ON ESTIMATING THE PARAMETERS OF TRUNCATED TRIVARIATE NORMAL DISTRIBUTIONS

M. N. BHATTACHARYYA, V. V. N. RAO & K. S. KAPOOR

Defence Science Laboratory, Delhi

(*Received* 17 *October* 1967; *revised* 18 *June* 1968)

Maximum likelihood estimates of the parameters of a trivariate normal distribution, with single truncation on two variates, have been derived in this paper. The information matrix has also been given from which the asymptotic variances and covariances might be obtained for the estimates of the parameters of the restricted variables. Numerical examples have been worked out.

The present paper deals with the problem of estimating the parameters of a trivariate normal distribution which is truncated (point or linear truncation) on two variates. The problem arises in analysing the selection and training results. Usually, the officer selection for the civil and military services is made through two successive screenings; a written test followed by a personality test. For selection, the candidates are required either to pass the two tests successively or their combined score should exceed a fixed minimum score whatever may be their performances in the individual tests. The selected candidates are imparted academic and service training before being absorbed in the cadre. The selection and training results are maintained, analysed and studied with a view to improve the selection procedure. The data thus collected may be considered as a random sample from a truncated trivariate normal distribution and the preliminary step in the analysis consists in estimating its parameters.

The same problem, in a much restricted sense, was dealt with by Votaw, Jr. *et al*[1] who assumed truncation on one of the variates only and obtained the estimates of some of the parameters on the assumption that others are known. In the present study no such assumption has been made and truncation on two of the variates has been considered.

The problem of estimating the parameters in truncated distributions has, of course, been studied extensively by many authors. Starting from singly and doubly truncated univariate normal distribution, Cohen[2-7] extended his investigation to the multinormal distribution, considering truncation on one of the variates only. Campbell[8] also considered the case of truncated bivariate distribution. One author of the present paper has investigated[9,10] bivariate case when the sample is doubly censored on both the variates as well as when the distribution is singly truncated on both the variates. But the problem considered in this paper does not appear to have been studied earlier.

MATHEMATICAL FORMULATION OF THE PROBLEM

Let

X_1 = Scores in the Written Test (at selection)

X_2 = Scores in the Personality Test (at selection)

X_3 = Scores in the Training Test

n = Number of candidates selected for training.

$X_{1\alpha}$, $X_{2\alpha}$ and $X_{3\alpha}$ ($\alpha = 1,2, \ldots\ldots n$) are the scores of the αth selected candidate at the Written Test, Personality Test and Training Test respectively. Since these are scores of selected candidates the conditions on $X_{1\alpha}$ and $X_{2\alpha}$ are either :

(i) $X_{1\alpha} \geqslant C_1$ and $X_{2\alpha} \geqslant C_2$, where C_1 & C_2 are pass marks in the Written Test and Personality Test respectively

or

(ii) $X_{1\alpha} + X_{2\alpha} \geqslant T$ where T represents pass marks for the combined Written Test and Personality Test,

depending on the method of selection. $X_{3\alpha}$ may take any value from zero to the maximum marks. The probability density function of the truncated trivariate normal distribution representing the scores of the selected candidates is

$$f(X_1, X_2, X_3) = \frac{1}{G\,(2\pi)^{3/2}\ \sigma_1\,\sigma_2\,\sigma_3\sqrt{R_{11}\,R_{33} - R_{13}^2}} \times \exp\left[-\frac{1}{2\,R_{33}}\left\{\frac{(X_1-m_1)^2}{\sigma_1^2} + \frac{(X_2-m_2)^2}{\sigma_2^2} - 2\rho_{12}\frac{(X_1-m_1)(X_2-m_2)}{\sigma_1\sigma_2}\right\}\right]$$
$$\times \exp\left[-\frac{1}{2\,(R_{11}\,R_{33} - R_{13}^2)}\left\{R_{33}\frac{(X_3-m_3)}{\sigma_3} + R_{13}\frac{(X_1-m_1)}{\sigma_1} + R_{23}\frac{(X_2-m_2)}{\sigma_2}\right\}^2\right] \quad (1)$$

where

m_i = mean of X_i

σ_i = s.d. of X_i

ρ_{ij} = correlation between X_i and X_j

$R_{11} = 1 - \rho_{23}^2$ $\qquad$ $R_{33} = 1 - \rho_{12}^2$

$R_{13} = \rho_{12}\,\rho_{23} - \rho_{13}$ $\qquad$ $R_{23} = \rho_{12}\,\rho_{13} - \rho_{23}$

$$G = \int_{X_2=C_2}\int_{X_1=C_1} \frac{1}{2\pi\,\sigma_1\,\sigma_2\sqrt{R_{33}}}\exp\left[-\frac{1}{2\,R_{33}}\left\{\frac{(X_1-m_1)^2}{\sigma_1^2} + \frac{(X_2-m_2)^2}{\sigma_2^2} - 2\rho_{12}\frac{(X_1-m_1)(X_2-m_2)}{\sigma_1\sigma_2}\right\}\right]dX_1\,dX_2$$

[for point truncation: $X_1 \geqslant C_1$ and $X_2 \geqslant C_2$]

or

$$G = \int_{y=T} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}\left(y - m\right)^2 \right]$$

where $y = X_1 + X_2$

$m = m_1 + m_2,$

$\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\,\rho_{12}\,\sigma_1\,\sigma_2$

[For linear truncation: $(X_1 + X_2) \geqslant T$].

The means, standard deviations and correlation coefficients in the above formulation are unknown parameters which are to be estimated.

MAXIMUM LIKELIHOOD ESTIMATES

The logarithm of the likelihood function for a sample of size n is

$$L = -\frac{3n}{2}\log 2\pi - n\sum_{i=1}^{3}\log\sigma_i - \frac{n}{2}\log\left(R_{11}R_{33} - R_{13}^2\right)$$

$$-\frac{1}{2R_{33}}\sum_{\alpha=1}^{n}\left[\frac{(X_{1\alpha}-m_1)^2}{\sigma_1^2} + \frac{(X_{2\alpha}-m_2)^2}{\sigma_2^2} - 2\rho_{12}\frac{(X_{1\alpha}-m_1)(X_{2\alpha}-m_2)}{\sigma_1\sigma_2}\right]$$

$$-\frac{1}{2(R_{11}R_{33}-R_{13}^2)}\sum_{\alpha=1}^{n}\left[\left\{R_{33}\frac{(X_{3\alpha}-m_3)}{\sigma_3} + R_{13}\frac{(X_{1\alpha}-m_1)}{\sigma_1}\right.\right.$$

$$\left.\left. + R_{23}\frac{(X_{2\alpha}-m_2)}{\sigma_2}\right\}^2\right] - n\log G \qquad (2)$$

The maximum likelihood estimates of the parameters m_i, σ_i and ρ_{ij} are obtained by equating to zero, the partial derivations of L with respect to these parameters and solving the resulting system of equations. Starting with the parameters connected with the untruncated variable (X_3), the estimating equations are

$$\frac{\partial L}{\partial m_3} = \frac{1}{\sigma_3(R_{11}R_{33}-R_{13}^2)}\sum_{i=1}^{3} R_{3i}\,C_{i0} = 0 \qquad (3)$$

$$\frac{\partial L}{\partial \sigma_3} = \frac{1}{\sigma_3(R_{11}R_{33}-R_{13}^2)}\sum_{i=1}^{3} R_{3i}\,C_{3i} - \frac{n}{\sigma_3} = 0 \qquad (4)$$

$$\frac{\partial L}{\partial \rho_{13}} = \frac{1}{(R_{11}R_{33}-R_{13}^2)^2}\sum_{i,j=1}^{3} C_{ij}\,R_{3i}\,R_{1j} - \frac{nR_{13}}{R_{11}R_{33}+R_{13}^2} = 0 \qquad (5)$$

$$\frac{\partial L}{\partial \rho_{23}} = \frac{1}{(R_{11}R_{33}-R_{13}^2)^2}\sum_{i,j=1}^{3} C_{ij}\,R_{3i}\,R_{2j} - \frac{nR_{23}}{R_{11}R_{33}-R_{13}^2} = 0 \qquad (6)$$

where

$$C_{i0} = \sum_{\alpha} \frac{(X_{i\alpha} - m_i)}{\sigma_i}$$

$$C_{ij} = \sum_{\alpha} \frac{(X_{i\alpha} - m_i)\,(X_{j\alpha} - m_j)}{\sigma_i\,\sigma_j}$$

From (4), $\Sigma\, C_{3i}\, R_{3i} = n\,(R_{11}\, R_{33} - R_{13}^2)$. Substituting this in (5) and (6) we obtain the equations

$$R_{11} \sum_{i=1}^{3} C_{i1}\, R_{3i} + R_{12} \sum_{i=1}^{3} C_{i2}\, R_{3i} = 0 \qquad (7)$$

$$R_{12} \sum_{i=1}^{3} C_{i1}\, R_{3i} + R_{22} \sum_{i=1}^{3} C_{i2}\, R_{3i} = 0 \qquad (8)$$

These lead to the equations

$$\Sigma\, C_{i1}\, R_{3i} = 0 \qquad (9)$$

$$\Sigma\, C_{i2}\, R_{3i} = 0 \qquad (10)$$

Using (9) and (10), the remaining partial derivations, equated to zero, are

$$\frac{\partial L}{\partial m_1} = \frac{1}{R_{33}\,\sigma_1}\left[C_{10} - \rho_{12}\, C_{20} \right] - \frac{n}{G}\,\frac{\partial G}{\partial m_1} = 0 \qquad (11)$$

$$\frac{\partial L}{\partial m_2} = \frac{1}{R_{33}\,\sigma_2}\left[C_{20} - \rho_{12}\, C_{10} \right] - \frac{n}{G}\,\frac{\partial G}{\partial m_2} = 0 \qquad (12)$$

$$\frac{\partial L}{\partial \sigma_1} = -\frac{n}{\sigma_1} + \frac{1}{R_{33}\,\sigma_1}\left[C_{11} - \rho_{12}\, C_{12} \right] - \frac{n}{G}\,\frac{\partial G}{\partial \sigma_1} = 0 \qquad (13)$$

$$\frac{\partial L}{\partial \sigma_2} = -\frac{n}{\sigma_2} + \frac{1}{R_{33}\,\sigma_2}\left[C_{22} - \rho_{12}\, C_{12} \right] - \frac{n}{G}\,\frac{\partial G}{\partial \sigma_2} = 0 \qquad (14)$$

$$\frac{\partial L}{\partial \rho_{12}} = \frac{n\rho_{12}}{R_{33}} - \frac{\rho_{12}}{R^2_{33}}\left[C_{11} + C_{22} - 2\rho_{12}\, C_{12} \right] + \frac{1}{R_{33}}\, C_{12} - \frac{n}{G}\,\frac{\partial G}{\partial \rho_{12}} = 0 \qquad (15)$$

It may be noted that the equations (11) to (15) are independent of the untruncated variable X_3 and the associated parameters m_3, σ_3, ρ_{13} and ρ_{23}. These equations lead to the following iterative equations for estimating the parameters m_1, m_2, σ_1 and σ_2 and ρ_{12}

$$\hat{m}_1 = \bar{X}_1 - (P_1 + \rho_{12}\, P_2)\,\sigma_1$$

$$\hat{m}_2 = \bar{X}_2 - (P_2 + \rho_{12}\, P_1)\,\sigma_2$$

$$\hat{\sigma}_1 = s_1 \left\{ (1 + \alpha_1 P_1) + \rho_{12}^2\,(P_3 + \alpha_2\, P_2) - (P_1 + \rho_{12}\, P_2)^2 \right\}^{-1/2}$$

$$\hat{\sigma}_2 = s_2 \left\{ (1 + \alpha_2 P_2) + \rho_{12}^2 (P_3 + \alpha_1 P_1) - (P_2 + \rho_{12} P_1)^2 \right\}^{-1/2}$$

$$\hat{\rho}_{12} = \frac{\gamma_{12} s_1 s_2}{\sigma_1 \sigma_2} \left[\left\{ 1 + \alpha_1 P_1 + \alpha_2 P_2 + P_3 \right\} - \left\{ (P_1 + \rho_{12} P_2)(P_2 + \rho_{12} P_1)/\rho_{12} \right\} \right]^{-1} \tag{16}$$

For point truncation on X_1 and X_2 ($X_1 \geqslant C_1$ and $X_2 \geqslant C_2$)

$$\alpha_1 = \xi_1 = \frac{(C_1 - m_1)}{\sigma_1}$$

$$\alpha_2 = \xi_2 = \frac{(C_2 - m_2)}{\sigma_2}$$

$$P_1 = \phi(\xi_1) I \left(\frac{\xi_2 - \rho_{12} \xi_1}{\sqrt{1 - \rho_{12}^2}} \right) \Big/ G$$

$$P_2 = \phi(\xi_2) I \left(\frac{\xi_1 - \rho_{12} \xi_2}{\sqrt{1 - \rho_{12}^2}} \right) \Big/ G$$

$$P_3 = \phi(\xi_1) \phi \left(\frac{\xi_2 - \rho_{12} \xi_1}{\sqrt{1 - \rho_{12}^2}} \right) \frac{\sqrt{1 - \rho_{12}^2}}{\rho_{12}} \Big/ G \tag{17}$$

where ϕ's are ordinates of the univariate normal distribution and

$$I(\xi_i) = \int_{\xi_i}^{\infty} \phi(t)\, dt$$

For linear truncation $(X_1 + X_2) \geqslant T$

$$P_1 = \frac{\sigma_1 \phi(\xi)}{\sigma} \Big/ G$$

$$P_2 = \frac{\sigma_2 \phi(\xi)}{\sigma} \Big/ G$$

$$P_3 = \frac{\xi \sigma_1 \sigma_2}{\sigma^2} \phi(\xi) \left[\frac{1 - \rho_{12}^2}{\rho_{12}} \right] \Big/ G \tag{18}$$

$$\alpha_1 = \frac{\xi (\sigma_1 + \sigma_2 \rho_{12})}{\sigma}$$

$$\alpha_2 = \frac{\xi (\sigma_2 + \sigma_1 \rho_{12})}{\sigma}$$

$$\xi = \frac{T - m}{\sigma}$$

$\bar{X}_1, \bar{X}_2, s_1, s_2, \gamma_{12}$ are the sample statistics uncorrected for truncation.

Now, solving the equations (3), (9) and (10) in terms of the above solutions, we have

$$\hat{m}_3 = \bar{X}_3 + \frac{s_3}{\bar{R}_{33}} \left[\bar{R}_{13} \frac{(\bar{X}_1 - \hat{m}_1)}{s_1} + \bar{R}_{23} \frac{(\bar{X}_2 - \hat{m}_2)}{s_2} \right]$$

$$\hat{\sigma}_3 = \frac{s_3}{\bar{R}_{33}} \left[\left\{ \bar{R}_{23} \hat{\rho}_{12} \frac{\hat{\sigma}_2}{s_2} + \bar{R}_{13} \frac{\hat{\sigma}_1}{s_1} \right\}^2 + \frac{\hat{R}_{33}^2 \bar{R}_{23}^2 \hat{\sigma}_2^2}{s_2^2} + \bar{R}_{33} \left\{ \bar{R}_{33} + \bar{R}_{13} \gamma_{13} + \bar{R}_{23} \gamma_{23} \right\} \right]^{1/2} \tag{19}$$

$$\hat{\rho}_{23} = - \frac{s_3}{\bar{R}_{33} \hat{\sigma}_3} \left[\bar{R}_{13} \hat{\rho}_{12} \frac{\hat{\sigma}_1}{s_1} + \bar{R}_{23} \frac{\hat{\sigma}_2}{s_2} \right]$$

$$\hat{\rho}_{13} = - \frac{s_3}{\bar{R}_{33} \hat{\sigma}_3} \left[\bar{R}_{23} \hat{\rho}_{12} \frac{\hat{\sigma}_2}{s_2} + \bar{R}_{13} \frac{\hat{\sigma}_1}{s_1} \right]$$

where $\hat{m}_1, \hat{m}_2, \hat{\sigma}_1, \hat{\sigma}_2$ and $\hat{\rho}_{12}$ are the estimates as obtained from the iterative equation (16). And $\bar{R}_{33}, \bar{R}_{13}, \bar{R}_{23}, \bar{X}_3, s_3, \gamma_{13}$ and γ_{23} are the sample statistics uncorrected for truncation.

ASYMPTOTIC VARIANCES AND COVARIANCES OF THE MAXIMUM LIKELIHOOD ESTIMATES

The asymptotic variances and covariances of the estimates worked out in the preceding sections are obtained from the information matrix of the likelihood. The elements of this matrix are the second partial derivatives of L with respect to these parameters, with sign reversed, at the values of the parameters equal to the maximum likelihood estimates. The expressions for these variances and covariances will be too unwieldy to be of any practical use. However, the variances and covariances of the estimates of the parameters, involved in the joint bivariate distributions of the two restricted variables, X_1 and X_2 in the present case, will be the same as obtained from the information matrix given in reference 10, when restrictions are of the nature of point-truncation. For linear truncation, the corresponding elements will be as given below, where L is the likelihood function of the marginal truncated bivariate distribution.

$$\frac{\partial^2 L}{\partial m_1^2} = - \frac{n}{\sigma_1^2 (1 - \rho_{12}^2)} \left[1 - (P_1^2 - \alpha_1 P_1)(1 - \rho_{12}^2) - P_3 \rho_{12}^2 \right]$$

$$\frac{\partial^2 L}{\partial m_1 \partial \sigma_1} = - \frac{n}{\sigma_1^2 (1 - \rho_{12}^2)} \left[(P_1 + \rho_{12} P_2) + P_1 (1 - \alpha_1 P_1)(1 - \rho_{12}^2) + \frac{P_1 \sigma_1}{\sigma^2} (1 - \rho_{12}^2) \left\{ \xi \sigma \alpha_1 - (\sigma_1 + \sigma_2 \rho_{12}) \right\} \right]$$

$$\frac{\partial^2 L}{\partial m_1 \partial m_2} = - \frac{n \rho_{12}}{\sigma_1 \sigma_2 (1 - \rho_{12}^2)} \left[P_3 - \frac{P_1 P_2}{\rho_{12}} (1 - \rho_{12}^2) - 1 \right]$$

$$\frac{\partial^2 L}{\partial m_1 \partial \sigma_2} = - \frac{n \rho_{12}}{\sigma_1 \sigma_2 (1 - \rho_{12}^2)} \left[\alpha_2 P_3 - (P_2 + \rho_{12} P_1) \right.$$

$$\left. - \frac{(1 - \rho_{12}^2) \alpha_2 P_1 P_2}{\rho_{12}} - \frac{P_1 \sigma_2 (\sigma_2 + \sigma_1 \rho_{12}) (1 - \rho_{12}^2)}{\rho_{12} \sigma^2} \right]$$

$$\frac{\partial^2 L}{\partial m_1 \partial \rho_{12}} = - \frac{n}{\sigma_2 (1 - \rho_{12}^2)} \left[P_2 - P_1 \rho_{12} - P_1 P_3 \rho_{12} + \frac{P_3 \alpha_1 \sigma_1 \rho_{12}}{(\sigma_1 + \sigma_2 \rho_{12})} \right.$$

$$\left. - \frac{P_1 \sigma_1 \sigma_2 (1 - \rho_{12}^2)}{\sigma^2} \right]$$

$$\frac{\partial^2 L}{\partial \sigma_1^2} = - \frac{n}{\sigma_1^2 (1 - \rho_{12}^2)} \left[2 - \rho_{12}^2 + \alpha_1 P_1 + (P_3 + \alpha_2 P_2) \rho_{12}^2 \right.$$

$$+ \alpha_1 P_1 (1 - \rho_{12}^2) (2 - \alpha_1 P_1) + \xi P_1 \sigma_1 (1 - \rho_{12}^2) (1 + \alpha_1^2)$$

$$\left. - \frac{3 \sigma_1 \alpha_1 P_1}{\sigma} (\sigma_1 + \sigma_2 \rho_{12}) (1 - \rho_{12}^2) \right]$$

$$\frac{\partial^2 L}{\partial \sigma_1 \partial m_2} = - \frac{n \rho_{12}}{\sigma_1 \sigma_2 (1 - \rho_{12}^2)} \left[\alpha_1 P_3 - (P_1 + \rho_{12} P_2) \right.$$

$$\left. - \frac{\alpha_1 P_1 P_2 (1 - \rho_{12}^2)}{\rho_{12}} - \frac{\sigma_1 P_2 (\sigma_1 + \sigma_2 \rho_{12}) (1 - \rho_{12}^2)}{\sigma^2 \rho_{12}} \right]$$

$$\frac{\partial^2 L}{\partial \sigma_1 \partial \sigma_2} = - \frac{n \rho_{12}}{\sigma_1 \sigma_2 (1 - \rho_{12}^2)} \left[\alpha_1 \alpha_2 (P_1 P_2 \rho_{12} + P_3) \right.$$

$$- \rho_{12} (1 + P_3 + \alpha_1 P_1 + \alpha_2 P_2) - \frac{\alpha_1 \alpha_2 P_1 P_2}{\rho_{12}} - \frac{P_3}{\sigma^2}$$

$$\left. \left\{ \sigma_1 \sigma_2 \rho_{12}^2 + 2 \sigma_1 \sigma_2 + 2 \rho_{12} (\sigma_1^2 + \sigma_2^2) \right\} \right]$$

$$\frac{\partial^2 L}{\partial \sigma_1 \partial \rho_{12}} = - \frac{n \rho_{12}}{\sigma_1 (1 - \rho_{12}^2)} \left[P_3 + \alpha_2 P_2 - \alpha_1 P_1 - 1 + \alpha_1 P_1 P_3 \right.$$

$$\left. + \left\{ \frac{\sigma_2 P_1 \xi (1 - \rho_{12}^2)}{\rho_{12} \sigma^3} \right\} \left\{ \sigma_2 (\sigma_2 + \sigma_1 \rho_{12}) + \sigma_1 (\sigma_1 + \sigma_2 \rho_{12}) \xi^2 \right\} \right]$$

$$\frac{\partial^2 L}{\partial m_2^2} = - \frac{n}{\sigma_2 (1 - \rho_{12}^2)} \left[1 - (P_2^2 - \alpha_2 P_2) (1 - \rho_{12}) - P_3 \rho_{12}^2 \right]$$

$$\frac{\partial^2 L}{\partial m_2 \, \partial \sigma_2} = - \frac{n}{\sigma_2^2 (1-\rho_{12}^2)} \Big[P_2 + \rho_{12} P_1 + P_2 (1-\alpha_2 P_2) (1-\rho_{12}^2)$$

$$+ \frac{P_2 \sigma_2 (1-\rho_{12}^2)}{\sigma_2} \Big\{ \xi \, \sigma \, \alpha_2 - (\sigma_2 + \sigma_1 \, \rho_{12}) \Big\} \Big]$$

$$\frac{\partial^2 L}{\partial m_2 \, \partial \rho_{12}} = - \frac{n}{\sigma_2 (1-\rho_{12}^2)} \Big[P_1 - \rho_{12} (P_1 + P_2 P_3)$$

$$+ \frac{P_3 \alpha_2 \sigma_2 \rho_{12}}{\sigma_2 + \sigma_1 \rho_{12}} - \frac{P_2 \sigma_1 \sigma_2 (1-\rho_{12}^2)}{\sigma^2} \Big]$$

$$\frac{\partial^2 L}{\partial \sigma_2^2} = - \frac{n}{\sigma_2^2 (1-\rho_{12}^2)} \Big[2 - \rho_{12}^2 + \alpha_2 P_2 + \rho_{12}^2 (P_3 + \alpha_1 P_1)$$

$$+ \alpha_2 P_2 (1-\rho_{12}^2)(2-\alpha_2 P_2) + \xi \, P_2 \sigma_2 (1-\rho_{12}^2)(1+\alpha_2^2)$$

$$- \frac{3 \sigma_2 P_2 \alpha_2}{\sigma} (\sigma_2 + \sigma_1 \rho_{12})(1-\rho_{12}^2) \Big].$$

$$\frac{\partial^2 L}{\partial \sigma_2 \, \partial \rho_{12}} = - \frac{n \, \rho_{12}}{\sigma_2 (1-\rho_{12}^2)} \Big[P_3 + \alpha_1 P_1 - \alpha_2 P_2 - 1 - \alpha_2 P_2 P_3$$

$$+ \frac{\xi \sigma_1 P_2 (1-\rho_{12}^2)}{\rho_{12} \, \sigma^3} \Big\{ \sigma_1 (\sigma_1 + \sigma_2 \rho_{12}) + \sigma_2 (\sigma_2 + \sigma_1 \rho_{12}) \, \xi^2 \Big\} \Big]$$

$$\frac{\partial^2 L}{\partial \rho_{12}^2} = - \frac{n}{(1-\rho_{12}^2)^2} \Big[1 + \rho_{12}^2 + (\alpha_1 P_1 + \alpha_2 P_2)(1-\rho_{12}^2)$$

$$- P_3 \rho_{12}^2 (4 + P_3) + \frac{(1-\rho_{12}^2)}{\sigma^2} \, \sigma_1 \sigma_2 P_3 \, \rho_{12} (\xi^2 - 3) \Big]$$

where various P, α and ξ are as in equation (18).

NUMERICAL EXAMPLE

The scores on a written test (X_1), personality test (X_2) and training test (X_3) of 50 selected trainees are available and summarised below to estimate the means, standard deviations and correlation coefficients between different tests.

Case I—For selection, the candidates were required to pass the written and personality test separately. The cutting scores (pass marks) for each of these tests were 360 and total marks were 900. The training test scores were recorded as obtained (unrestricted) by the trainees. The uncorrected sample statistics are

$$\bar{X}_1 = 484 \cdot 67 \qquad \bar{X}_2 = 484 \cdot 55 \qquad \bar{X}_3 = 495 \cdot 62$$

$$s_1 = 60 \cdot 47 \qquad s_2 = 74 \cdot 81 \qquad s_3 = 82 \cdot 15$$

$$\gamma_{12} = \cdot 41 \qquad \gamma_{13} = \cdot 53 \qquad \gamma_{23} = \cdot 48$$

Using the iterative equation (16) the estimates of the parameters of the restricted variables (X_1, X_2) are (for numerical computation—see ref 10) :

$$\hat{m}_1 = 473 \cdot 51, \quad \hat{m}_2 = 463 \cdot 46, \quad \hat{\sigma}_1 = 67 \cdot 94, \quad \hat{\sigma}_2 = 91 \cdot 33, \quad \hat{\rho}_{12} = \cdot 55$$

Using these in (19) the estimates of the parameters connected with the unrestricted variable (X_3) are

$$\hat{m}_3 = 482 \cdot 24, \quad \hat{\sigma}_3 = 87 \cdot 94, \quad \hat{\rho}_{13} = \cdot 59, \quad \hat{\rho}_{23} = \cdot 62$$

Case II—For selection, the candidates were required to pass the written and the personality test jointly; the combined scores had to be more than 700. The uncorrected sample statistics are

$$\bar{X}_1 = 480 \cdot 21 \quad \bar{X}_2 = 480 \cdot 25 \quad \bar{X}_3 = 498 \cdot 52$$

$$s_1 = 63 \cdot 25 \quad s_2 = 76 \cdot 35 \quad s_3 = 83 \cdot 25$$

$$\gamma_{12} = \cdot 43 \quad \gamma_{13} = \cdot 55 \quad \gamma_{23} = \cdot 50$$

Starting with the uncorrected sample statistics as the first approximations and using the iterative equation (16) the estimates of the parameters of the restricted variables (X_1, X_2) after a few cycles of iterations, are

$$\hat{m}_1 = 477 \cdot 37, \quad \hat{m}_2 = 476 \cdot 56, \quad \hat{\sigma}_1 = 65 \cdot 74, \quad \hat{\sigma}_2 = 79 \cdot 82, \quad \hat{\rho}_{12} = .48$$

Using these in (19) the estimates of the remaining parameters are

$$\hat{m}_3 = 487 \cdot 68, \quad \hat{\sigma}_3 = 84 \cdot 34, \quad \hat{\rho}_{13} = \cdot 44, \quad \hat{\rho}_{23} = \cdot 40$$

ACKNOWLEDGEMENTS

The authors are grateful to Dr. N. K. Chakravarti, Asstt. Director for suggesting the problem and Dr. Kartar Singh, Director, Defence Science Laboratory, for permissoin to publish the paper.

REFERENCES

1. VOTAW, D. F. Jr., RAFFERTY, J. A. & DEEMER, W. L., "Estimation of Parameters in a Truncated Trivariate Normal Distribution ", *Psychometrika*, 15 (1950), 339.

2. COHEN, A. C. Jr., "On Estimating the Mean and Standard Deviation of Truncated Normal Distribution" *J. Amer. Stat. Assoc.*, 44 (1949), 518.

3. COHEN, A. C. Jr., "Estimating the Mean and Variance of Normal Population from Singly Truncated and Doubly Truncated Samples", *Ann. Math. Stat.*, 21 (1950), 557.

4. COHEN, A. C. Jr., "Estimation in Truncated Bivariate Normal Distribution", University of Georgia, Mathematical Technical Report No. 2 (June 1953).

5. COHEN, A. C. Jr., "Estimation in Truncated Multivariate Normal Distributions", University of Georgia, Mathematical Technical Report No. 3 (Aug. 1953).

6. COHEN, A. C. Jr., "Restriction & Selection in Samples from Bivariate Normal Distributions", *J. Amer. Stat. Assoc.*, **50** (1955), 884.

7. COHEN, A. C. Jr, "Restriction & Selection in Multinormal Distributions", *Annals. Math. Stat.*, **28** (1957), 731.

8. CAMPBELL, F. L., "A Study of Truncated Bivariate Normal Distributions" Doctoral Dissertation, University of Michigan (June 1945).

9. BHATTACHARYYA, M. N., "Estimation from Censored Bivariate Samples", *J. Ind. Soc. Agri. Stat.*, **1** (1954), 83.

10. BHATTACHARYYA, M. N., "On Estimating the Parameters of Truncated Bivariate Normal Distribution" Unpublished.