·93
	25	5	12·07	3·42	23·28	11·74
40	20	20	29·23	12·24	56·92	33·26
	30	10	21·92	7·34	42·69	21·87
	32	8	18·71	5·57	36·43	17·50
	35	5	12·79	2·94	24·90	10·51
50	25	25	36·73	15·37	71·94	42·01
	40	10	23·51	6·84	46·04	21·07
	30	20	35·26	14·28	69·06	39·36
	45	5	13·22	2·55	25·90	9·31
60	30	30	44·24	18·51	86·95	50·76
	50	10	24·58	6·30	48·31	19·80
	40	20	39·32	14·84	77·29	41·76
	45	15	33·18	10·84	65·21	31·69

APPENDIX III

Values of T'_1, T'_2, T'_3 and T'_4 for a few values of m and n

(a) Spacings 2 and less:

$$\mu'_1 = 2mn/(m+n)$$
$$\mu_2 = 2mn(2mn - m - n)/N^2(N-1)$$

(b) Spacings 3 and less:

$$\mu'_1 = 2(2N-3)\frac{mn}{N(N-1)}$$

$$\mu_2 = 2(8N-19)\frac{mn}{N(N-1)}$$

$$+\frac{(16N^2 - 104N + 176)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)}$$

$$-\left\{2(2N-3)\frac{mn}{N(N-1)}\right\}$$

(c) Spacings 4 and less:

$$\mu'_1 = 6(N-2)mn/N(N-1)$$
$$\mu_2 = 4(9N-29)\frac{mn}{N(N-1)}$$

$$+\frac{(36N^2 - 276N + 584)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{6(N-2)mn/N(N-1)\}^2$$

(d) Spacings 5 and less:

$$\mu'_1 = 4(2N-5)\frac{mn}{N(N-1)}$$

$$\mu_2 = (64N-260)\frac{mn}{N(N-1)}$$

$$+\frac{(64N^2 - 560N + 1400)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)}$$

$$-\{4(2N-5)mn/N(N-1)\}^2$$

APPENDIX III—contd.

			(a)		(b)		(c)		(d)	
			Expected value	Variance	Expected value	Variance	Expected value	Variance	Expected value	Variance
N	m	n	μ'_1	μ_2	μ'_1	μ_2	μ'_1	μ_2	μ'_1	μ_2
20	10	10	10·00	4·74	19·47	8·18	28·42	10·45	36·84	11·71
	15	5	7·50	2·57	14·60	4·73	21·32	6·68	27·63	8·50
	12	8	9·60	4·35	18·69	7·57	27·28	9·79	35·37	11·18
24	12	12	12·00	5·74	23·48	10·18	34·43	13·45	44·87	15·64
	16	8	10·67	4·48	20·87	8·12	30·61	11·05	39·88	13·39
	18	6	9·00	3·13	17·61	5·86	25·83	8·37	33·65	10·76
30	15	15	15·00	7·24	29·48	13·20	43·45	17·96	56·90	21·60
	20	10	13·33	5·67	26·21	10·49	38·62	14·60	50·57	18·12
	24	6	9·60	2·85	18·87	5·56	27·81	8·33	36·41	11·30
	25	5	8·33	2·11	16·38	4·23	24·14	6·59	31·61	9·30
40	20	20	20·00	9·74	39·49	18·26	58·46	25·46	76·92	31·56
	25	15	18·75	8·53	37·02	16·04	54·81	22·62	72·12	28·37
	30	10	15·00	5·38	29·62	10·36	43·85	15·11	57·69	19·78
	32	8	12·80	3·87	25·27	7·61	37·42	11·39	49·23	15·41
	35	5	8·75	1·74	17·28	3·64	25·58	5·90	33·65	8·70
50	25	25	25·00	12·24	49·49	23·22	73·47	32·97	96·94	41·54
	40	10	16·00	4·90	31·67	9·64	47·02	14·46	62·04	19·52
	30	20	24·00	11·27	47·51	21·42	70·53	30·54	93·06	38·70
	45	5	9·00	1·47	17·81	3·14	26·45	5·35	34·90	7·67
60	30	30	30·00	14·75	59·49	28·23	88·47	40·47	116·95	51·53
	50	10	16·67	4·43	33·05	8·86	49·15	13·51	64·97	18·60
	40	20	26·67	11·60	52·88	22·35	78·64	32·39	103·95	41·79
	45	15	22·50	8·20	44·62	15·99	66·36	23·54	87·71	31·03

APPENDIX IV

Values of W'_1, W'_2, W'_3 and W'_4 for a few values of m and n

(a) Blocks of 2:

$$\mu'_1 = 2mn/(m+n)$$

$$\mu_2 = 2mn(2mn - m - n)/N^2(N-1)$$

(b) Blocks of 3:

$$\mu'_1 = 6(N-2)mn/N(N-1)$$

$$\mu_2 = 4(9N-26)\frac{mn}{N(N-1)} + \frac{(36N^2-268N+512)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \{6(N-2)mn/N(N-1)\}^2$$

(c) Blocks of 4:

$$\mu'_1 = 12(N-3)mn/N(N-1)$$

$$\mu_2 = 36(4N-17)\frac{mn}{N(N-1)} + \frac{(144N^2-1384N+3536)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \{12(N-3)mn/N(N-1)\}^2$$

(d) Blocks of 5;

$$\mu'_1 = 20(N-4)mn/N(N-1)$$

$$\mu^2 = 80(5N-28)\frac{mn}{N(N-1)} + \frac{400N^2-4680N+14760)m(m-1)n(n-1)}{N(N-1)(N-2)(N-3)} - \{20(N-4)mn/N(N-1)\}^2$$

APPENDIX IV—contd.

N	m	n	(a) Expected value μ'_1	(a) Variance μ_2	(b) Expected value μ'_1	(b) Variance μ_2	(c) Expected value μ'_1	(c) Variance μ_2	(d) Expected value μ'_1	(d) Variance μ_2
20	10	10	10·00	4·74	28·43	19·74	53·68	45·38	84·21	77·94
	15	5	7·50	2·56	21·32	12·05	40·26	34·72	63·16	79·40
	12	8	9·60	4·34	27·28	18·42	51·54	43·75	80·84	79·12
24	12	12	12·00	5·74	34·43	24·78	65·74	59·44	104·35	107·56
	16	8	10·67	4·48	30·61	20·16	58·43	51·91	92·75	104·23
	18	6	9·00	3·13	25·83	15·03	49·30	42·84	78·26	97·02
30	15	15	15·00	7·24	43·45	32·32	83·79	80·52	134·48	152·31
	20	10	13·33	5·67	38·62	26·10	74·48	68·56	119·54	140·02
	24	6	9·60	2·85	27·81	14·54	53·63	44·52	86·07	108·27
	25	5	8·33	2·11	24·14	11·38	46·55	37·22	74·71	96·18
40	20	20	20·00	9·74	58·46	44·87	113·85	115·62	184·62	227·22
	25	15	18·75	8·53	54·81	39·75	106·73	104·54	173·08	211·58
	30	10	15·00	5·38	43·85	26·27	85·38	74·39	138·46	165·82
	32	8	12·80	3·87	37·42	19·64	72·86	58·85	118·15	139·78
	35	5	8·75	1·74	25·58	9·94	49·81	34·41	80·77	93·38
50	25	25	25·00	12·24	73·47	57·40	143·88	150·68	234·69	302·19
	40	10	16·00	4·90	47·02	24·75	92·08	73·18	150·20	170·92
	30	20	24·00	11·26	70·53	53·10	138·12	140·78	225·31	286·36
	45	5	9·00	1·47	26·45	8·68	51·80	31·00	84·49	86·30
60	30	30	30·00	14·75	88·47	69·91	173·90	185·73	284·74	377·21
	50	10	16·67	4·43	49·15	22·89	96·61	69·59	158·19	167·22
	40	20	26·67	11·60	78·64	55·77	154·58	151·65	253·11	318·36
	45	15	22·50	8·20	66·36	40·32	130·42	113·79	213·56	250·75