SHEAR FLOW OF AN ELASTICO-VISCOUS COMPRESSIBLE FLUID PAST A POROUS FLAT PLATE

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Shear flow of an elastico-viscous compressible liquid past a porous flat plate has been studied and a perturbed solution has been obtained, assuming the elastic number to be small. The variations in the axial velocity have been investigated through graphs. The skin friction and the rate of heat transfer at the plate are found to be not affected by the elasticity of the liquid. The displacement thickness and the normal stress difference have also been studied.

The problem of shear flow of a viscous compressible fluid past a porous flat plate has been studied by Young[1]. Gupta[2] has extended the same problem to an Oldroyd type elastico-viscous compressible fluid possessing a single relaxation time parameter.

In the present paper, a perturbed solution has been obtained for the same flow problem taking Walters liquid B'' which possesses a very short memory and is governed by the constitutive equations[3] :

$$p'^{ik} = 2\,\eta_0\, e^{(1)ik} - 2\,k_0 \frac{\delta}{\delta t}\, e^{(1)ik} \tag{1}$$

where $\eta_0 = \left[\int_0^\infty N(\tau)\, d\tau\right]$ is the limiting viscosity at small rates of shear, $k_0\left[= \int_0^\infty \tau N(\tau)\, d\tau\right]$ is the short memory coefficient; $N(\tau)$ being the relaxation spectrum as introduced by Walters[4]. p'^{ik} is the partial stress tensor and the rate of strain rate tensor $\frac{\delta}{\delta t} e^{(1)ik}$ is defined as

$$\frac{\delta}{\delta t}\, e^{(1)ik} \equiv \frac{\partial}{\partial t}\, e^{(1)ik} + v^j\, e^{(1)ik}{}_{,j} - v^k{}_{,j}\, e^{(1)ij} - v^i{}_{,j}\, e^{(1)jk} \tag{2}$$

where the strain rate tensor $e^{(1)ik}$ is given by

$$\overset{(1)}{e}_{ik} = \frac{1}{2}\left[v_{i,k} + v_{ki} - \frac{2}{3}\, vp, p\, \delta\, ik\right] \tag{3}$$

This liquid is a valid approximation of Walters[4] liquid B', taking very short memory into account so that terms involving $\int_0^\infty \tau^n\ N(\tau)\, d\,\tau\ (n \geqslant 2)$ have been dropped out.

Implicit in the derivation of (1) is also the fact that second order terms in k_0 have been neglected.

FORMULATION AND SOLUTION OF THE PROBLEM

Taking x-axis along the plate and y-axis normal to it, the equations of momentum, continuity, energy (neglecting elastico-viscous dissipation) and state respectively are

$$\rho\left\{ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right\} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\lambda \triangle\right) + \frac{\partial p'(xx)}{\partial x} + \frac{\partial p'(xy)}{\partial y} \tag{4}$$

$$\rho\left\{ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right\} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left(\lambda \triangle\right) + \frac{\partial p'(yx)}{\partial x} + \frac{\partial p'(yy)}{\partial y} \tag{5}$$

$$\frac{\partial}{\partial x}\left(\rho u\right) + \frac{\partial}{\partial y}\left(\rho v\right) = 0 \tag{6}$$

$$\rho C_p \left\{ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right\} = u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\lambda_1 \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda_1 \frac{\partial T}{\partial y}\right) \tag{7}$$

and

$$p = R \rho T^{\dagger} \tag{8}$$

where p is the pressure, $\triangle = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ the dilatation, λ the coefficient of bulk viscosity, C_p the specific heat at constant pressure and λ_1 is the thermal conductivity.

As the plate is infinite, it can be assumed that all the physical variables are functions of y alone. Then from equation (6) we get

$$\rho v = \text{const.} = -k \text{ (say)} \tag{9}$$

Using equations (1), (9) and the fact that the flow conditions depend on y alone, equations (4), (5) and (7) reduce to

$$-k \frac{du}{dy} = \frac{d}{dy}\left[\eta_0 \frac{du}{dy} - k_0 \left\{ v \frac{d^2u}{dy^2} - \frac{7}{3} \frac{dv}{dy} \cdot \frac{du}{dy} \right\} \right] \tag{10}$$

$$-k \frac{dv}{dy} = -\frac{dp}{dy} + \frac{d}{dy}\left(\lambda \frac{dv}{dy}\right) + \frac{d}{dy}\left[\frac{4}{3} \eta_0 \frac{dv}{dy} - \frac{4}{3} k_0 \left\{ v \frac{d^2v}{dy^2} - 2\left(\frac{dv}{dy}\right)^2 \right\} \right] \tag{11}$$

and

$$-k C_p \frac{dT}{dy} = v \frac{dp}{dy} + \frac{d}{dy}\left(\lambda_1 \frac{dT}{dy} \right) \tag{12}$$

†From the kinetic theory, it is well known that with a deviation from the Newtonian stress-strain relationship the physical equations also change. However, as a first approximation, the elasticity effect in the physical equations have been neglected.

Following Young, we make the usual boundary layer approximation, viz.

$$\left.\begin{aligned} &u = 0\,(1), \quad v = 0\,(\delta), \quad y = 0\,(\delta), \quad x = 0\,(1) \\ &\lambda = 0\,(\delta^2), \quad \eta_0 = 0\,(\delta^2) \text{ and } \rho = 0\,(1) \end{aligned}\right\} \tag{13}$$

where δ is the thickness of the boundary layer.

The above approximations give $k = 0(\delta)$ and then the left hand side of equation (10) becomes of order (1). Hence in case elasticity of the fluid is to contribute we find that the order of k_0 is (δ^2).

With the above complete boundary layer approximations equation (11) gives

$$\frac{dp}{dy} = 0\,(\delta) \tag{14}$$

Taking δ to be small equation (14) shows that pressure can be assumed to be constant within the boundary layer region. Hence

$$p = \text{const.} = p_\infty \text{ (say)} \tag{15}$$

The energy equation (12) becomes

$$-k\,C_p \frac{dT}{dy} = \lambda_1 \frac{d^2 T}{dy^2} \tag{16}$$

The equation of state (8) and (9) give

$$\frac{\rho}{\rho_\infty} = \frac{T_\infty}{T} = \frac{V_\infty}{v} \tag{17}$$

which shows that v is proportional to T. Hence we may take

$$v = \alpha\, T \tag{18}$$

where α is the constant of proportionality.

From the conditions at the plate it is found that

$$\alpha = -\frac{V}{T_\omega} \tag{19}$$

and then

$$k = \frac{V\, T_\infty\, \rho_\infty}{T_\omega} \tag{20}$$

Now, introducing the non-dimensional transformations

$$\bar{v} = \frac{v}{U}, \quad \bar{u} = \frac{u}{U}, \quad y = \frac{\eta_0}{\rho_\infty}\,\eta, \; R_s = \frac{V}{U} \text{ (suction parameter)},$$

$$k^* = \frac{k_0\, U^2\, \rho_\infty}{\eta_0} \text{ (elastic number)}$$

$$\theta = \frac{T - T_\infty}{T_\omega - T_\infty}, \quad \theta_0 = \frac{T_\omega}{T_\infty}, \text{ and } P_r = \frac{\eta_0\, C_p}{\lambda_1} \text{ (Prandtl number)} \tag{21}$$

Equations (16), (18) and (20) respectively reduce to

$$\frac{d^2\theta}{d\eta^2} + R_s\, P_r \theta_0 \frac{d\theta}{d\eta} = 0 \tag{22}$$

$$\bar{v} = -R_s \left[\theta_0 + \left(1 + \theta_0 \right) \theta \right] \tag{23}$$

and

$$-R_s \frac{d\bar{u}}{d\eta} = \frac{d}{d\eta}\left[\frac{d\bar{u}}{d\eta} - k^* \left\{ \bar{v}\, \frac{d^2\bar{u}}{d\eta^2} - \frac{7}{3}\, \frac{d\bar{v}}{d\eta} \cdot \frac{d\bar{u}}{d\eta} \right\} \right] \tag{24}$$

The modified boundary conditions for θ and $\bar{u}$ become

$$\theta = 1 \quad at\ \eta = 0; \quad \theta \to 0 \quad as\ \eta \to \infty \tag{25}$$

and

$$\bar{u} = 0 \ at\ \eta = 0 \,;\ u \to 1\ as\ \eta \to \infty \tag{26}$$

The solution of equation (22) subject to the boundary conditions (25) is obtained as

$$\theta = e^{-R_s\, \theta_0\, P_r \eta} \tag{27}$$

Hence

$$\bar{v} = -R_s \left\{ \theta_0 + \left(1 - \theta_0 \right) e^{-R_s\, \theta_0\, P_r \eta} \right\} \tag{28}$$

Then equation (24) becomes

$$-R_s\, \theta_0 \frac{d\bar{u}}{d\eta} = \frac{d}{d\eta}\left[\frac{d\bar{u}}{d\eta} + k^*\, R_s \left\{ < \theta_0 + \left(1 - \theta_0 \right) e^{-R_s\, \theta_0\, P_r \eta} \right. \right.$$

$$\left. \left. > \frac{d^2\bar{u}}{d\eta^2} + \frac{7}{3}\, R_s\, P_r\, \theta_0 \left(1 - \theta_0 \right) e^{-R_s\, \theta_0 P_r \eta}\, \frac{d\bar{u}}{d\eta} \right\} \right] \tag{29}$$

Integrating once the above equation, we get

$$B - R_s\, \theta_0\, \bar{u} = \left[\frac{d\bar{u}}{d\eta} + k^* R_s \left\{ < \theta_0 + \left(1 - \theta_0\right) e^{-R_s\, \theta_0\, P_r\, \eta} > \frac{d^2\bar{u}}{d\eta^2} \right.\right.$$

$$\left.\left. + \frac{7}{3} R_s P_r\, \theta_0 \left(1 - \theta_0\right) e^{-R_s\, \theta_0\, P_r\, \eta} \frac{d\bar{u}}{d\eta} \right\}\right] \tag{30}$$

This second degree ordinary differential equation does not admit an exact solution. Now, as implicit in the derivation of the constitutive equations (1) that squares and higher powers of k_0 are neglected, we may take a perturbed solution of the form

$$\bar{u} = u_0 + k^* u_1 + 0\,(k^{*2}) \tag{31}$$

with

$$B = B_0 + k^* B_1 + 0\,(k^{*2}) \tag{32}$$

The boundary conditions on u_0 and u_1 become

$$u_0 = 0 \text{ at } \eta = 0 \quad ; \quad u_0 \to 1 \text{ as } \eta \to \infty \tag{33}$$

and

$$u_1 = 0 \text{ at } \eta = 0 \quad ; \quad u_1 \to 0 \text{ as } \eta \to \infty \tag{34}$$

Substituting from (31) and (32) in (30) and solving the resulting equations for u_0 and u_1 subject to the above-mentioned conditions, we get

$$\bar{u} = 1 - e^{-R_s\, \theta_0\, \eta} + k^* R_s\, \theta_0^2 \left[\theta_0\, \eta\, e^{-R_s\, \theta_0\, \eta} + \frac{(1 - \theta_0)}{R_s \theta_0} \cdot \frac{(3 - 7P_r)}{3P_r} \right.$$

$$\left. \left\{ e^{-R_s\, \theta_0\, \eta} - e^{-R_s\, \theta_0\, (1 + P_r)\eta} \right\} \right] + 0\,(k^{*2}) \tag{35}$$

In the incompressible case, however, $\bar{u}$ and θ are obtained as[5]

$$\bar{u} = \left(1 - e^{-m_1 n}\right) \tag{36}$$

and

$$\theta = e^{-R_s\, P_r\, \eta} \tag{37}$$

where

$$m_1 = \frac{(1 - \sqrt{1 - 4\, k^* R_s^2})}{2\, k^* R_s} \tag{38}$$

For small value of k_0 equation (36) gives

$$\bar{u} = 1 - e^{-R_s \eta} + k^* R_s^3 \eta e^{-R_s \eta} + 0\left(k^{*2}\right) \tag{39}$$

DISCUSSION

(1) *Study of the velocity profile*

For rough illustration, Fig. 1 is drawn, keeping $\theta_0 = \frac{1}{2}$ and $P_r = 2$ and varying R_s and η for $k^* = 0 \cdot 0$ and $0 \cdot 01$. It shows that for small values of η the elastico-viscous values of $\bar{u}$ are less than those in the corresponding viscous case for all values of R_s. However, the difference between the elastico-viscous values and the viscous values decreases with an increase in the transverse distance η till it vanishes; the point where the elastico-viscous and the viscous values become equal, shifts towards the plate with an increase in R_s.

Fig. 2 has been drawn for the corresponding incompressible case. It shows that the elastico-viscous values of $\bar{u}$ in the incompressible case are always greater than the corresponding viscous case values.

(2) *Point of separation*

No point of separation occurs in the viscous case. However, a point of separation exists in the elastico-viscous case for

$$k^* = \frac{3}{\theta_0 \, [7 \, P_r (1 - \theta_0) - 3]} \tag{40}$$

(3) *Shear stress and the skin friction*

From the momentum equation (4) an explicit expression for the shear stress in the dimensionless form is obtained as

$$\frac{p(xy)}{\rho_\infty \, U^2} = \theta_0 \, R_s (1 - \bar{u}) \tag{41}$$

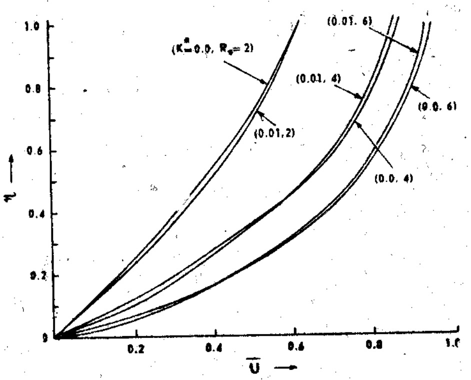

Fig. 1—Variations of $\bar{u}$ *vs* η with R_s increasing in the compressible case ($\theta_0 = \frac{1}{2}$, $Pr = 2$)

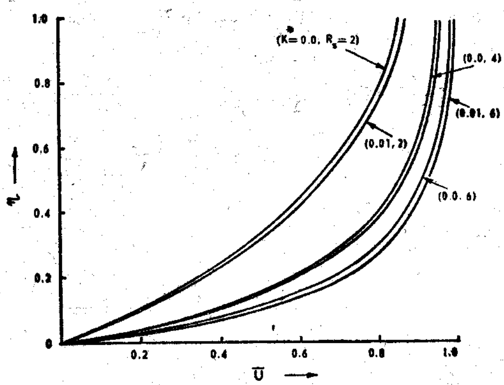

Fig. 2—Variations of $\bar{u}$ *vs* η with R_s increasing in the incompressible case ($\theta_0 = \frac{1}{2}$, $Pr = 2$)

Hence the coefficient of skin friction at the plate becomes

$$\tau_0 = \frac{p\ (xy)}{\rho_\infty\ U^2}\bigg|_{\eta = 0} = \theta_0\ R_s \tag{42}$$

which is independent of k^*, the elastic number. This result contradicts the findings of Gupta[2] who seems to have calculated the expression incorrectly; for the model considered by him, the same result should have been obtained.

(4) *Normal stress difference*

The only normal stress difference in non-dimensional form is obtained as

$$\frac{p\ (xx) - p\ (yy)}{\rho_\infty\ U^2} = -2\frac{\partial \bar{v}}{\partial \eta} + 2k^* \left\{ -\bar{v}\ \frac{\partial^2 \bar{v}}{\partial \eta^2} + \left(\frac{\partial \bar{u}}{\partial \eta}\right)^2 - \frac{4}{3}\left(\frac{\partial \bar{v}}{\partial \eta}\right)^2 \right\} \tag{43}$$

It may be noted with interest that in the compressible case the normal stress difference does not vanish even in the viscous flow case; however, in the incompressible case it exists only in the elastico-viscous case.

(5) *Displacement thickness*

The displacement thickness δ^* defined as

$$\delta^* = \int_0^\infty (1 - \bar{u})\ dy$$

is obtained as

$$\frac{\delta^*}{[\eta_0/\rho_\infty U]} = \frac{1}{R_s}\left[\frac{1}{\theta_0} + \frac{k^*}{3\ (1 + P_r)}\left\{(7\ Pr - 3) - 10\ P_r\ \theta_0\right\}\right] \tag{44}$$

Elastic elements increase, do not affect or decrease it according as

$$P_r >, = \text{Or} < \frac{3}{(7 - 10\ \theta_0)} \tag{45}$$

(6) *Rate of heat transfer*

The rate of heat transfer at the plate is obtained as

$$Q = -\lambda_1 \frac{dT}{dy}\bigg|_{y=0} = C_P\ V\ \rho_\infty\ \theta_0\ (Tw - T_\infty) \tag{46}$$

A coefficient of heat transfer may then be defined as

$$\frac{Q}{UC_p\ \rho_\infty\ \theta_0\ (Tw - T_\infty)} = R_s \tag{47}$$

which is independent of the elastic number k^*.

However, in the incompressible case the same coefficient may be defined as

$$\frac{Q}{UC_p \ \ \rho \, (Tw - T_\infty)} = R_s \tag{48}$$

It may be pointed out that for the Oldroyd type model Gupta[2] obtained that the rate of heat transfer increases with elasticity. As he remarked himself that although his solution in which p, $p\,(xx)$ etc. vary as $\exp\,(-y/\bar{\tau}\, v_0)$ [$\bar{\tau}$ being the relaxation time parameter] exists, that appeared to be rather artificial.

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