A NEW GENERALISATION OF HANKEL TRANSFORM*

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(*Received* 28 *October* 1968)

In this paper a new generalisation of Hankel transform of two variables has been made. The theory of self-reciprocal functions in two variables has also been developed on the basis of Reed's theory of double Mellin transform with some theorems.

The object of this paper is to generalise an integral transform and to develop the theory of self-reciprocal functions in two variables on the basis of Reed's theory of double Mellin transform and latter some theorems analogous to Agarwal's theorems[1].

A generalised Hankel transform of two complex variables is defined on the basis of Bhise[2] by the relation :

$$F(x,y)=\int_0^\infty\int_0^\infty G^{1,1,1,2,2}_{2,[2:2],0,[4:4]}\left[\begin{matrix} xu \\ yu' \end{matrix}\right|$$

$$\left|\begin{matrix} K+K'-m-m'-1-\frac{v}{2}-\frac{v'}{2}, -K-K'+m'+m+1 \\ +\frac{v}{2}+\frac{v'}{2} -K+m+\frac{1}{2}+\frac{v}{2}, K-m+\frac{3}{2}-\frac{v}{2}; \\ -K'+m'+\frac{1}{2}+\frac{v'}{2}, K'-m'+\frac{3}{2}-\frac{v'}{2}\ \frac{v}{2}-\lambda-m, \\ \frac{v}{2}-\lambda+m, -\frac{v}{2}+\lambda+m, -\frac{v}{2}+\lambda-m; \\ \frac{v'}{2}-\lambda'-m', \frac{v'}{2}-\lambda'+m', -\frac{v'}{2}+\lambda'+m', -\frac{v'}{2}+\lambda'-m' \end{matrix}\right] f(u,v')\,du\,du', \tag{1}$$

Where m, m' are neither integers nor zeros

and

$$G^{n,\,v_1,\,v_2,\,m_1,\,m_2}_{p,\,[t:t'],\,s,\,[q:q']}\left[\begin{matrix} x \\ \\ y \end{matrix}\left|\begin{matrix} (\epsilon_p) \\ (\gamma_t);\ (\gamma'_{t'}) \\ (\delta_s) \\ (\beta_q);\ (\beta'_{q'}) \end{matrix}\right.\right]=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Phi(\xi+\eta)\,\psi(\xi,\eta)\,x^{\xi}\,y^{\eta}\,d\xi d\eta,$$

*This work is supported by the Council of Scientific and Industrial Research, New Delhi and Dr. H. M. Srivastava as the supervisor.

where

$$\Phi(\xi+\eta)=\prod_{j=1}^{n}\Gamma(1-\epsilon_j+\xi+\eta)\Big/\prod_{j=n+1}^{p}\Gamma(\epsilon_j-\xi-\eta)\prod_{j=1}^{s}\Gamma(\delta_j+\xi+\eta),$$

$$\psi(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-\xi)\prod_{j=1}^{v_1}\Gamma(\gamma_j+\xi)\prod_{j=1}^{m_2}\Gamma(\beta'_j-\eta)\prod_{j=1}^{v_2}\Gamma(\gamma'_j+\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-\beta_j+\xi)\prod_{j=v_1+1}^{t}\Gamma(1-\gamma_j-\xi)\prod_{j=m_2+1}^{q'}\Gamma(1-\beta'_j+\eta)\prod_{j=v_2+1}^{t'}\Gamma(1-\gamma'_j-\eta)}$$

and

$$0\leqslant m_1\leqslant q,\ 0\leqslant m_2\leqslant q',\ 0\leqslant v_1\leqslant t,\ 0\leqslant v_2\leqslant t',\ 0\leqslant n\leqslant p,$$

is Meijer's G-function of two complex variables due to Agarwal[3].

As this is similar to Bhise[2], we call a function $f(x,y)$ of two complex variables x and y to be self-reciprocal in the generalised Hankel transform of order (ν,ν') if it satisfies the integral equation by writing $f(x,y)=F(x,y)$ in (1).

Drawing the analogy from the case of Bhise[2] we shall denote such a function by the symbol

$$R_{\nu;\,\nu'}(K,\lambda,m\,;K',\lambda',m').$$

Conditions of self-reciprocity

If $M(r,s)$ denotes the double Mellin transform of $f(x,y)$, then we have

$$M(r,s)=\int_0^\infty\int_0^\infty f(x,y)\,x^{r-1}\,y^{s-1}\,dx\,dy. \tag{2}$$

Substituting the expression for $f(x,y)$ in equation (2) from Bhise[2] case and changing the order of integration, we get

$$M(r,s)=\frac{\Gamma\left(3-K-K'+m+m'+\frac{v}{2}+\frac{v'}{2}-r-s\right)}{\Gamma\left(\frac{v}{2}+\frac{v'}{2}-K-K'+m+m'+1+r+s\right)}$$
$$\times\frac{\Gamma\left(\frac{1}{2}-K+m+\frac{v}{2}-r\right)\Gamma\left(\frac{1}{2}-K'+m'+\frac{v'}{2}-s\right)}{\Gamma\left(\frac{v}{2}-K+m-\frac{1}{2}+r\right)\Gamma\left(\frac{v'}{2}-K'+m'-\frac{1}{2}+s\right)}$$
$$\times\frac{\Gamma_*\left(\frac{v}{2}-\lambda\pm m+r\right)\Gamma_*\left(\frac{v'}{2}-\lambda'\pm m'+s\right)}{\Gamma_*\left(\frac{v}{2}+1-\lambda\pm m-r\right)\Gamma_*\left(\frac{v'}{2}+1-\lambda'\pm m'-s\right)}$$
$$\times M(1-r,1-s)\ , \tag{3}$$

it being assumed that (1) is absolutely convergent. Hence,

$$f(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma_* \left(\frac{v}{2} - \lambda \pm m + r\right) \Gamma_* \left(\frac{v'}{2} - \lambda' \pm m' + s\right)}{\Gamma \left(\frac{v}{2} - K + m - \frac{1}{2} + r\right) \Gamma \left(\frac{v'}{2} - K' + m' - \frac{1}{2} + s\right)}$$

$$\times \frac{\psi(r, s)\, x^{-r}\, y^{-s}}{\Gamma \left(\frac{v}{2} + \frac{v'}{2} - K - K' + m + m'\right)} \, dr\, ds \,, \tag{4}$$

where

$$\psi(r, s) = M(r, s)\, \Gamma \left(\frac{v}{2} + \frac{v'}{2} - K - K' + m + m' + 1 + r + s\right)$$

$$\times \Gamma \left(\frac{v}{2} - K + m - \frac{1}{2} + r\right) \Gamma \left(\frac{v'}{2} - K' + m' - \frac{1}{2} + s\right) \tag{5}$$

and satisfies the functional equation

$$\psi(r, s) = \psi(1 - r, 1 - s) \tag{6}$$

Assuming the restriction imposed on $M(r, s)$ and $f(x, y)$ in the Reed's theorem[4] we find that for the change in the order of integration of the double integrals to be valid it is sufficient if similar conditions as in the case of Agarwal are satisfied.

THEOREM 1

If $f(x, y)$ is $R\rho;\rho'$ $(h, \mu, n; h', \mu', n')$ and

$$K(x, y) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c'-i\infty}^{c'+i\infty} \frac{\Gamma_*(\rho/2 - \mu \pm n + r)\; \Gamma_*(\rho'/2 - \mu' \pm n' + s)}{\Gamma(\rho/2 - h + n - \frac{1}{2} + r)\; \Gamma(\rho'/2 - h' + n' - \frac{1}{2} + s)}$$

$$\times \frac{\Gamma_*\left(\frac{v}{2} - \lambda \pm m + r\right) \Gamma_*\left(\frac{v'}{2} - \lambda' \pm n' + s\right)}{\Gamma\left(\frac{v}{2} - K + m - \frac{1}{2} + r\right) \Gamma\left(\frac{v'}{2} - K' - \frac{1}{2} + s + m'\right)}$$

$$\times \{\Gamma(\rho/2 + \rho'/2 - h - h' + n + n' + 1 + r + s)\}^{-1}$$

$$\times \left\{\Gamma\left(\frac{v}{2} + \frac{v'}{2} - K - K' + m + m' + 1 + r + s\right)\right\}^{-1}$$

$$\times \chi(r, s)\, x^{-r}\, y^{-s}\, dr\, ds \tag{7}$$

where $\chi(r, s) = \chi(1 - r, 1 - s)$,

then $$g(x, y) = \int_0^\infty \int_0^\infty K(xu, yu') f(u, u')\, du\, du' ,$$

is $R_{\nu;\, \nu'}(K, \lambda, m ; K', \lambda', m')$.

Proof :

We have

$$g(x, y) = \int_0^\infty \int_0^\infty K(xu, yu') f(u, u')\, du\, du'.$$

Substituting the value of $f(x, y)$ from (4), changing the order of integration of double integral as justified due to

$$\psi(r, s) = 0\left(e^{(\pi/2 - \alpha + \eta)\,[\,|t| + |\tau|\,]}\right),$$

for every positive η however small $0 < \alpha \leqslant \pi$ and from the third case of Reed's[4] theorem II, we establish the theorem after the application of double Mellin inversion formula.

THEOREM 2

If $f(x, y)$ is $R_{\rho;\, \rho'}(h, \mu, n ; h', \mu', n')$

$$K(x, y) = \frac{1}{(2\pi i)^2} \int_{c - i\infty}^{c + i\infty} \int_{c' - i\infty}^{c' + i\infty} \Gamma\left(3 - K - K' + m + m' + \frac{v}{2} + \frac{v'}{2} - r - s\right)$$

$$\times \Gamma\left(3 - h - h' + n + n' + \frac{\rho}{2} + \frac{\rho'}{2} - r - s\right)$$

$$\times \frac{\Gamma_*(\rho/2 - \mu \pm n + r)\, \Gamma_*(\rho'/2 - \mu' \pm n' + s)}{\Gamma(\rho/2 - h + n - \frac{1}{2} + r)\, \Gamma(\rho'/2 - h' + n' - \frac{1}{2} + s)}$$

$$\times \frac{\Gamma_*\left(\frac{v}{2} - \lambda \pm m + 1 - r\right) \Gamma_*\left(\frac{v'}{2} - \lambda' \pm m' + 1 - s\right)}{\Gamma\left(\frac{v}{2} - K + m + \frac{1}{2} - r\right) \Gamma\left(\frac{v'}{2} - K' + m' + \frac{1}{2} - s\right)}$$

$$\times \chi(r, s)\, x^{-r}\, y^{-s}\, dr\, ds , \tag{8}$$

where $\chi(r, s) = \chi(1 - r, 1 - s)$, then

$$g(x, y) = \int_0^\infty \int_0^\infty K(u, u') f(xu, yu')\, du\, du'$$

$$R_{\nu;\, \nu'} (K, \lambda, m ; K', \lambda', m').$$

The proof of the theorem follows as in the previous theorem.

Example

To illustrate the theorem let us put

$$\chi(r, s) = \frac{\Gamma\left(\frac{\gamma}{2} + r\right) \Gamma\left(1 + \frac{\gamma}{2} - r\right)}{\Gamma\left(\frac{\delta}{2} + s\right) \Gamma\left(1 + \frac{\delta}{2} - s\right)},$$

in equation (8). Then

$$K(x, y) = G^{2,3,2,3,2}_{2,[4:4],0,[4:4]} \left[\begin{array}{c|l} & -2 + K + K' - m - m' - \frac{\nu}{2} - \frac{\nu'}{2}, -2 + h \\ x & + h' - n - n' - \frac{\rho}{2} - \frac{\rho'}{2}\ \frac{v}{2} - \lambda \pm m + 1, 1 \\ & + \frac{\gamma}{2}, 1 - \frac{\rho}{2} + h - n - \frac{1}{2}; \\ & \frac{v'}{2} - \lambda' \pm m' + 1, 1 - \frac{\delta}{2}, 1 - \frac{\rho'}{2} + h' - n' - \frac{1}{2} \\ y & \frac{\rho}{2} - \mu \pm n, \frac{\gamma}{2}, 1 - \frac{v}{2} + K - m - \frac{1}{2}; \\ & \frac{\rho'}{2} - \mu' \pm n', 1 - \frac{v'}{2} - K' - m' - \frac{1}{2}, - \frac{\delta}{2} \end{array} \right]$$

is a kernel transforming $R_{\rho;\, \rho'} (h, \mu, n ; h', \mu', n')$ into $R_{\nu;\, \nu'} (K, \lambda, m ; K', \lambda', m')$

provided that $R(\rho - 2\mu + 2n) > 0$, $R(\rho - 2\mu - 2n) > 0$, $R(\gamma) > 0$, $R(\delta) > 0$,

$$R(\rho' - 2\mu' + 2n') > 0,\ R\left(3 - h - h' + n + n' + \frac{\rho}{2} + \frac{\rho'}{2}\right) > 0,$$

$$R\left(3 - K - K' + m + m' + \frac{v'}{2} + \frac{v}{2}\right) > 0,$$

$2m$, $2m'$, $2n$ and $2n'$ are neither integers nor zeros.

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