LIAPUNOV FUNCTION OF CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD AND FOURTH ORDER

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(*Received* 22 *January* 1968; *revised* 27 *December* 1968)

Liapunov functions have been chosen for certain non-linear differential equations of the third and fourth order, thus giving the possible situations under which these equations would give stable solutions.

Successful attempts are being made since Barbashin[1] and Simanov[2] to generalize Routh-Hurwitz criteria for the stability of solutions of non-linear differential equations of the third[3–5], fourth[6] and higher order[7] by the use of Liapunov functions.

Liapunov functions for a few non-linear differential equations of the third and fourth order are constructed here to study their stability properties by means of Routh-Hurwitz criterion.

LIAPUNOV FUNCTION

By a Liapunov function is meant a function $V(x_1, x_2, \ldots \ldots, x_n)$ defined in some region that contains the unperturbed solution $x_1 = x_2 = \ldots \ldots x_n = 0$ of the motion considered and which, together with its derivative, satisfies certain conditions given below :—

(*i*) The real function $V(x_1, x_2, \ldots \ldots \ldots \ldots, x_n)$ of the variables $x_1, x_2, \ldots \ldots \ldots, x_n$ is assumed to be continuous in some region Ω of n— dimensional space.

(*ii*) $V(0, 0, \ldots \ldots \ldots, 0) = 0$.

(*iii*) The function $V(x_1, x_2, \ldots \ldots \ldots, x_n)$ is positive and definite, i.e. for all $x_1, x_2, \ldots \ldots \ldots, x_n$ in the region, $x_1 \neq 0,\ x_2 \neq 0, \ldots \ldots \ldots, x_n \neq 0$, $V(x_1, x_2, \ldots \ldots \ldots, x_n) > 0$, and

(*iv*) Its time derivative $\dot{V}(x_1, x_2, \ldots \ldots \ldots, x_n)$ following the motion is negative and semi-definite, i.e. the inequality $\dot{V}(x_1, x_2, \ldots \ldots \ldots, x_n) \leqslant 0$ holds for all $x_1, x_2, \ldots \ldots, x_n$ in the region.

If such a function could be formed for the motion, it is then easy to prove that the null solution is asymptotically stable for arbitrary initial perturbations.

THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

Case 1 :

Let us first consider third order non-linear differential equation

$$\dddot{\xi} + f(\ddot{\xi}) + g(\dot{\xi}) + a\,\xi = 0 \quad \left(\cdot = \frac{d}{dt}\right), \tag{1}$$

where the functions f, g depend only on the arguments explicitly displayed by them and it is assumed that the functions are continuous and satisfy Lipschitz' condition.

To investigate its stability, the differential equation (1) is differentiated with respect to t and is replaced by the system (2) as follows :

Put $\dot{\xi} = x_1$, and then

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= - f'(x_2)\, x_3 - g'(x_1)\, x_2 - a\, x_1 \end{aligned} \tag{2}$$

$$\left(f'(x_2) = \frac{\partial f}{\partial x_2} \, , \; g'(x_1) = \frac{\partial f}{\partial x_1} \right)$$

Routh's type conditions :—Let $a > 0$ and let $f(x_2)$, $g(x_1)$ be differentiable and are such that

$$\left.\begin{array}{ll} (i)\; f(0) = 0 \, , & \\ (ii)\; f'(x_2) \geqslant \delta_1 > 0 & \text{for all } x_2, \\ (iii)\; g'(x_1) \geqslant \delta_2 > 0 & \text{for all } x_1, \\ \text{where } \delta_1, \delta_2 \text{ satisfy the relation } \delta_1 \delta_2 - a > 0 \text{, and} & \\ (iv)\; g''(x_1) \text{ is continuous,} & \end{array}\right\} \tag{3}$$

then, the null solution of equation (1) is asymptotically stable for arbitrary initial perturbations, i.e. every solution $x_1(t)$ of (1) satisfies $x_1(t) \to 0$, $\dot{x}_1(t) \to 0$, $\ddot{x}_1(t) \to 0$ as $t \to \infty$ provided that

$$| \, g''(x_1)\, x_2 \, | < \delta_2 \, , \tag{4}$$

for all x_1, x_2 considered.

The condition for the asymptotic stability is the existence of a Liapunov function $V(x_1, x_2, x_3)$, positive definite in x_1, x_2, x_3 such that $-\dot{V}(x_1, x_2, x_3)$ along with the solution of (1) is positive semi-definite in x_1, x_2, x_3.

Since $(\delta_1 \delta_2 - a) > 0$, a number $\alpha > 0$ can be chosen such that

$$\frac{\delta_2}{a} > \alpha > \frac{1}{\delta_1} \tag{5}$$

Now consider the function

$$2\, V(x_1, x_2, x_3) = \alpha^{-1} (\alpha\, x_3 + x_2)^2 + a\, (\alpha\, x_2 + x_1)^2 + \alpha \Big[g'(x_1) - a\, \alpha \Big] x_2^2 + \int_0^{x_2} \left[f'(\eta) - \frac{1}{\alpha} \right] \eta \, d\eta \tag{6}$$

which is obviously positive definite by (3).

Hence

$$\begin{aligned} & V(0, 0, 0) = 0, \\ & V(x_1, x_2, x_3) > 0 \quad \text{for } (x_1^2 + x_2^2 + x_3^2) > 0, \\ & V(x_1, x_2, x_3) \to \infty \quad \text{as } (x_1^2 + x_2^2 + x_3^2) \to \infty. \end{aligned} \tag{7}$$

The time derivative of $V(x_1, x_2, x_3)$ is

$$- \dot{V}(x_1, x_2, x_3) = \alpha \left\{ f'(x_2) - \frac{1}{\alpha} \right\} x_3^2$$

$$+ \left\{ g'(x_1) - a\,\alpha - \frac{1}{2}\,\alpha\, g''(x_1) \times x_2 \right\} x_2^2. \tag{8}$$

Therefore $\dot{V}(x_1, x_2, x_3) < 0$

and $\dot{V}(x_1, x_2, x_3) = 0$ only when $x_2 = 0 = x_3$ (9)

Thus $V(x_1, x_2, x_3)$ given by (6) is the required Liapunov function.

Case 2:

Let us now consider equation

$$\dddot{\xi} + f(\dot{\xi})\,\ddot{\xi} + g(\dot{\xi}) + h(\xi) = 0 \tag{10}$$

or with the equivalent system

$$\left.\begin{aligned} \xi &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \end{aligned}\right\} \tag{11}$$

$$\dot{x}_3 = -f(x_2)\,x_3 - g(x_2) - h(x_1).$$

The function $f(x_2)$, $g(x_2)$, $h(x_1)$ are differentiable for all real x_1 ; x_2 ; $h'(x_1)$ is continuous for all x_1.

Routh's type conditions : Suppose

$$\left.\begin{array}{l} (i)\ g(0) = h(0) = 0 \\ (ii)\ f(x_2) \geqslant \delta_0 > 0 \quad \text{for all } x_2 \\ (iii)\ g(x_2)/x_2 \geqslant \delta_1 > 0 \quad (x_2 \neq 0) \\ \qquad h(x_1)/x_1 \geqslant \delta_2 > 0 \quad (x_1 \neq 0) \\ (iv)\ h'(x_1) \leqslant k \quad \text{for all } x_1, \text{ where } \delta_0\,\delta_1 - k > 0 \end{array}\right\} \tag{12}$$

then the null solution is asymptotically stable for arbitrary initial perturbations.

Again, choosing a number $\beta > 0$ such that $\dfrac{\delta_1}{k} > \beta > \dfrac{1}{\delta_0}$, the Liapunov function could be written as

$$2\,V(x_1, x_2, x_3) = \beta^{-1}(\beta\,x_3 + x_2)^2 + 2\,\beta\,x_2\,h(x_1) + 2\int_0^{x_2} \left\{ f(\eta) - \beta^{-1} \right\} \eta\, d\eta$$

$$+ 2\int_0^{x_2} g(\eta)\, d\eta + 2\int_0^{x_1} h(\zeta)\, d\zeta \tag{13}$$

which is positive definite ;

for,

$$2\,V(x_1, x_2, x_3) \geqslant \beta^{-1}(\beta x_3 + x_2)^2 + 2\int_0^{x_2}\left\{f(\eta) - \beta^{-1}\right\}\eta\, d\eta + \beta\,\delta_1^{-1}\left\{\delta_1 x_2 + h(x_1)\right\}^2$$

$$+ 2\int_0^{x_1}\left\{1 - \beta\,\delta_1^{-1} h'(\zeta)\right\} \times h(\zeta)\, d\zeta$$

$$\geqslant \beta^{-1}(\beta x_3 + x_2)^2 + (\delta_0 - \beta^{-1})\,x_2^2 + \beta\,\delta_1^{-1}\left\{\delta_1\; x_2 + h\;(x_1)\right\}^2$$

$$+ \delta_2\,(1 - \beta\, k\, \delta_1^{-1})\, x_1^2$$

Differentiation gives

$$-\dot V(x_1, x_2, x_3) = \left\{x_2\, g(x_2) - \beta\, h'(x_1)\right\} x_2^2 + \beta\left\{f(x_2) - \beta^{-1}\right\} x_3^2$$

$$-\dot V(x_1, x_2, x_3) \geqslant (\delta_1 - \beta k)\; x_2^2 + \beta\,(\delta_0 - \beta^{-1})\; x_3^2 \tag{14}$$

Thus $\quad \dot V(x_1, x_2, x_3) \leqslant 0$ (15)

Thus expression (13) gives the required Liapunov function. The rest of the proof is similar to that given in references 1 and 2.

Let $R_0\ (x_1^0, x_2^0, x_3^0)$ be an arbitrary point of the phase space of (2), from which let an arbitrary trajectory τ: $x_1 = x_1(t)$, $x_2 = x_2(t)$, $x_3 = x_3(t)$ be issuing. What is to be proved is to show that for $t > 0$ all trajectories of system (2) remain inside the bounded region defined by $V < V_0$. Which is true for,

$$V(t) \equiv V\left\{x_1(t), x_2(t), x_3(t)\right\} \leqslant V_0\,(x_1^0, x_2^0, x_3^0),\ t \geqslant 0 \tag{16}$$

Also, $V(t)$ is non-increasing and non-negative and tends to a non-negative limit $V(\infty)$, say as $t \to \infty$. If $V(\infty) = 0$, the required result follows.

If $\quad V(\infty) \neq 0$, let $\quad V(\infty) > 0$. (17)

Since the set of points (x_1, x_2, x_3) for which $V(x_1, x_2, x_3) \leqslant V_0\,(x_1^0, x_2^0, x_3^0)$ is bounded from (16) the surface $V(x_1, x_2, x_3) = V(\infty)$ contains all the limiting points of $\{x_1(t), x_2(t), x_3(t)\}$.

In particular, if R is a limiting point, evidently $\dot V \equiv 0$ on τ_R, the half-trajectory issuing from R. Hence $\dot V = 0$ at all points of τ_R, and this is only possible if $x_2 = 0$ and $x_3 = 0$ and hence $x_1 =$ constant for any (x_1, x_2, x_3) on τ_R, which contradicts equations (2) for case 1 and equations (11) for case 2.

FOURTH ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

Case 1 :

Let us consider the fourth order differential equation of the form

$$\overset{(4)}{\xi} + a_1 \dddot{\xi} + f(\ddot{\xi}) + a_3 \dot{\xi} + a_4 \xi = 0 \qquad (18)$$

Differentiating equation (18) with respect to t we have the system with $\xi = x_1$, as follows :

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -a_1 x_4 - f'(x_2)\, x_3 - a_3 x_2 - a_4 x_1 \end{aligned}\right\} \qquad (19)$$

with

$$\begin{aligned} 2\,V(x_1, x_2, x_3, x_4) &= a_1 a_3^2 \left(x_4 + a_1 x_3 + \frac{a_1 a_4}{a_3} x_2\right)^2 \\ &+ a_3^3 \left(x_3 + a_1 x_2 + \frac{a_1 a_4}{a_3} x_1\right)^2 + a_3 \left\{ a_1 a_3 f'(x_2) - a_3^2 - a_1 a_4 \right\} x_3^2 \\ &+ 2\, a_1^2 a_3 a_4 \int_0^{x_2} \eta f'(\eta)\, d\eta - a_1 a_4 (a_3^2 + a_1^2 a_4)\, x_2^2 \end{aligned} \qquad (20)$$

as a possibility for Liapunov function.

Routh's type conditions :

Let us suppose that

$$\left.\begin{array}{l} (i)\ a_1, a_3, a_4 \text{ are all positive. The function } f(x_2) \text{ is such that} \\ (ii)\ f'(x_2) \text{ is positive and } f''(x_2) \text{ is continuous} \\ (iii)\ A_4(x_2) \equiv a_1 a_3 f'(x_2) - a_3^2 - a_1^2 a_4 \geqslant \delta > 0 \end{array}\right\} \qquad (21)$$

for all x_2.

Then every solution of (19) tends to the trivial solution $x_4 = 0 = x_3 = x_2 = x_1$ of (19) as $t \to \infty$ provided that

$$| f''(x_2)\, x_3 | < \delta / a_3, \qquad (22)$$

for all x_2, x_3 considered.

Obviously $V(x_1, x_2, x_3, x_4)$ is positive definite; for

$$\begin{aligned} 2\,V(x_1, x_2, x_3, x_4) &\geqslant a_1 a_3^2 \left(x_4 + a_1 x_3 + \frac{a_1 a_4}{a_3} x_2\right)^2 \\ &+ a_3^3 \left(x_3 + a_1 x_2 + \frac{a_1 a_4}{a_3} x_1\right)^2 + a_3 \left\{ a_1 a_3 f'(x_2) - a_3^2 - a_1^2 a_4 \right\} x_3^2 \\ &+ 2\, a_1 a_4 \int_0^{x2} \left\{ a_1 a_3 f'(\eta) - a_3^2 - a_1^2 a_4 \right\} \eta\, d\eta \end{aligned}$$

and

$$-\dot{V}(x_1, x_2, x_3, x_4) = a_1 a_3 \left\{ A_4 (x_2 x_4) - \tfrac{1}{2} a_3 f''(x_2)\, x_3 \right\} x_3^2 \qquad (23)$$

which shows that $\dot{V}(x_1, x_2, x_3, x_4) < 0$, and $\dot{V}(x_1 x_2, x_3, x_4) = 0$, only when $x_3 = 0$

Hence the expression given by (20) is the required Liapunov function.

Case 2 :

Let us now consider the differential equation

$$\xi^{(4)} + f(\ddot{\xi})\dddot{\xi} + a_2 \ddot{\xi} + a_3 \dot{\xi} + a_4 \xi = 0 \tag{24}$$

or rather the system. $\xi = x_1$,

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -f(x_3)\, x_4 - a_2 x_3 - a_3 x_2 - a_4 x_1 \end{aligned}\right\} \tag{25}$$

and the Liapunov function

$$\begin{aligned} 2\,V(x_1, x_2, x_3, x_4) = {} & 2\left\{ x_4 + F(x_3) + \frac{a_2}{2} x_2 + \frac{a_3}{2} x_1 \right\}^2 \\ & + a_2 \left(x_3 + \frac{a_3}{a_2} x_2 + \frac{2\,a_4}{a_2} x_1\right)^2 + \tfrac{1}{2}\left(a_3 x_1 + \frac{a_2^2 - 4\,a_4}{a_2} x_2\right)^2 \\ & + \frac{a_3}{a_2}\left\{ f(x_3)\, a_2 - a_3 \right\} x_2^2 + \frac{2\,a_4}{a_2^2}(a_2^2 - 4\,a_4)^2\, x_2 \\ & + \frac{a_4}{a_2}(a_2^2 - 4\,a_4)\, x_1^2 \end{aligned} \tag{26}$$

Routh's type conditions :

$$\left.\begin{array}{l} (i)\ a_2,\ a_3,\ a_4 \text{ are all positive. The function } f(x_3) \text{ is such that} \\ (ii)\ f(0) = 0, \\ (iii)\ f(x_3) \geq \delta > 0,\ \ x_4 f'(x_3) \mid \leq \delta_1 < 0 \\ (iv)\ f(x_3)\, a_2\, a_3 - a_3^2 - a_4\, f^2(x_3) > 0 \\ \\ F(x_3) = \displaystyle\int_0^{x_3} f(\zeta)\, d\zeta \end{array}\right\} \tag{27}$$

condition (iv) implies two obvious conditions

$$\left.\begin{array}{l} (a)\ \ \left\{ f(x_3) - \dfrac{a_3}{a_2} \right\} > 0 \\ (b)\ \ (a_2^2 - 4\,a_4) > 0 \end{array}\right\} \tag{28}$$

Under these conditions, (26) is positive definite whose time derivative following (25) is

$$\begin{aligned} -\dot{V}(x_1, x_2, x_3, x_4) = {} & a_3\, a_4\, x_1^2 + a_2 \left\{ f(x_3) - \frac{a_3}{a_2} \right\} x_3^2 \\ & + 2\, a_4 f(x_3) \times x_1 x_3 - a_3\, x_2^2 f(x_3) \end{aligned} \tag{29}$$

which shows that $\dot{V}(x_1, x_2, x_3, x_4) \leqslant 0$ or expression (26) gives the required Liapunov function.

EXAMPLES

Let us consider a few illustrative examples :

Example 1 :

We first consider a third order non-linear control system as represented[8] in Fig. 1. The dynamic behaviour of such a system is governed by the differential equation :

$$\dddot{\xi} + \left(\frac{1}{T_m} + \frac{1}{T}\right)\ddot{\xi} + \frac{1}{T_m T}\dot{\xi} + F(\dot{\xi}) + \left(\frac{1}{T_m} + \frac{1}{T}\right)\frac{1}{T_m T}\xi = 0 \quad (30)$$

When $F(\dot{\xi}) = 0$, the system is unstable according to Routh's stability criterion. Now we proceed to find a suitable value for $F(\dot{\xi})$ so that the system becomes stable.

Applying the conditions (3) to (30) we get

(*i*) $f(0) = 0$

(*ii*) $f(\ddot{\xi}) = \dfrac{1}{T_m} + \dfrac{1}{T} = \delta_1 > 0$

(*iii*) $g'(\dot{\xi}) = \dfrac{1}{T_m T} + F'(\dot{\xi}) = \delta_2 > 0$

and the condition

$$\left(\frac{1}{T_m} + \frac{1}{T}\right)\left(\frac{1}{T_m T} + F'(\dot{\xi})\right) - \frac{1}{T_m T} > 0$$

implies that $F'(\dot{\xi}) > 0$ for all $\dot{\xi}$

(*iv*) $g''(\dot{\xi}) = F''(\dot{\xi})$ is continuous.

Thus the function $F(\dot{\xi})$ must be such that $F'(\dot{\xi}) > 0$ for all $\dot{\xi}$ and $F''(\dot{\xi})$ is continuous and bounded. A possible form of $F(\dot{\xi})$ is therefore given by $F(\dot{\xi}) = k\,\dot{\xi}^3$, k being a positive constant.

Example 2 :

The second system considered is shown[9] in Fig. 2.

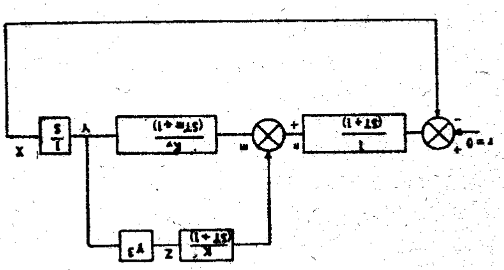

Fig. 1—Third order non-linear control first system.

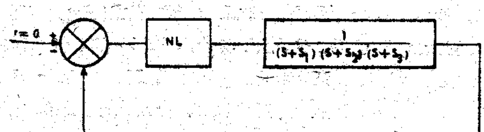

Fig. 2—Third order non-linear control second system

This is governed by the differential equation

$$\dddot{\xi} + (s_1 + s_2 + s_3)\,\ddot{\xi} + (s_1 s_2 + s_2 s_3 + s_3 s_1)\,\dot{\xi} + s_1 s_2 s_3\,\xi + F(\xi) = 0 \quad (31)$$

Where $F(\xi)$ is a non-linear gain function. The system is asymptotically stable for any single valued continuous non-linearity satisfying

$$k_1 < \frac{F(\xi)}{\xi} < k_2 \quad (32)$$

where

$$k_1 = - s_1 s_2 s_3 ,$$
$$k_2 = (s_1 + s_2 + s_3)(s_1 s_2 + s_2 s_3 + s_3 s_1) - s_1 s_2 s_3$$

Application of the conditions (12) to the above system gives

$$g(\dot{\xi}) = (s_1 s_2 + s_2 s_3 + s_3 s_1)\,\dot{\xi}$$
$$h(\xi) = s_1 s_2 s_3\,\xi + F(\xi)$$
$$f(\dot{\xi}) = s_1 + s_2 + s_3$$

with

(i) $g(0) = h(0) = 0$

(ii) $f(\dot{\xi}) = s_1 + s_2 + s_3 = \delta_0 > 0$ for all $\dot{\xi}$

(iii) $g(\dot{\xi})/\dot{\xi} = s_1 s_2 + s_2 s_3 + s_3 s_1 = \delta_1 > 0$ for $\dot{\xi} \neq 0$

$h(\xi)/\xi = s_1 s_2 s_3 + F(\xi)/\xi = \delta_2 > 0$ for $\xi \neq 0$

(iv) $h'(\xi) = s_1 s_2 s_3 + F'(\xi)$.

Now $h(\xi)/\xi > 0$ implies that $F(\xi)/\xi > - s_1 s_2 s_3$ and the condition $(\delta_0 \delta_1 - k) > 0$ implies that

$$\{ (s_1 + s_2 + s_3)(s_1 s_2 + s_2 s_3 + s_3 s_1) - F'(\dot{\xi}) - s_1 s_2 s_3 \} > 0$$
$$\{ (s_1 + s_2 + s_3)(s_1 s_2 + s_2 s_3 + s_3 s_1) - s_1 s_2 s_3 \} > F'(\xi).$$

If $F(\xi)$ is such that $F'(\xi) > F(\xi)/\xi$ then $k_2 > F(\xi)/\xi > k_1$ is satisfied. Again, one can choose $F(\xi)$ to be simply $k\,\xi^3$, $k > 0$ and constant.

ACKNOWLEDGEMENTS

The author is grateful to Dr. R. R. Aggarwal for his keen interest in the work, to Dr. C. D. Ghildyal for helpful suggestions and to the Director, Defence Science Laboratory, Delhi for permission to publish this work.

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