KINEMATIC PROPERTIES OF PSEUDOSTATIONARY HYDROMAGNETIC FLUID FLOWS

G. PURUSHOTHAM & D. N. MURTHY

Govt. Nagarjunasagar Engineering College, Hyderabad

By assigning unidirection to the magnetic field and using geometrical configuration of spatial curves of congruences formed by the streaklines, principal normals and their binormals, kinetic and kinematic properties of pseudostationary hydromagnetic fluid flows are established. It has been observed that the surfaces and the momentum simultaneously cannot be minimal and conserved along a streakline respectively. Also the hydromagnetic pressure remains uniform along an individual binormal to a streakline and decreases along the latter, if the velocity remains uniform. The elegant expressions for vorticity components are obtained and the compatibility conditions to be satisfied by the velocity of the streakline are derived.

BASIC EQUATIONS

This paper is concerned with an electrically conducting, ideal, compressible fluid, devoid of viscosity and thermal conductivity. It is assumed further that the mean electric charge is zero, so that the medium is essentially neutral, that the displacement currents may be neglected and that the electrical conductivity of the fluid is infinite. Under these assumptions, the basic equations governing the fluid flow can be written[1] as

$$\frac{\partial \rho}{\partial t} + div\,(\rho \vec{q}\,) = 0 \tag{1}$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}.\nabla)\,\vec{q} = -\frac{1}{\rho}\nabla p + \frac{\mu_e}{\rho}\vec{J}\wedge\vec{H} \tag{2}$$

$$\frac{\partial \vec{H}}{\partial t} + curl\,(\vec{H}\wedge\vec{q}) = 0 \tag{3}$$

$$\frac{dS}{dt} = 0 \tag{4}$$

$$p = \rho^{\gamma}\, e^{S/Je_v} \tag{5}$$

$$div\,\vec{H} = 0 \tag{6}$$

where t denotes the time, p the fluid pressure, ρ the density, $\vec{q}$ the velocity vector, $\vec{H}$ the magnetic field vector, S the entropy, γ the adiabatic exponent, μ_e the magnetic permeability, and J the Joule constant respectively.

Equation (1) is the continuity equation of fluid dynamics, relations (2) are the equations of motion, the last two terms on the left hand side of (2) together represents the Lorentz force per unit mass. The last two Maxwell equations which do not involve current and charge have been written as (3) and (6). Equations (4) and (5) express the energy and the state relations to be satisfied by flow.

PSEUDOSTATIONARY TRANSFORMATION

Considering the co-ordinate system $\vec{r} = i\,x + j\,y + k\,z$ where $\vec{r}$ is the position vector in space, referring to triply orthogonal cartesian co-ordinate system, which can be transformed into new system as $\vec{R} = i\,\frac{x}{t} + j\,\frac{y}{t} + k\,\frac{z}{t}$ (where t is the time). Further we consider the flow quantities as homogeneous functions of zero degree in (x, y, z, t).

By Euler's theorem for homogeneous functions we have

$$\frac{\partial f}{\partial t} = -\left(\frac{\vec{R}}{t} \cdot \nabla'\right) f \tag{7}$$

where ∇' is the gradient operator in new co-ordinate system.

Also differentiating the position vector $\vec{r}$ with respect to time t we get

$$\frac{d\,\vec{r}}{dt} = \vec{q} = \vec{R} + t\,\frac{d\vec{R}}{dt} = \vec{R} + \vec{Q} \tag{8}$$

where $\vec{Q}$ is the velocity vector of the streakline flow.

The basic equations (1) to (6) in pseudostationary vector field $\vec{Q}$ simplify to

$$3\rho + \nabla' . (\rho\,\vec{Q}) = 0 \tag{9}$$

$$\vec{Q} + (\vec{Q} . \nabla')\,\vec{Q} = -\frac{1}{\rho}\,\nabla' p + \frac{\mu_e}{\rho}\vec{J} \wedge \vec{H} \tag{10}$$

$$2\vec{H} + curl\ (\vec{H} \wedge \vec{Q}) = 0 \tag{11}$$

$$\vec{Q} . \nabla' S = 0 \tag{12}$$

$$\nabla' . \vec{H} = 0 \tag{13}$$

Introducing the velocity of sound 'c' and eliminating pressure 'p' from (10) we obtain

$$\vec{Q} + (\vec{Q} . \nabla')\,\vec{Q} + \frac{2c}{\gamma - 1}\,\nabla' c = \frac{c^2}{J c_p (\gamma - 1)}\nabla' S + \frac{\mu_e}{\rho}\,curl'\,\vec{H} \wedge \vec{H} \tag{14}$$

This can further be written as

$$\vec{Q} + curl' \vec{Q} \wedge \vec{Q} + \nabla'\left(\frac{Q^2}{2} + \frac{c^2}{\gamma - 1} + \frac{\mu_e}{2\rho} H^2\right) + \frac{\mu_e H^2}{\rho(\gamma - 1)} \nabla' \log c =$$

$$\frac{1}{Jc_v(\gamma - 1)} \left(\frac{c^2}{\gamma} + \frac{\mu_e H^2}{2\rho}\right) \nabla' S + \frac{\mu_e}{\rho} \left(\vec{H} . \nabla'\right) \vec{H} \qquad (15)$$

DECOMPOSITION IN INTRINSIC FORM

In this section it is proposed to transform the basic equations into intrinsic form and study some of the kinetic and kinematic properties of hydromagnetic fluid flows.

Using the geometric results established earlier[2] we can decompose the continuity equation (9) in intrinsic form as

$$\frac{3\rho + \frac{d}{ds}\left(\rho Q\right)}{\rho Q} = K' + K'' = -\mu \qquad (16)$$

where $\mu = -(K' + K'')$ is the mean curvature; this vanishes when the surfaces are minimal[5]. Also it is observed that the surfaces and the flux per unit mass simultaneously cannot be minimal and conserved along a streakline which is independent of the magnetic field.

Assigning constant unidirection $\vec{h}$ to the magnetic field $\vec{H}$, the conservation of magnetic field can be written as

$$\frac{dH}{dh} = 0 \qquad (17)$$

This can be expressed as

$$\cos \alpha \frac{dH}{dS} + \cos \beta \frac{dH}{dn} + \cos \gamma \frac{dH}{db} = 0 \qquad (18)$$

where $\frac{d}{dh}, \frac{d}{ds}, \frac{d}{dn}, \frac{d}{db}$ are directional derivatives along the magnetic lines of force and the streaklines, principal normals and binormals respectively. Also α, β, γ, are the inclinations of the magnetic field with the streaklines, principal normals and binormals respectively. Also α, β and γ are the inclinations of the magnetic field with the streaklines, principal normals and binormals respectively.

Introducing the magnetic pressure $P = p + \mu_e \frac{H^2}{2}$ the momentum equation (10) can be decomposed in intrinsic form as

$$Q\left(1 + \frac{dQ}{ds}\right) = -\frac{1}{\rho} \frac{dP}{ds} \qquad (19)$$

$$K Q^2 = -\frac{1}{\rho} \frac{dP}{dn} \qquad (20)$$

$$\frac{dP}{db} = 0 \qquad (21)$$

where 'K' is the curvature of the streakline.

From (19) we observe that the hydromagnetic pressure and the velocity of the streakline cannot be uniform simultaneously along an individual streakline. From (20) it is evident that the hydromagnetic pressure remains uniform along normals to streaklines, if the streaklines are straight and vice versa.

Field equation (11) can be decomposed into intrinsic form as

$$2 H \cos \alpha + Q \cos \alpha \frac{dH}{ds} - HQ \cos \alpha (K' + K'')$$

$$- H \left(\cos \beta \frac{dQ}{dn} + \cos \gamma \frac{dQ}{db} \right) = 0 \tag{22}$$

$$\cos \beta \frac{d}{ds} (HQ) + 2 H \cos \beta - HQ (K'' \cos \beta + K \cos \alpha + \sigma'' \cos \gamma) = 0 \tag{23}$$

$$\cos \gamma \frac{d}{ds} (HQ) + 2 H \cos \gamma - HQ (\sigma' \cos \beta + K' \cos \gamma) = 0 \tag{24}$$

where (K, K', K''), $(\tau, \sigma', \sigma'')$ are the curvatures and torsions of the streaklines, the principal normals, the binormals respectively.

Eliminating HQ from (23) and (24) we obtain

$$\cos \beta \cos \gamma (K' - K'') + \sigma' \cos^2 \beta = \sigma'' \cos^2 \gamma + K \cos \gamma \cos \alpha \tag{25}$$

which is independent of the field and the velocity of a streakline.

Forming the scalar products of (15) by $\vec{t}, \vec{n}$, and $\vec{b}$ successively we obtain the following equations :

$$Q + \frac{dX}{ds} + \frac{\mu_e H^2}{\rho (\gamma - 1)} \frac{d}{ds} \log c = 0 \tag{26}$$

$$Q\zeta_b + \frac{dX}{dn} + \frac{\mu_e H^2}{\rho (\gamma - 1)} \frac{d}{dn} \log c = \frac{1}{Jc_v (\gamma - 1)} \left(\frac{c^2}{\gamma} + \frac{\mu_e}{2\rho} H^2 \right) \frac{dS}{dn} \tag{27}$$

$$- Q\zeta_n + \frac{dX}{db} + \frac{\mu_e H^2}{\rho (\gamma - 1)} \frac{d}{db} \log c = \frac{1}{Jc_v (\gamma - 1)} \left(\frac{c^2}{\gamma} + \frac{\mu_e}{2\rho} H^2 \right) \frac{dS}{db} \tag{28}$$

where $X = \frac{\mu_e H^2}{2\rho} + \frac{Q^2}{2} + \frac{c^2}{\gamma - 1}$ and ζ_n, ζ_b are the resolved parts of the vorticity along principal normals and binormals respectively.

These give us the momentum equations along a streakline, principal normal and binormal respectively.

COMPATIBILITY CONDITIONS

The necessary conditions to be satisfied by a velocity vector of the streakline, hydromagnetic flow in which the magnetic field has assigned unidirection are obtained in this section. These have been transformed into intrinsic form and the kinetic and kinematic properties of flows studied.

The momentum equation (2) can be written as

$$\dot{\vec{Q}} + (\vec{Q} \cdot \nabla')\vec{Q} = -\frac{1}{d} \nabla' P \tag{29}$$

Denoting left hand side expression by $\vec{A}$ and operating curl′ on (29) and then forming the scalar product by $\vec{A}$ we obtain

$$\vec{A} \cdot curl\ \vec{A} = 0 \tag{30}$$

This shows that the acceleration field is complex-lamellar, and the converse is also true.

Operating curl on (29) and forming the vector product by $\vec{Q}$ and using (9) we obtain

$$curl \left\{ \frac{\vec{Q} \wedge curl\ \vec{A} + (3 + \nabla' \cdot \vec{Q})\vec{A}}{\vec{Q} \cdot \vec{A}} \right\} = 0 \tag{31}$$

The conditions (30) and (31) are to be satisfied by a velocity vector of a streakline flow in which the magnetic line has assigned unidirection, which are independent of the magnetic field.

The condition (30) in intrinsic form simplifies to

$$\frac{dQ}{ds} \left\{ (\sigma' - \sigma'') \left(2 + \frac{dQ}{ds}\right) - \frac{d}{db}(KQ) \right\} + (\sigma' - \sigma'')$$
$$+ \frac{dQ}{dn}(\tau + \sigma'')(1 - Q)\left(\frac{dQ}{dn} - 2KQ\right) - \frac{d}{db}(KQ)$$
$$+ KQ \frac{d}{db}\left(\frac{dQ}{ds}\right) - K^2 Q^3 (\tau + \sigma'') = 0 \tag{32}$$

This is the condition that the acceleration field to be complex lamellar.

The condition (31) yields

$$(\sigma' - \sigma'') \left\{ \frac{3}{Q} + \frac{d}{ds} \log Q - (K' + K'') \right\}$$
$$- \left[\frac{d^2}{dbdn} \log\left(Q + Q\frac{dQ}{ds}\right) + \frac{d}{db}\left\{ \frac{2KQ - KK''Q^2 - \frac{d}{ds}(KQ^2)}{Q\left(1 + \frac{dQ}{ds}\right)} \right\} \right]$$
$$+ \left[\frac{d^2}{dndb} \log\left(Q + Q\frac{dQ}{ds}\right) - \frac{d}{dn}\left\{ \frac{KQ(\tau + \sigma'')}{1 + \frac{dQ}{ds}} \right\} \right] = 0 \tag{33}$$

$$\frac{d^2}{dbds}\log Q - \frac{d}{db}\left(K' + K''\right) - \frac{3}{Q^2}\frac{dQ}{db}$$

$$-(\tau + \sigma'')\left\{\frac{d}{dn}\log\left(Q + Q\frac{dQ}{ds}\right) + \frac{2KQ - KK''Q^2 - \frac{d}{ds}(KQ^2)}{Q\left(1 + \frac{dQ}{ds}\right)}\right\}$$

$$+ K''\left\{\frac{d}{db}\log\left(Q + Q\frac{dQ}{ds}\right) - \frac{KQ(\tau + \sigma'')}{1 + \frac{dQ}{dS}}\right\}$$

$$-\frac{d^2}{dsdb}\log\left(Q + Q\frac{dQ}{ds}\right) + \frac{d}{ds}\left\{\frac{KQ(\tau + \sigma'')}{1 + \frac{dQ}{ds}}\right\} = 0 \qquad (34)$$

$$\frac{3K}{Q} + K\frac{b}{ds}\log Q - K(K' + K'') + \frac{3}{Q^2}\frac{dQ}{dn} + \frac{d}{dn}(K' + K'')$$

$$-\frac{d^2}{dnds}\log Q - K'\frac{d}{dn}\log\left(Q + Q\frac{dQ}{ds}\right)$$

$$-\frac{K'}{Q\left(1 + \frac{dQ}{ds}\right)}\left\{2KQ - KK''Q^2 - \frac{d}{ds}\log(KQ^2)\right\}$$

$$-\frac{d^2}{dsdn}\log\left(Q + Q\frac{dQ}{ds}\right) - \frac{d}{ds}\left\{\frac{2KQ - KK''Q^2 - \frac{d}{ds}(KQ^2)}{Q\left(1 + \frac{dQ}{ds}\right)}\right\}$$

$$+(\sigma' - \tau)\left\{\frac{d}{db}\log\left(Q + Q\frac{dQ}{ds}\right) - \frac{KQ(\tau + \sigma'')}{1 + \frac{dQ}{ds}}\right\} = 0 \qquad (35)$$

These conditions also hold for non-magnetic field as well[4].

From this investigation the results for non-magnetic pseudostationary flows considered by Purushotham & Sadiqa[3] can be deduced as a special case.

REFERENCES

1. PAI, S.I., Magnetogasdynamics and Plasma Dynamics (Prentice Hall, New York), 1962, 36.
2. PURUSHOTHAM, G., *App. Sci. Res.*, 15A (1965), 23.
3. PURUSHOTHAM, G. & SADIQA FAROOQUI, *Viswakarma*, 8 (1967), 20.
4. PURUSHOTHAM, G. & MURTHY, D.N., *Tensor Society*, Japan), (1969), 20.
5. WEATHERBURN, C.E., Differential Geometry of Three Dimensions. (Oxford Univ. Press Ltd., Oxford), I (1955), 15.