UNSTEADY FLOW OF GASES THROUGH A POROUS MEDIUM OF PRESSURE—DEPENDENT VISCOSITY

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In the isothermic state, gas flows in an infinite strip of a porous medium of finite length in the unsteady condition. The pressure inside the medium is dependent on the viscosity of the gas. The medium is sealed at one end, so that no flow occurs across it. An analytical solution for the pressure declines at the sealed end is obtained when there is a slight decrease in pressure at the other end.

Since the study of transient flow of gases through porous medium involves a type of a non-linear equation[1] for which there are no analytical methods, attempts have been made by Jain[2] and others to find the approximate solution of such equations. However, the modern trend of studying the flow of gases through porous medium is to take the physical properties such as permeability, viscosity and density as variables. In view of the variable nature of the physical quantities, Jain[3] discussed the unsteady flow of gases through porous medium of pressure-dependent permeability by analytical method.

In the present paper, unsteady flow of gases through porous medium of pressure-dependent viscosity has been discussed analytically. Perturbation method[4] has been employed to seek the approximate solution of the problem, when there is a slight decrease in pressure at one end of the medium. It has been found that as the viscosity of the gas increases, the dimensionless pressure ratio decreases.

BASIC EQUATIONS FOR GAS FLOWS IN POROUS MEDIUM

In the motion of a gas in a porous medium, the equation of state is

$$\rho = \rho_0 \left(\frac{P}{P_0} \right)^m \tag{1}$$

For isothermic state, we have $m = 1$.

The equation of motion[1] is

$$\text{div} \left(\rho \, \frac{K}{\mu} \, \text{grad} \, P \right) = f \, \frac{\partial \rho}{\partial t} \tag{2}$$

where K and f are respectively the permeability and porosity of the medium, and μ is the viscosity of the gas. In equation (2), it is assumed that the external forces are ignored.

STATEMENT OF THE PROBLEM

We consider an infinite strip of porous medium of pressure-dependent viscosity, through which there takes place the transient flow of a gas under isothermal change of state. The medium is of finite length L. The initial and boundary conditions to be considered may be described as follows. Let P_0 represent the constant initial fluid pressure inside the medium. The pressure at the end $x = 0$ is slightly decreased to P_1 and the other end of the medium is sealed so that no flow occurs across the plane $x = L$. The problem is to determine the pressure declines at the sealed end at any instant of time.

Assuming the viscosity of the gas as a linear function of the pressure in the form $\mu = nP + d$ and taking the axis of x along the length of the medium, the fundamental equation (2) reduces to the form

$$\frac{\partial}{\partial x}\left[\frac{P}{(nP+d)}\,\frac{\partial P}{\partial x}\right] = \frac{f}{K}\,\frac{\partial P}{\partial t} \tag{3}$$

if equation (1) is used with $m = 1$.

SOLUTION

Introducing a new variable $F(T)$ defined by

$$\left.\begin{aligned} P(x,t) &= P_0 + (P_1 - P_0)\,F(T) \\ \text{where}\quad T &= \frac{KP_0\,t}{f\,d\,x^2} \end{aligned}\right\} \tag{4}$$

Using substitution (4), the initial and boundary conditions take the forms

$$F(0) = 0 \text{ and } F(\infty) = 1 \tag{5}$$

and equation (3) reduces to

$$\begin{aligned} &4T^2\left[\{1 + \alpha(1 + bF)\}(1 + bF)F'' + bF'^2\right] \\ &+ \left[(6T - \alpha)(1 + bF) - 1\right]\left[1 + \alpha(1 + bF)\right]F' = 0 \end{aligned} \tag{6}$$

where $\alpha = \frac{nP_0}{d}$ and $b = (P_1 - P_0)$

To seek the solution of the equation (6), we employ Perturbation Method[4]. Let the solution may be expressed in the form

$$F = F_0 + bF_1 + b^2F_2 + \ldots\ldots\ldots\ldots \tag{7}$$

where F_0, F_1, ...are continuous differentiable functions of T, to be determined, and $-1 < b < 0$ is the perturbation parameter. Substituting (7) in (6) and then equating coefficients of b^r to zero for $r = 0, 1, \ldots\ldots\ldots\ldots$, we obtain

$$4T^2F_0'' + (6T - \alpha - 1)F_0' = 0 \tag{8}$$

$$4T^2[(1+\alpha)F_1''+(1+2\alpha)F_0F_0''+F_0'^2]$$
$$+[(6T-\alpha-1)(1+\alpha)F_1'+\{6T(1+2\alpha)-2\alpha(1+\alpha)\}F_0F_0']=0 \quad (9)$$

and so on.

Now, the equation (8) may be written as

$$\frac{F_0''}{F_0'}=\frac{(1+\alpha)-6T}{4T^2} \quad (10)$$

whose solution is

$$F_0=A\int^{T} T^{-3/2}e^{-(1+\alpha)/4T}\,dT+B \quad (11)$$

where A and B are constants of integration, and their values are obtained with the help of conditions (5). Using the properties of error function[5], we then obtain

$$A=\left(\frac{1+\alpha}{4\pi}\right)^{\frac{1}{2}} \text{ and } B=-\left(\frac{1+\alpha}{4\pi}\right)^{\frac{1}{2}}\int^{T=0} T^{-3/2}e^{-(1+\alpha)/4T}\,dT \quad (12)$$

so the required solution of the equation (8) is

$$F_0(T)=\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}} \quad (13)$$

Since

$$F_0(T)=\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}=1-\frac{2}{\sqrt{\pi}}\int_0^{\omega=\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}} e^{-\omega^2}\,d\omega$$

differentiation with respect to T gives,

$$\left.\begin{aligned} F_0'(T)&=\left(\frac{1+\alpha}{4\pi}\right)^{\frac{1}{2}} T^{-3/2}e^{-(1+\alpha)/4T} \\ \text{and} \quad & \\ F_0''(T)&=\left(\frac{1+\alpha}{4\pi}\right)^{\frac{1}{2}}\left[-\frac{3}{2}T^{-5/2}+\frac{(1+\alpha)}{4}T^{-7/2}\right]e^{-(1+\alpha)/4T}\end{aligned}\right\} \quad (14)$$

Substituting (13) and (14), the equation (9) reduces to

$$4T^2F_1''+(6T-\alpha-1)F_1'=-\left[\left(\frac{1+\alpha}{4\pi}\right)^{\frac{1}{2}}T^{-3/2}e^{-(1+\alpha)/4T}\right.$$
$$\left.\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}+\frac{1}{\pi}T^{-1}e^{-(1+\alpha)/4T}\right] \quad (15)$$

This is a linear equation in F_1' whose solution may be obtained in the usual manner

Thus, we obtain

$$F_1(T)=\frac{1}{2\pi^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}}\,T^{-1/2}\,e^{-(1+\alpha)/4T}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}$$
$$-\frac{1}{\pi(1+\alpha)}\,e^{-(1+\alpha)/2T}-\frac{1}{2(1+\alpha)}\left[\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right]^2$$
$$+C\left(\frac{4\pi}{1+\alpha}\right)^{\frac{1}{2}}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}+C_1 \qquad (16)$$

where C and C_1 are constants of integration.

Substituting values of F_0 and F_1 from (13) and (16) in (7) and retaining only terms containing b only, we obtain

$$F(T)=\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}+b\left[\frac{1}{2\pi^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}}\,T^{-\frac{1}{2}}\,e^{-(1+\alpha)/4T}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right.$$
$$-\frac{1}{\pi(1+\alpha)}\,e^{-(1+\alpha)/2T}-\frac{1}{2(1+\alpha)}\left\{\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right\}^2$$
$$\left.+C\left(\frac{4\pi}{1+\alpha}\right)^{\frac{1}{2}}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}+C_1\right]. \qquad (17)$$

With the help of the properties of error function[5], the values of C and C_1 are obtained from the conditions in (5). Thus, the solution of the equation (17) is expressed as

$$F(T)=\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}+b\left[\frac{1}{2\pi^{\frac{1}{2}}(1+\alpha)^{\frac{1}{2}}}\,T^{-\frac{1}{2}}\,e^{-(1+\alpha)/4T}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right.$$
$$-\frac{1}{\pi(1+\alpha)}\,e^{-(1+\alpha)/2T}-\frac{1}{2(1+\alpha)}\left\{\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right\}^2$$
$$\left.+\frac{(\pi+2)}{2\pi(1+\alpha)}\operatorname{erfc}\left(\frac{1+\alpha}{4T}\right)^{\frac{1}{2}}\right]. \qquad (18)$$

This provides the required approximate solution for different values of α, when b is negative and small such that $-1 < b < 0$.

DISCUSSION

For giving a definite idea of the values of the function $F(T)$ as given by equation (18), we consider a particular case when

$$b = -0{\cdot}1 \qquad (19)$$

which implies that P_1, the pressure at the end $x = 0$ is reduced to the pressure of amount $0{\cdot}9\ P_0$ and maintained at this value at all instances for which $t > 0$.

The numerical values of $F(T)$ thus obtained for $\alpha = 0, 0{\cdot}2, 0{\cdot}6, 1{\cdot}0$ and $2{\cdot}0$ have been graphically represented in Fig. 1 for increasing values of T starting from $T = 0$. The ordinate is the dimensionless pressure ratio $F(T)$, where

$$F(T) = \frac{P(L, t) - P_0}{P_1 - P_0} \tag{20}$$

and the abscissa is the dimensionless parameter T, where

$$T = \frac{K P_0 t}{f d L^2}. \tag{21}$$

Evidently $F(T)$ is the ratio of the pressure change at the sealed end to the pressure change at the end $x = 0$.

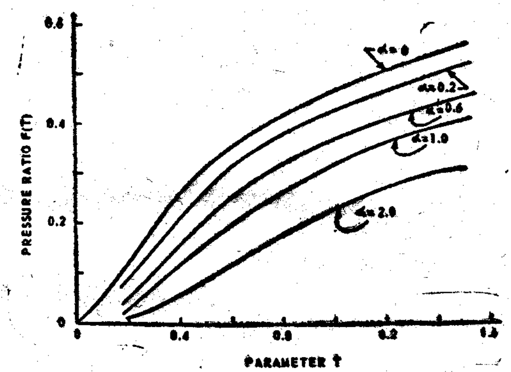

Fig. 1—Curves of pressure ratio $F(T)$ vs parameter T for different values of α.

The curves for various values of α in Fig. 1 show a systematic shift for increasing values of α, which implies that as the viscosity of the gas increases, the dimensionless pressure ratio $F(T)$ decreases at any instant of time. The curve for $\alpha = 0$ represents the case of an ideal gas of constant viscosity. In this case $F(T)$ takes the form

$$F(T) = \operatorname{erfc}\left(\frac{1}{2T^{\frac{1}{2}}}\right) + b\left[\frac{1}{2\pi^{\frac{1}{2}}} T^{-\frac{1}{2}} e^{-1/4T} \operatorname{erfc}\left(\frac{1}{2T^{\frac{1}{2}}}\right) - \frac{1}{\pi} e^{-1/2T} - \frac{1}{2}\left\{\operatorname{erfc}\left(\frac{1}{2T^{\frac{1}{2}}}\right)\right\}^2 + \left(\frac{1}{2} + \frac{1}{\pi}\right) \operatorname{erfc}\left(\frac{1}{2T^{\frac{1}{2}}}\right)\right]. \tag{22}$$

This result has been discussed in a previous paper[3].

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