A NOTE ON ELASTIC CATENARY

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Seth's generalized relation between stress and strain has been used to obtain the equation of the curve that an elastic heavy string makes when hung freely at two fixed points on the same horizontal line. The parametric equation of the curve has been obtained.

The equation of the curve made by a uniform heavy elastic string hanging freely under the action of gravity at two fixed points has been derived in the book by Ramsey[1]. In the derivation of such an equation Hooke's law has been used to obtain a relation between tension and stretch. But, the linear relation Hooke's law cannot explain many facts observed in practice. In 1935, Seth[2] in his memoir on finite deformation has given a non-linear relation between tension and stretch in the form

$$T = \frac{1}{2} E \left[1 - \frac{1}{(1+s)^2} \right] \tag{1}$$

and used this relation for solving a few problems[3,4]. Later, Seth[5] generalized eqn. (1) and gave the stress-strain relation in the form

$$T = \frac{E}{\alpha} \left[1 - \frac{1}{(1+s)^{\alpha}} \right] \tag{2}$$

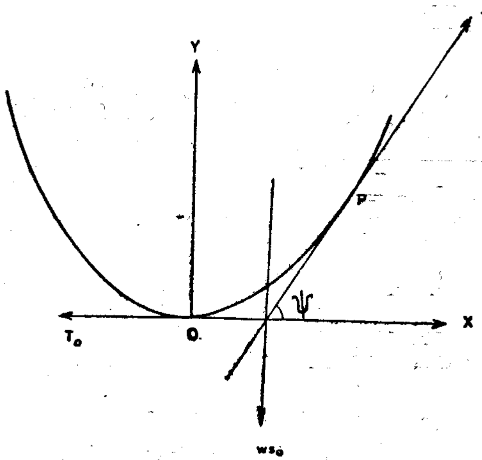

Fig. 1—Equilibrium position of the string.

We see that $T = \frac{E}{\alpha}$ leads to an infinite elongation and therefore, roughly corresponds to the yield point. Moreover, since for infinite contraction $s = -1$ with $T \to -\infty$, it shows that no amount of compression can reduce the length to zero. These results cannot be given by ordinary Hooke's law $T = Es$ which comes as a particular case of the relation (1) or (2) when second and higher powers of 's' are neglected. It will therefore, be of interest to determine the equation of the curve that an elastic heavy string makes when hung freely at two fixed points at the same horizontal line if the strain caused is finite.

Using Fig. 1 to represent the equilibrium position of the string and taking 'w' as the weight of unit length of unstretched string, let $OP = s$ and let s_0 be the unstretched length of OP so that 'ws_0' is its weight. Then by resolving horizontally and vertically for OP, we have

$$T \cos \psi = T_0 = wc \text{ (say)} \tag{3}$$

and

$$T \sin \psi = ws_0 \tag{4}$$

so that by squaring and adding

$$T^2 = w^2 (c^2 + s_0^2) \tag{5}$$

Also by Seth's law[2]

$$T = \frac{E}{\alpha} \left[1 - \left(\frac{ds_0}{ds} \right)^{\alpha} \right] \tag{6}$$

Now from the relation (3)

$$\frac{dx}{ds} = \cos \psi = \frac{wc}{T} \tag{7}$$

and therefore

$$x = wc \int \frac{ds}{T} = wc \int \frac{ds_0}{T \left(1 - \frac{\alpha T}{E} \right)^{1/\alpha}} \tag{8}$$

From equations (5) and (8) we get

$$x = c \int \frac{ds_0}{(c^2 + s_0^2)^{\frac{1}{2}} \left\{ 1 - \frac{\alpha w}{E} (c^2 + s_0^2)^{\frac{1}{2}} \right\}^{1/\alpha}} \tag{9}$$

Similarly, from equation (4)

$$\frac{dy}{ds} = \sin \psi = \frac{ws_0}{T}$$

and therefore,

$$y = w \int \frac{s_0 \, ds}{T} = w \int \frac{s_0 \, ds_0}{T \left(1 - \frac{\alpha T}{E} \right)^{1/\alpha}} \tag{10}$$

From the equations (5) and (10), we get

$$y = \int \frac{s_0 \, ds_0}{(c^2 + s_0^2)^{\frac{1}{2}} \left\{ 1 - \frac{\alpha w}{E} (c^2 + s_0^2)^{\frac{1}{2}} \right\}^{1/\alpha}} \tag{11}$$

If we choose the origin at the lowest point of the string, then integration of (11) gives

$$y = \frac{E}{w(\alpha - 1)} \left[\left\{ 1 - \frac{\alpha wc}{E} \right\}^{\frac{\alpha - 1}{\alpha}} - \left\{ 1 - \frac{\alpha w}{E} \left(c^2 + s_0^2 \right)^{\frac{1}{2}} \right\}^{\frac{\alpha - 1}{\alpha}} \right] \tag{12}$$

From equations (9) and (11) we can obtain an integral relation between x and y in the form

$$cy = \int s_0 \, dx + A,$$

where A is a constant of integration. Equation (9) can however, be integrated in terms of elliptic function. Putting

$$z = (c^2 + s_0^2)^{\frac{1}{2}}$$

we get

$$x = c \int \frac{dz}{\left(z^2 - c^2\right)\left(1 - \frac{\alpha w z}{E}\right)^{1/\alpha}} \tag{13}$$

which is an elliptic integral and from a standard table we can obtain different values of 'x' for different values for s_0. Equations (12) and (13) give the equation to the curve in terms of a single parameter s_0. It is also convenient to obtain the equation of the curve by using certain approximations as described below :—

It has been pointed out earlier that the relation (2) gives a qualitative value of the yield stress and that is $\frac{E}{\alpha}$. In practice, the stress developed at any point of the string is sufficiently less than the yield stress, and therefore, it is reasonable to assume that $T/E << 1$. We expand the expression in the kernel of the integrals (8) and (10) in ascending powers of T/E and retain terms up to squares of T/E. Therefore integral (8) gives

$$x = wc \int \left[\frac{1}{T} + \frac{1}{E} + \frac{(\alpha + 1)}{2!} \frac{T}{E^2} + \frac{(1 + \alpha) \; (1 + 2\alpha)}{3!} \frac{T^2}{E^3} + \dots\right] ds_0 \tag{14}$$

Substitution of the value of T from (5) and integration gives

$$x = c \sinh^{-1} \frac{s_0}{c} + \frac{wcs_0}{E} + \frac{(\alpha + 1)}{2!} \frac{w^2 c}{2E^2}\left[s_0 \; (c^2 + s_0^2)^{\frac{1}{2}} + c^2 \sinh^{-1} \frac{s_0}{c}\right]$$
$$+ \frac{(1 + \alpha)(1 + 2\alpha)}{3!} \frac{w^3 c}{3E^3}\left[\; s_0 \; (3c^2 + s_0^2) \;\right] \tag{15}$$

Similarly, integral (10) gives

$$y + c = (c^2 + s_0^2)^{\frac{1}{2}} + \frac{w}{2E} s_0^2 + \frac{w^2}{2!} \frac{(1 + \alpha)}{3E^2} (c^2 + s_0^2)^{\frac{3}{2}}$$
$$+ \frac{(1 + \alpha)(1 + 2\alpha)}{3!} \frac{w^3 s_0^2}{4E^3} (2c^2 + s_0^2) - \frac{(1 + \alpha) \; w^2 c^3}{2E^2} \cdot \tag{16}$$

Equations (15) and (16) express the coordinates of a point on the curve in terms of the parameter s_0. It can be easily verified that the first two terms in the right hand side of (15) and (16) give the corresponding results by applying ordinary Hooke's law. If the string is perfectly inextensible, E is infinitely large and from (15) and (16) we obtain

$$s_0 = c \sinh \; (x/c)$$

and

$$(y + c)^2 = c^2 + s_0^2$$

which are the equations of a common catenary.

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