MAXIMUM ATMOSPHERIC ENTRY ANGLE FOR SPECIFIED RETROFIRE IMPULSE

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Maximum atmospheric entry angles for vehicles initially moving in elliptic orbits are investigated and it is shown that tangential retrofire impulse at the apogee results in the maximum entry angle. Equivalence of maximizing the entry angle and minimizing the retrofire impulse is also established.

Galman[1] has investigated the retrofire alignment angle that produces the steepest atmospheric entry angle for a given instantaneous velocity increment. Baker, Baxter & Arthur[2] have discussed minimization of the retrofire impulse with respect to altitude for a given entry angle. They have also shown that minimizing the retrofire impulse with respect to retrofire angle is equivalent to maximizing the atmospheric entry angle. In the aforementioned analyses the vehicle is assumed moving in a circular orbit.

The present paper analyses the problem of maximizing the entry angle for specified retrofire impulse in case of vehicle moving initially in elliptic orbit. Galman's result follows therefrom as a particular case. It is further shown that for initial elliptic orbit maximizing the atmospheric entry angle with respect to retrofire alignment angle and true anomaly is equivalent to minimizing the retrofire impulse. Numerical examples show that maximum entry occurs when retrofire impulse is directed tangentially at the apocentre of the initial orbit.

ANALYSIS

Suppose the vehicle is initially moving in the elliptic orbit

$$l = r\,(1 + e \cos \theta) \tag{1}$$

where l, e represent semi-latus rectum and eccentricity and r, θ are radius vector and true anomaly of the orbit.

Let a retrofire impulsive velocity decrement $\triangle V$ at an alignment angle β be applied at the point (r_0, θ_0) changing the vehicle's velocity from V_0 to V_1. If the vehicle makes atmospheric entry at an angle γ_E at a radial distance r_E from the force centre, by conservation of angular momentum

$$V_1\, r_0 \cos \gamma_1 = V_E\, r_E \cos \gamma_E \tag{2}$$

where γ is angle between velocity vector and the local horizontal and suffixes 1 and E denote conditions just after the retrofire impulse and at the entry point respectively.

Energy equation yields

$$V_E^2 - 2V_{CE}^2 = V_1^2 - 2V_{CO}^2 \tag{3}$$

where V_{CE} and V_{CO} are circular velocities at r_E and r_0 respectively.

Also if γ_0 is the value of γ just before the retrofire impulse

$$V_1 \cos(\gamma_0 - \gamma_1) = V_0 - \triangle V \cos\beta \tag{4}$$

$$V_1^2 = V_0^2 - 2\triangle V\, V_0 \cos\beta + (\triangle V)^2 \tag{5}$$

Equations (2) to (5) yield

$$\cos\gamma_E = \frac{\left[\sqrt{\mu l} - \triangle V\, r_0 \cos(\gamma_0 + \beta)\right]}{r_E\left[\mu\left(\frac{2}{r_E} - \frac{1-e^2}{l}\right) - 2\triangle V\, V_0 \cos\beta + (\triangle V)^2\right]^{\frac{1}{2}}} \tag{6}$$

It can be shown that

$$\tan\gamma_0 = \frac{e\sin\theta_0}{1 + e\cos\theta_0} \tag{7}$$

$$V_0 = \left[\frac{\mu}{l}\,(1 + e^2 + 2e\cos\theta_0)\right]^{\frac{1}{2}} \tag{8}$$

where μ is gravitational parameter.

Obviously from equation (6) by virtue of equations (1), (7) and (8) γ_E is a function of two variables β and θ_0 for specified $\triangle V$. For extreme values of γ_E

$$\frac{\partial\gamma_E}{\partial\beta} = 0 \tag{9}$$

$$\frac{\partial\gamma_E}{\partial\theta_0} = 0 \tag{10}$$

Using equation (6), conditions (9) and (10) give

$$r_0\left[\sqrt{\mu l} - \triangle V\, r_0 \cos(\gamma_0 + \beta)\right]\sin(\gamma_0 + \beta) = V_0\, r_E^{\,2}\cos^2\gamma_E \sin\beta \tag{11}$$

$$r_0\left[\sqrt{\mu l} - \triangle V\, r_0 \cos(\gamma_0 + \beta)\right]\left[\mu(e + \cos\theta_0)\sin(\gamma_0 + \beta) - r_0 V_0^2 \sin\theta_0 \cos(\gamma_0 + \beta)\right]$$
$$= \mu V_0\, r_E^2 \cos^2\gamma_E \cos\beta \sin\theta_0 \tag{12}$$

Equations (11) and (12) can be solved for the two unknowns β, θ_0 which on substituting in (6) will give extreme values of γ_E

Equations (11) and (12) give

$$\tan\beta = \frac{e\sin\theta_0\sqrt{\frac{l}{r_0}}}{\frac{l}{r_0}\left[2 \mp \sqrt{\frac{l}{r_0}}\right] - (1 - e^2)} \tag{13}$$

Eliminating β between equations (13) and (11) we got an equation in single unknown θ_0 which was solved numerically. Having known θ_0 equation (13) gave β and then from equation (6) extremum γ_E was evaluated.

This procedure has been adopted to solve the following two particular cases to obtain maximum entry angle at $r_E = 4032$ miles in earth's gravitational field

Case I

$$e = 0{\cdot}2$$
$$l = 5280 \text{ miles}$$
$$\triangle V = 1500 \text{ ft./sec.}$$

Case II

$$e = 0{\cdot}8$$
$$l = 7920 \text{ miles}$$
$$\triangle V = 3000 \text{ ft./sec.}$$

For both cases the value comes out to be $\theta_0 = 0$ or π and $\beta = 0$. But $\theta_0 = 0$ and $\beta = 0$ when substituted in equation (6) give $\cos \gamma_E > 1$ and hence they are inadmissible.

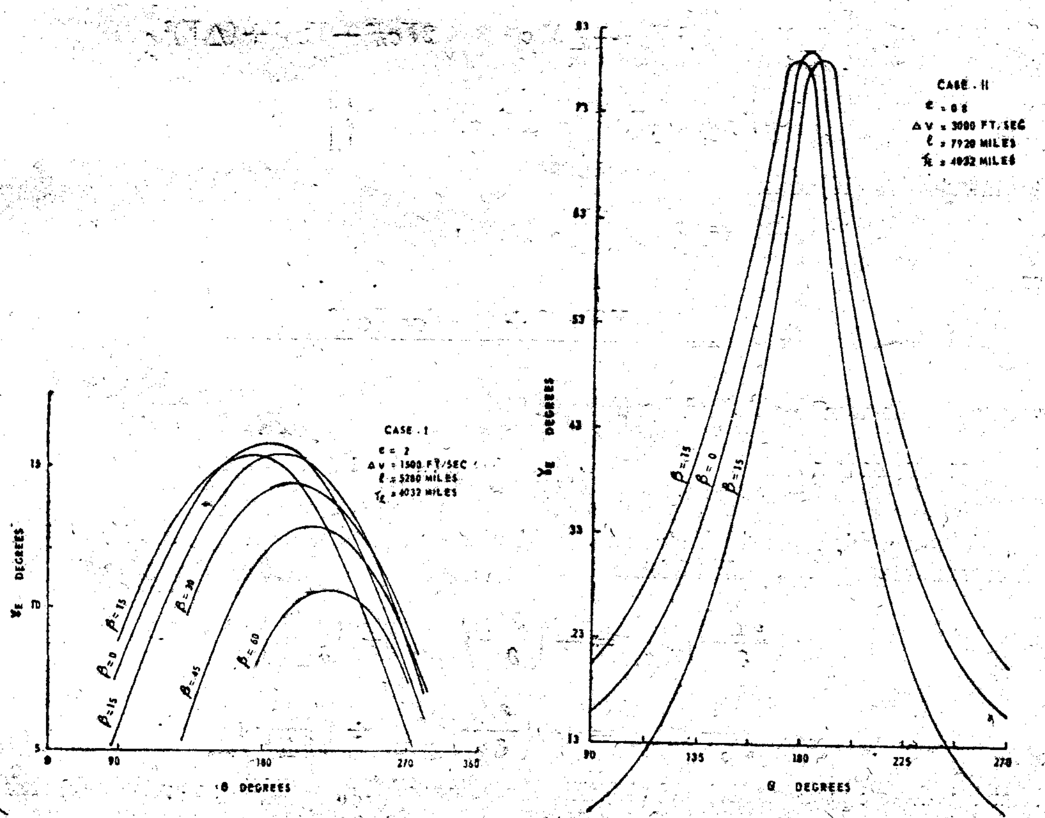

Fig. 1 (Case I and Case II)—Variation of atmospheric entry angle with vectorial angle.

Thus extreme γ_E occurs when retrofire impulse is applied tangentially at the apogee of the initial orbit.

Extremum γ_E for

$$\left.\begin{array}{l} \textit{Case I} = 15°, 51' \\ \textit{Case II} = 78°, 58' \end{array}\right\} \tag{14}$$

To decide whether the extreme values of γ_E correspond to the maximum or the minimum, we have plotted the variation of the atmospheric entry angle γ_E with respect to the vectorial angle for various values of β (Fig. 1). A study of the figure shows that $\theta_0 = \pi$ and $\beta = 0$ corresponds to the maximum value of γ_E and the values of extremum γ_E given in (14) are maximum values of γ_E in the two cases.

If the initial orbit is a circle of radius r_0 equation (6) becomes

$$\cos \gamma_E = \frac{r_0 (V_{CO} - \triangle V \cos \beta)}{r_E [2V_{CE}^2 - V_{CO}^2 - 2\triangle V\, V_{CO} \cos \beta + (\triangle V)^2]^{\frac{1}{2}}} \tag{15}$$

and equation (11) with the help of equation (15) can be written as

$$\sin \beta \Big[(\triangle V \cos \beta)^2 V_{OO} - \triangle V \cos \beta \left\{ 2V_{CE}^2 - V_{CO}^2 + (\triangle V)^2 \right\}$$

$$+ V_{CO} \left\{ 2 (V_{CE}^2 - V_{CO}^2) + (\triangle V)^2 \right\} \Big] = 0 \tag{16}$$

From equation (16) either

$$\sin \beta = 0$$

or

$$\cos \beta = \frac{(\triangle V/V_{CO})^2 + 2 [(V_{CE}/V_{CO})^2 - 1]}{\triangle V/V_{CO}}$$

which are the results obtained by Galman.

MAXIMUM ENTRY ANGLE AND MINIMUM RETROFIRE IMPULSE

For unspecified retrofire impulse it is evident from equation (6) that γ_E is a function of three variables $\triangle V$, θ_0 and β. Hence

$$\left(\frac{\partial \triangle V}{\partial \beta}\right)_{\gamma_E \theta_0} = - \left(\frac{\partial \gamma_E}{\partial \beta}\right)_{\triangle V \theta_0} \div \left(\frac{\partial \gamma_E}{\partial \triangle V}\right)_{\beta \theta_0} \tag{17}$$

$$\left(\frac{\partial \triangle V}{\partial \theta_0}\right)_{\gamma_E \beta} = - \left(\frac{\partial \gamma_E}{\partial \theta_0}\right)_{\triangle V \beta} \div \left(\frac{\partial \gamma_E}{\partial \triangle V}\right)_{\beta \theta_0} \tag{18}$$

Now it is from equations (17) and (18) that the values of β, θ_0 satisfying equations (9) and (10) giving maximum γ_E for specified $\triangle V$ will also satisfy equations

$$\left(\frac{\partial \triangle V}{\partial \beta}\right)_{\gamma_E \theta_0} = 0$$

$$\left(\frac{\partial \triangle V}{\partial \theta_0}\right)_{\gamma_E \beta} = 0$$

giving minimum retrofire impulse for specified entry angle provided that

$$\left(\frac{\partial \gamma_E}{\partial \triangle V}\right)_{\beta \theta_0} \neq 0 \tag{19}$$

Now from equation (6) $\left(\frac{\partial \gamma_E}{\partial \triangle V}\right)_{\beta \theta_0} = 0$ gives

$$[\sqrt{\mu l} - \triangle V \, r_0 \cos(\gamma_0 + \beta)] \, r_0 \cos(\gamma_0 + \beta) = r_E^2 \cos^2 \gamma_E \, (V_0 \cos \beta - \triangle V) \tag{20}$$

Dividing equation (20) by equation (11)

$$\cot(\gamma_0 + \beta) = \cot \beta - \frac{\triangle V}{V_0 \sin \beta}$$

which yields

either (i) $\sin \beta = 0$ (21)

or (ii) $\beta = \sin^{-1}\left(\sqrt{\frac{\mu}{l}} \cdot \frac{e \sin \theta_0}{\triangle V}\right) - \gamma_0$ (22)

Substituting $\beta = 0$ in equation (11) gives $\gamma_0 = 0$ and hence by equation (7) $\theta_0 = 0$ or π. Putting $\beta = 0$ and $\theta_0 = 0$ or π, $(\gamma_0 = 0)$ in equation (20) we have

$$\triangle V = V_0 + \frac{2\mu}{(r_0 V_0 - \sqrt{\mu l})} \left(\frac{r_0}{r_E} - 1\right)$$

which cannot hold good for arbitrary choice of r_E and $\triangle V$. Hence $\beta = 0$ and $\theta_0 = 0$ or π do not satisfy equation (20) and therefore equation (21) is invalid. Evidently then β and θ_0 satisfying equations (11) and (20) are related by equation (22), but since equation (22) differs from equation (13), therefore β and θ_0 satisfying equations (11) and (12) will not satisfy equation (20) and hence relation (19) is true. Therefore β and θ_0 given by equations (11) and (12) for maximum γ_E will also yield minimum $\triangle V$ for fixed γ_E.

CONCLUSIONS

(*i*) For a vehicle moving initially in an elliptic orbit, maximizing entry angle with respect to alignment angle and true anomaly is equivalent to minimizing the retrofire impulse.

(*ii*) Numerical results show that maximum entry angle at a given entry altitude occurs for tangential retrofire impulse at the apogees.

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