LANCHESTER'S EQUATIONS APPLIED TO ASSAULT ON BRESKENS

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ABSTRACT

The validity of the applicability of the Lanchester's equations of the type

$$\frac{dM}{dt} = P(t) - AN$$

$$\frac{dN}{dt} = -BM$$

(where M(t) and N(t) are the effective attackers' and defenders' strength at time t ; P (t) is the attackers' reinforcement rate ; and A and B are the respective loss rates, per opposing combatant) to the situation at Breskens Pocket is investigated. It is concluded, on the basis of a series of calculations, that the above type of equations fairly describe the said engagement if we assume that the defenders' strength is reduced by about 40 percent by the time the battle ends.

Introduction

Lanchester[1], in 1916, proposed a set of differential equations, from which it is possible to predict the expected result of an engagement. These equations are formulated under the assumption that the casualty producing rate of an entire force is proportional to the number of troops in the force. J. H. Engel[2] has verified the validity of these equations to the combat situation where the U.S. forces captured Iwo Jima. It is the object of this paper to test the validity of a certain type of Lanchester's equations to the situation at Breskens Pocket.

Analysis

Let M(t) and N(t) be the number of effective attacker and defender troops at time t ; P(t) and Q(t) be the respective rates at which the combatants are being reinforced ; t is the time measured from the start of the battle ; A and B are the attacker and defender casualties per opposing combatant and finally let C and D be the operational loss rates for the two sides. Then the engagement is described by the following set of differential equations :

$$\left.\begin{aligned} \frac{dM}{dt} &= P - AN - CM \\ \frac{dN}{dt} &= Q - BM - DN \end{aligned}\right\} \quad (1)$$

These are the most general set of Lanchester's equations.

It is not possible to test the validity of the applicability of equations (1) to the situation at Breskens because of the large number of parameters involved or, to be more correct, due to insufficient information regarding the engagement. We can make one simple assumption, which is quite valid for most of the battles, that is, the losses due to operational causes are negligible as compared to those due to enemy activity so that C=D=O. Further, for lack of any information regarding the defenders, suppose that the defender troops were neither reinforced nor withdrawn during the course of the battle. Under these circumstances, the situation at Breskens is described by the following two equations :

$$\frac{dM}{dt} = P - AN \tag{2a}$$

$$\frac{dN}{dt} = -BM \tag{2b}$$

For further treatment it is convenient to make the following substitutions:—

$AN(o)=A_1$, $B/N(o) = B_1$, $\sqrt{A_1 B_1}=\mu$
and $N(t)/N(o) = E(t)$

where N(o) is the initial defender strength. With these substitutions, equations (2) transform to

$$\frac{dM}{dt} = P - A_1 E \tag{3a}$$

$$\frac{dE}{dt} = -B_1 M \tag{3b}$$

For arbitrary but integrable P(t), these equations admit of the following solution (see Appendix I)—

$$M(t) = M(o) \cosh \mu t - \frac{A_1}{\mu} \sinh \mu t + \int_o^t \cosh \mu(t-s)\, P(s)\, ds \tag{4a}$$

$$E(t) = \cosh \mu t - \frac{\mu}{A_1} M(o) \sinh \mu t - \frac{\mu}{A} \int_o^t \sinh \mu\, (t-s)\, P(s)\, ds \tag{4b}$$

where M(o) is the initial attacker strength engaged in the battle.

The following information regarding assault on Breskens was available (See appendix 2, Table I).

(*i*) The total number of attacker troops entering the engagement on each day, their number being zero at the start of the battle (M(o)=o).

(*ii*) The total number of attacker casualties during each day of the battle.

(*iii*) The time for which the engagement lasted.

Two points are to be noted. Firstly, nothing is known about the rate at which the defender forces are being reinforced or withdrawn. This rate, however, we have assumed to be zero. Secondly, we do not know about the initial strength of the defenders, and neither do we know their strength at the end of engagement. It is precisely for lack of information regarding the second point that we choose to deal with the quantities E, A_1 and B_1, as occuring in equations (3a, 3b), rather than with N, A and B, occuring in equations (2a, 2b).

Now solution (4) which we have derived involves the continuous time variable "t" whereas we know P(t), the reinforcement rate for the attackers only for discreet values of t. We choose one day as the unit of time and make a simplifying assumption that the attacker troops are entering the engagement at a constant rate during each day. Thus, if a reinforcement P(r) arrives at the beginning of the r-th day, then we define P(t) as

$$P(t)=P(r) \quad \text{for} \quad r \leqslant t < r+1$$

where r is a non-negative integer.

With this understanding, integrals in equations (4) can be replaced by summations as follows :

$$M(t) = M(o)\cosh \mu t_1 - \frac{A_1}{\mu}\sinh \mu t + \frac{1}{\mu}\sum_{r=o}^{t-1} P(r)\Big(\sinh \mu(t-r) - \sinh \mu(t-r-1)\Big) \qquad (4\,a')$$

$$E(t) = \cosh \mu t - \frac{\mu}{A_1}\sinh \mu t - \frac{1}{A\pi}\sum_{r=o}^{t-1} P(r)\Big(\cosh \mu(t-r) - \cosh \mu(t-r-1)\Big) \qquad (4\,b')$$

These equations hold for positive integral t. These formulae are not convenient for computational purposes. For calculation purposes, we write them in a different manner :

$$M(t+1) = \cosh \mu\, M(t) - \left[E(t) - \frac{P(t)}{A_1} \right]\frac{A_1}{\mu}\sinh \mu \qquad (5a)$$

$$E(t+1) = \cosh \mu\, E(t) + \frac{P(t)}{A_1}\Big(1 - \cosh \mu\Big) - \frac{\mu \operatorname{Sinh} \mu}{A_1} \qquad (5b)$$

where E (o)=O and M(o) is equal to the observed effective attacker's strength at the start of the battle. For the situation at Breskens (M (o)=O).

Whether or not the situation at Breskens is described by the equations (4a, 4b) or (4a′, 4b′) or (5a′, 5b′) depends upon the fact whether it is possible to assign values to A and B (which is the same thing as μ and A_1 since we are dealing with them) such that calculated values of M(t) and E(t) are in good agreement with the observed values *$\overline{M}(t)$ and $\overline{E}(t)$. (In our case the question of agreement between $\overline{E}(t)$ and E(t) does not arise since E(t) is not known).

We will now briefly describe a method, due to J. H. Engel[2], for rough estimation of A_1 and B_1 and then obtain a better estimate from them.

Let the battle last for T days. Integrating (3b) from 0 to T and replacing integrals by summation, we have

$$B_1 = \left\{ 1 - \overline{E}(T) \right\} \Big/ \left\{ \sum_{r=o}^{t} \overline{M}(t) \right\}$$

Now integrating (3b) from 0 to t, we get

$$\left.\begin{aligned} E'(t) &= 1 - B_1 \sum_{r=1}^{t} \overline{M}(r) \text{ for } r > o \\ &\text{and } E(o)=1 \end{aligned}\right\} \qquad \ldots \qquad \ldots \quad (7)$$

*In what follows bar (——) over a variable will imply the observed value of that quantity.

where $E'(t)$ denotes the approximate theoretical value of the fraction of the defender strength at time t. We designate it approximate theoretical value since it has been calculated from (3b) by using observed values of the effective attacker strength on different days.

Now integrating (3a) from 0 to s, where 's' is some fixed time at or near the termination of the engagement, replacing integrals by summations, and using (7) we get

$$A_1 = \left\{ \sum_{r=0}^{s} P(t) - \bar{M}(s) \right\} \Big/ \sum_{t=0}^{s} E'(t) \qquad \ldots \qquad \ldots \qquad (8)$$

Having obtained A_1 and B_1, the approximate theoretical values of the attacker strength are given by

$$M'(t) = \sum_{k=0}^{t} P(k) - A_1 \sum_{k=0}^{t} E'(k) \qquad \ldots \qquad \ldots \qquad \ldots \qquad (9)$$

The choice of A_1 assures that $\bar{M}(o)=M(o)$ and $M(s)=\bar{M}(s)$ and that of B_1 assures that $E(o)=\bar{E}(o)$ and $E(T)=\bar{E}(T)$.

The introduction of 's', some fixed time at or near the termination of the engagement requires some explanation. What happens in an actual battle is that A and B, the casualty producing rates per combatant of the two sides do not remain constant throughout the period of the battle. They are subject to the influence of many factors such as the morale, the training, the experience of the soldiers; the nature of the weapons used, the ratio of the troops in battle to that in support, and last but not the least to the terrain. It is not possible to consider the influence of these factors. But, besides all these factors, during later stages of the battle, when the battle becomes sporadic, the rate of combat may be markedly affected. Roughly, 's' may be taken as the time when the attackers are able to hold the ground securely. For the situation at Breskens we have taken $s=T$, the length of the battle. This is justified by observing the number of day to day casualties of the attackers which remain quite high even till the end of the engagement.

From formulae (6) and (8), we find that the determination of A_1 and B_1 involves the knowledge of $\bar{E}(T)$, the observed fractional strength of the defenders at the end of the battle. Since we do not know $E(T)$, we make estimates of A_1 and B_1 by taking $\bar{E}(T)=0{\cdot}25$, $0{\cdot}4$, $0{\cdot}5$ and $0{\cdot}6$.

Having obtained a rough estimate of A_1 and B_1, we can now obtain their exact estimate. For $t=T$, taking $M(T)$ equal to the observed attacker strength at the end of battle and $E(T)$ equal to $0{\cdot}25$, $0{\cdot}4$, $0{\cdot}5$ and $0{\cdot}6$ in succession, we get from (4a) and (4b) two equations in two unknowns μ and A_1. Let us write them as

$$f(\mu, A_1)=0$$

$$\text{and } \flat(\mu, A_1)=0$$

We know a rough estimate of A_1 and μ by the method given above. Then, by Newton's method of successive approximation for finding the root, a better approximation to the roots will be A_1+h and μ_1+k where h and k are the roots of the simultaneous equations

$$\left.\begin{array}{l} \left(\frac{\partial f}{\partial A_1}\right)_{A_1,\mu} h + \left(\frac{\partial f}{\partial \mu}\right)_{A_1,\mu} k + f(\mu, A_1) = 0 \\ \text{and} \\ \left(\frac{\partial \phi}{\partial A_1}\right)_{A_1,\mu} h + \left(\frac{\partial \phi}{\partial \mu}\right)_{A_1,\mu} k + \phi(\mu, A_1) = 0 \end{array}\right\} \quad .. \quad (10)$$

It was found for the case in question that this process of successive approximation has not to be employed more than once. The choice of A_1 and B_1 by this method ensures that (i) the total calculated number of attackers' casualties is equal to the total observed number of attackers' casualties, and (ii) the observed fractional defenders' strength at the end of the battle is equal to the calculated one. Appendix III gives two sets of calculations, one based on the rough estimate of A_1 and B_1 from formulae (6) and (8) and then calculating M(t) from (9). This we call as rough theoretical estimate of M(t). The other one is based on the exact determination of A_1 and B_1 from (10) and then calculating M(t) from (5a, 5b). This we designate as exact theoretical estimate of M(t). Tables III and IV give the effective attacker strength and attacker casualties on different days. On going through the calculations we observe that for $E(T)=0\cdot6$, the agreement between calculated and observed values of M(t) is quite close.

Conclusion derived is that the Lanchester type equations (2a, 2b) are valid for the engagement at Breskens, if we assume that the enemy strength is reduced by about 40 per cent by the time the battle ends.

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References

1. F. W. Lanchester, Aircraft in Warfare; the Dawn of the Fourth Arm, (Constable and Co., Ltd., London, 1916.)

2. J. H. Engel, "A Verification of Lanchester's Law", Journal of the operations Research Society of America, 2, 163—171, 1954.

3. Measurement of Opposition to an Attack; Report No. 2, Military Operational Research Unit.

APPENDIX I

Solution of Equations (2a) and (2b)

$$\frac{dM}{dt} = P(t) - AN \tag{1a}$$

$$\frac{dN}{dt} = -BM \tag{1b}$$

Taking Laplace Transforms of these equations with respecte to 't', we get

$$p\overline{M} + M(0) = \overline{P} - A\overline{N} \tag{2a}$$

and $$p\overline{N} + N(0) = -B\overline{M} \tag{2b}$$

where 'p' is the transform parameter and a bar denotes the L.T. of the respective variable.

Solving (2a) and (2b) for $\overline{M}$ and $\overline{N}$, we have

$$\overline{M} = \frac{p}{p^2 - AB}\overline{P} + \frac{p}{p^2 - AB}M(0) - \frac{AN(0)}{p^2 - AB} \tag{3a}$$

$$\overline{N} = -\frac{B}{p^2 - AB}\overline{P} + \frac{p}{p^2 - AB}N(0) - \frac{BM(0)}{p^2 - AB} \tag{3b}$$

Now, since $\overset{-1}{L}\left(\frac{p}{p^2 - AB}\right) = \cosh(\sqrt{AB}\,t)$

$$\overset{-1}{L}\left(\frac{\sqrt{AB}}{p^2 - AB}\right) = \sinh(\sqrt{AB}\,t)$$

and $$\overset{-1}{L}\left(\bar{f}(p).\,\bar{g}(p)\right) = \int_0^t f(t)\,g(t-a)\,da$$

$$= \int_0^t f(t-a)\,g(t)\,da$$

we immediately get

$$M(t) = M(0)\cosh(\sqrt{AB}\,t) - \sqrt{\frac{A}{B}}\,N(0)\sinh(\sqrt{AB}\,t)$$

$$+ \int_0^t \cosh\sqrt{AB}\,(t-s).\,P(s)\,ds,$$

$$N(t) = N(0)\cosh(\sqrt{AB}\,t) - \sqrt{\frac{A}{B}}\,M(0)\sinh(\sqrt{AB}\,t)$$

$$-\sqrt{\frac{B}{A}}\int_0^t \sinh\sqrt{AB}\,(t-s).\,P(s).\,ds$$

APPENDIX II

TABLE I

Data available

Time in days	Observed attacker casualties	P(t) (men)	M(t)
0	—	2700*	0
1	64	1800	2636
2	53	0	4383
3	44	1800	4339
4	50	0	6089
5	52	0	6037
6	26	0	6011
7	47	0	5964
8	61	0	5903
9	54	0	5849
10	46	0	5803
11	18	0	5785
12	40	0	5743

*Fighting strength of a brigade has been taken to be 2700 men.

TABLE II

Approximate and exact values of A_1 and B_1 for different values of $\overline{N}(T)/N(o)$

	N(T)/N(o)=0·25		N(T)/N(o)=0·4		N(T)/N(o)=0·5		N(T)/N(o)=0·6	
	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact
A_1	67·71	69·21	61·96	63·05	58·63	59·43	55·66	56·22
10^5 B_1	1·16	1·12	0·93	0·98	0·78	0·81	0·62	0·65

TABLE III

Effective Attacker strength

Time in days	Observed	N(T)/N(o)=0·25		N(T)/N(o)=0·4		N(T)/N(o)=0·5		N(T)/No=0·6	
		Approx calc.	Exactly calc.	Approx. calc.	Exactly calc.	Approx. calc.	Exactly calc.	Approx. calc.	Exactly calc.
0	0	0	0	0	0	0	0	0	0
1	2636	2632	2631	2638	2637	2641	2641	2644	2644
2	4383	4367	4364	4378	4376	4384	4383	4390	4389
3	4339	4304	4301	4320	4317	4328	4327	4336	4335
4	6089	6046	6042	6064	6062	6075	6073	6085	6083
5	6037	5992	5988	6012	6009	6024	6022	6035	6033
6	6011	5942	5938	5964	5961	5976	5974	5987	5986
7	5964	5898	5894	5919	5916	5931	5929	5942	5940
8	5903	5858	5854	5877	5874	5889	5886	5899	5897
9	5849	5823	5820	5839	5837	5849	5847	5857	5856
10	5803	5792	5790	5805	5803	5811	5810	5818	5817
11	5785	5766	5766	5773	5772	5777	5776	5780	5780
12	5745	5745	5745	5745	5745	5745	5745	5745	5745

TABLE IV

Attacker Casualties

Time in days	Observed	N(T)/N(o)=0·25		N(T)/N(o)=0·4		N(T)/N(o)=0·5		N(T)/N(o)=0·6	
		Approx. calc.	Exactly calc.	Approx. calc.	Exactly calc.	Approx. calc.	Exactly calc.	Approx. calc.	Exactly calc.
1	64	68	69	62	63	59	59	56	56
2	53	65	67	60	61	57	58	55	55
3	44	63	63	58	59	56	56	53	54
4	50	50	59	56	56	53	54	52	52
5	52	54	55	52	52	51	51	50	50
6	26	50	49	48	49	48	48	48	48
7	47	44	44	45	45	45	45	46	45
8	61	40	39	42	41	42	43	43	43
9	54	35	34	38	38	40	39	41	41
10	46	31	30	34	34	38	37	39	39
11	18	26	25	32	31	34	34	37	37
12	40	21	21	28	27	31	31	35	35