UNSTEADY FLOW BETWEEN TWO OSCILLATING PLATES

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The flow of a viscous liquid between two harmonically oscillating infinite plates has been analysed when a body force is applied at $t = 0$, in the direction of motion of the plates. An expression for the velocity distribution has been obtained by the use of Laplace-transform technique.

The motion of a viscous fluid in contact with vibrating solids was first studied by Stokes. He discussed the motion of a fluid in contact with a harmonically vibrating plane. The motion of a liquid over an infinite oscillating plate has been discussed by several workers. In the present paper we have discussed the motion of an incompressible viscous liquid between two infinite harmonically oscillating plates when a constant body force X is applied at $t = 0$ in the direction of motion of the plates. The Laplace-transform technique is used in solving the problem.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

Let us suppose that the two plates situated at $z = \pm h$ execute harmonic oscillations in a direction parallel to themselves. The space between them is occupied by an incompressible viscous liquid. The liquid is supposed to be at rest initially and a uniform body force X is applied at $t = 0$ in the direction of motion of the plates.

The equation of motion is

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \operatorname{grad} \vec{u} = \vec{X} - \frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \vec{u}, \tag{1}$$

where $\vec{u}$ is the velocity vector, p the pressure, ν the kinematic viscosity of the liquid and ρ its density.

If we take x-axis along the direction of motion of the plates, the x-component of equation (1) yields

$$\frac{\partial u}{\partial t} = X + \nu \frac{\partial^2 u}{\partial z^2}, \tag{2}$$

where $u = u(z,t)$ is the x-component of $\vec{u}$ and the pressure is supposed to be constant throughout the field.

Equation (2) is to be solved subject to the boundary conditions

$$u = 0 \text{ at } t = 0 \tag{3}$$

$$u = U_0 \sin \omega t \text{ at } z = \pm h,\ t > 0 \tag{4}$$

SOLUTION OF THE PROBLEM

To solve the problem we introduce Laplace-transform $\bar{u}(p)$ of the function $u(t)$ defined by

$$\bar{u}(p) = \int_0^\infty e^{-pt} u(t)\, dt, \qquad Rl(p) > 0$$

and the inverse is given by

$$u(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \bar{u}(\lambda)\, d\lambda,$$

where γ is a constant greater than the real part of the singularities of the function $\bar{u}(\lambda)$.

Applying Laplace-transform to equation (2) we find

$$p\bar{u} = \frac{X}{p} + \nu \frac{d^2\bar{u}}{dz^2}$$

which can be written as

$$\frac{d^2\bar{u}}{dz^2} - \frac{p}{\nu}\bar{u} = -\frac{X}{p\nu}, \tag{5}$$

Since $\bar{u} = \dfrac{U_0\,\omega}{p^2 + \omega^2}$ when $z = \pm h$, we obtain

$$\bar{u} = \frac{X}{p^2}\left[1 - \frac{\cosh(z\sqrt{p/\nu})}{\cosh(h\sqrt{p/\nu})}\right] + \frac{U_0\,\omega}{p^2+\omega^2} \cdot \frac{\cosh(z\sqrt{p/\nu})}{\cosh(h\sqrt{p/\nu})}$$

as the solution of equation (5).

Inverting,

$$u = \frac{X}{2\nu}(h^2 - z^2) - \frac{16h^2X}{\nu\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left\{\frac{(2n+1)\pi z}{2h}\right\} e^{-\left(\frac{\nu\pi^2}{h^2}\right)\left(n+\frac{1}{2}\right)^2 t}$$

$$+ 2U_0\,\nu\pi\omega h^2 \sum_{n=0}^{\infty}\frac{(-1)^n (n+\frac{1}{2})\cos\left\{\frac{(2n+1)\pi z}{2h}\right\}}{\nu^2(n+\frac{1}{2})^4 + \omega^2 h^4}\, e^{-\left(\frac{\nu\pi^2}{h^2}\right)(n+\frac{1}{2})^2 t}$$

$$+ U_0\left\{\left[\cosh\left(z\sqrt{\frac{\omega}{2\nu}}\right)\cosh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(z\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(h\sqrt{\frac{\omega}{2\nu}}\right) + \sinh\left(z\sqrt{\frac{\omega}{2\nu}}\right)\right.\right.$$

$$\sinh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(z\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(h\sqrt{\frac{\omega}{2\nu}}\right)\Bigg] \div \left[\left\{\cosh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(h\sqrt{\frac{\omega}{2\nu}}\right)\right\}^2 + \right.$$

$$\left.\left.\left\{\sinh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(h\sqrt{\frac{\omega}{2\nu}}\right)\right\}^2\right]\right\}\sin\omega t$$

$$+ U_0\left\{\left[\sinh\left(z\sqrt{\frac{\omega}{2\nu}}\right)\cosh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(h\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(h\sqrt{\frac{\omega}{2\nu}}\right) - \cosh\left(z\sqrt{\frac{\omega}{2\nu}}\right)\right.\right.$$

$$\sinh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(z\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(h\sqrt{\frac{\omega}{2\nu}}\right)\Bigg] \div \left[\left\{\cosh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\cos\left(h\sqrt{\frac{\omega}{2\nu}}\right)\right\}^2 + \right.$$

$$\left.\left.\left\{\sinh\left(h\sqrt{\frac{\omega}{2\nu}}\right)\sin\left(h\sqrt{\frac{\omega}{2\nu}}\right)\right\}^2\right]\right\}\cos\omega t$$

If $U_0 = 0$, the value of u agrees with that given by Carslaw & Jaeger[1].

DISCUSSION

Curves have been drawn showing the variation of u with z and t separately for the general case and for the two particular cases :

(*i*) when the body force X is absent,

(*ii*) when the plates are at rest.

The constants appearing in the value of u are assumed to be

$$X = 0\cdot5, \qquad U_0 = 5, \qquad h = 5, \qquad \nu = 1\cdot1$$

in C.G.S. system of units.

The graphical representation of u, as depicted in Fig. 1, shows that when X and U_0 are both different from zero, the velocity is least at the plates. It increases rapidly as the distance from the plates increases and attains the zero value at $z \simeq \pm 4$. After that it becomes positive and increases and finally attains its greatest value at $z \simeq \pm 1\cdot2$. It then decreases, the velocity curve having a minimum at $z = 0$.

When the body force is absent, the pattern of variation of u with z is almost the same as that for the general case. The increase in velocity in this case is, however, not so rapid and the velocity is zero for $z \simeq \pm 2\cdot9$ and $z \simeq \pm 0\cdot6$.

For the case (*ii*), i.e. when $U_0 = 0$ and $X \neq 0$, the velocity is zero at the plates and increases gradually with the distance from the plates. The velocity profile curve is nearly parabolic.

The curves are symmetrical about $z = 0$. Further, it is obvious from the graphs that the effect of body force is to increase the velocity and that this effect is greatest at $z = 0$.

Fig. 2 shows the variation of velocity with time in the central region between the plates. It is seen that the velocity is zero when $t = 0$. When U_0 and X are both different from zero,

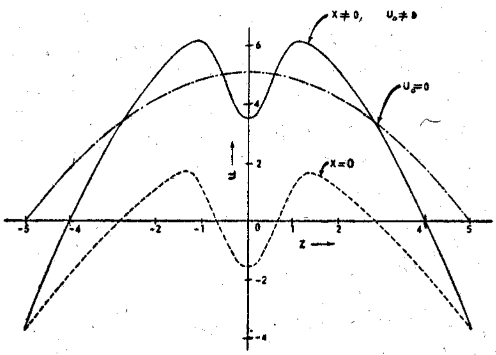

Fig. 1—Variation of u with z (t=20)

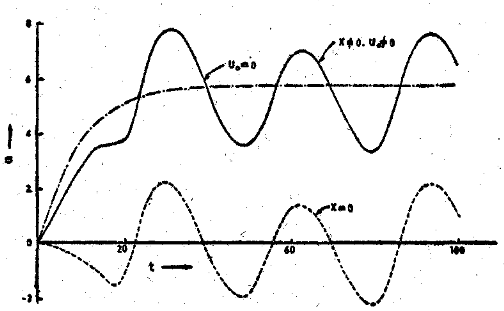

Fig. 2—Variation of u with t (z=0)

the velocity increases rapidly as t increases from $t = 0$ to $t \simeq 14$. After that it remains stationary for a short while and then shoots up rapidly to follow an oscillatory pattern. The oscillations are permanent in nature and almost alike.

In the particular case when $X = 0$, the velocity remains negative for some time and thereafter follows an oscillatory pattern, increasing and then decreasing as t increases.

When $U_0 = 0$ and $X \neq 0$, the velocity increases rapidly from zero value at $t = 0$ and after some time ($t \simeq 40$ in this case) attains a value which subsequently remains nearly constant.

We thus infer that the growth of velocity is rapid in the early stages, but soon it tends to follow a steady pattern.

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