UNSTEADY FLOW OF A DUSTY VISCOUS LIQUID THROUGH CIRCULAR CYLINDER

P. SAMBA SIVA RAO

Evening College, Warangal (A.P.)

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The laminar flow of an unsteady viscous liquid with uniform distribution of dust particles through a circular cylinder under the influence of exponential pressure gradient has been investigated. Two interesting cases have been discussed and analytical expressions for velocities of fluid and dust particles are obtained.

In the recent years the attention of researchers in fluid dynamics has been diverted towards study of the influence of dust particles on the motion of fluids. Saffman[1] has discussed the stability of the laminar flow of a dusty gas in which the dust particles are uniformly distributed. Michael[2] has considered the Kelvin-Helmohz instability of the dusty gas. Later, Michael & Miller[3] have investigated the motion of dusty gas with uniform distribution of the dust particles occupied in the semi-infinite space above a rigid plane boundary. Two cases when the plane moves parallel to itself, (*i*) Simple Harmonic Motion and (*ii*) impulsively from rest with uniform velocity, have been discussed. In view of such interest in this aspect of study of the subject the laminar flow of an unsteady viscous liquid with uniform distribution of dust particles through circular cylinder, under the influence of exponential pressure gradient, has been discussed. Analytical expressions for the velocities of the fluid and dust particles in two different cases are obtained.

EQUATIONS OF MOTION

The equations of motion of a dusty, unsteady, viscous, and incompressible fluid are[3] :

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla \vec{u}) = - \frac{1}{\rho} \nabla p + \gamma \nabla^2 \vec{u} + \frac{KN}{\rho} (\vec{v} - \vec{u}) \tag{1}$$

$$m \left[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = K (\vec{u} - \vec{v}) \tag{2}$$

$$\operatorname{div} \vec{u} = 0 \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} (N \vec{v}) = 0 \tag{4}$$

Where $\vec{u}$, $\vec{v}$ denote the velocity vectors of fluid and dust particles respectively, p the fluid pressure, m the mass of dust particles, N the number density, K the Stokes resistance coefficient (for spherical particles of radius ϵ is $\sigma \pi \mu \epsilon$), μ the viscocity of fluid, ρ the density and ν the kinematic coefficient of viscocity.

FORMULATION AND SOLUTION OF THE PROBLEM

We shall investigate the laminar flow of an unsteady viscous liquid with uniform distribution of dust particles, through a circular cylinder of radius a, under the influence of exponential pressure gradient. Since both the dust and fluid particles move along the length of the cylinder, the motion is symmetrical along the axis and the distribution of dust particles is uniform. The velocity distributions of fluid and dust particles are defined respectively as :

$$u_1 = 0\,, \quad u_2 = 0\,, \quad u_3 = w_1\,(r\,,\,t) \tag{5}$$

$$v_1 = 0\,, \quad v_2 = 0\,, \quad v_3 = w_2\,(r\,,\,t) \tag{6}$$

$$N = N_0 \quad \text{a constant} \tag{7}$$

where $(u_1\,,\,u_2\,,\,u_3)$ and $(v_1\,,\,v_2\,,\,v_3)$ are velocity components of fluid and dust particles.

Using (r, θ, z) coordinates, the equations (3) and (5), (6) and (7) and equations (1) and (2) can be expressed as:

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{8}$$

$$\frac{\partial w_1}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r}\right) - \frac{KN_0}{\rho}\,(w_2 - w_1) \tag{9}$$

$$m\,\frac{\partial w_2}{\partial t} = K\,(w_1 - w_2) \tag{10}$$

Taking $R = \frac{r}{a}$ and eliminating w_2 from (9) and (10), we get :

$$\frac{\partial^2 w_1}{\partial t^2} = -\frac{1}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right) + \frac{\nu}{a^2}\frac{\partial}{\partial t}\,(\nabla_1^2\,w_1) - \frac{KN_0}{\rho}\frac{\partial w_1}{\partial t} - \frac{K}{m}\left(\frac{\partial w_1}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\nu}{a^2}\,\nabla_1^2\,w_1\right) \tag{11}$$

where $\nabla_1^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R}$ and a is radius of the cylinder.

From (8) and (9) we have

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = \phi\,(t)$$

Since we have assumed pressure gradient to be exponential we can take

$$\varphi\,(t) = \alpha\,e^{-\lambda^2 t} \tag{12}$$

$$w_1\,(R, t) = f\,(R)\,e^{-\lambda^2 t} \tag{13}$$

$$w_2\,(R, t) = g\,(R)\,e^{-\lambda^2 t} \tag{14}$$

where α and λ are real constants. Since both the fluid and dust have no slip at the wall of the cylinder and their velocities are finite on the axis of cylinder, we have the following boundary conditions :

$$f\,(1) = 0 \tag{15}$$

$$g\,(1) = 0 \tag{16}$$

and both f and g are finite on the axis of cylinder. Introducing following dimentionless parameters,

$$n^2 = \frac{a^2\,\lambda^2}{\nu}\left(1 + \frac{1}{1 - \tau\lambda^2}\right)$$

$$\Omega = \frac{\alpha}{\lambda^2}\left(\frac{1 - \tau\lambda^2}{l + 1 - \tau\lambda^2}\right)$$

where $\tau = \dfrac{m}{K}$ is the relaxation time of dust particles ; $l = \dfrac{N_0 m}{\rho}$ is the mass concentration of the dust particles.

Using relations (12) to (14), equations (10) and (11) are simplified to

$$g\,(1 - \tau\lambda^2) = f \tag{17}$$

$$\frac{d^2 f}{dR^2} + \frac{1}{R}\frac{df}{dR} + n^2\,(f + \Omega) = 0 \tag{18}$$

The solution of equation (18) is

$$f\,(R) = A\,J_0\,(nR) + BY_0\,(nR) - \Omega$$

where J_0 and Y_0 are the first and second kind of Bessal functions of order zero[4] and A, B are constants to be determined, subject to the boundary conditions mentioned above.

Using boundary conditions we have

$$f\,(R) = \Omega\left[\frac{J_0\,(nR)}{J_0\,(n)} - 1\right] \tag{19}$$

Now we shall discuss two interesting cases of very small and very large value of n.

Case I: When $|\,n\,|$ *is very small.*

We have following asymptotic values[4]

$$J_0\;(nR) \approx \left(1 - \frac{n^2\;R^2}{4}\right)$$

$$J_0\,(n) \simeq \left(1 - \frac{n^2}{4}\right) \tag{20}$$

By virtue of these equations, (19) is expressed as

$$f\,(R) = \Omega\,\frac{n^2\,(1 - R^2)}{(4 - n^2)}$$

Therefore w_1 and w_2 can be expressed as

$$w_1\,(R, t) = \left(\frac{\Omega\;n^2}{4 - n^2}\right)(1 - R^2)\,e^{-\lambda^2 t} \tag{21}$$

$$w_2(R,t) = \frac{\Omega\, n^2 (1 - R^2)\, e^{-\lambda^2 t}}{(4 - n^2)(1 - \tau\lambda^2)} \qquad (22)$$

These are the expressions for velocities of fluid and dust particles in this case.

Case II : When | n | is very large

Taking $n^2 = -n'^2$ the solution of differential equation (18) can be expressed as

$$f(R) = C I_0(n'R) + D K_0(n'R) - \Omega \qquad (23)$$

where I_0 and K_0 are modified Bessel functions of 1st and 2nd kind of order zero[4], C and D are constants to be determined.

Using boundary conditions (23) can be written as

$$f(R) = \Omega \left[-\frac{I_0(n'R)}{I_0(n)} - 1 \right] \qquad (24)$$

Since $| n' |$ is very large the asymptotic value of $I_0(n'R)$ and $I_0(n')$ are given by[4]

$$I_0(n'R) = \frac{e^{n'R}}{\sqrt{2\pi n'R}}$$

$$I_0(n') = \frac{e^{n'}}{\sqrt{2\pi n'}} \qquad (25)$$

In this case the velocity components are obtained as

$$w_1(R,t) = \frac{\Omega}{\sqrt{R}} \left[e^{n'(R-1)} - \sqrt{R} \right] e^{-\lambda^2 t}$$

$$w_2(R,t) = \frac{\Omega}{\sqrt{R}\,(1 - \tau\lambda^2)} \left[e^{n'(R-1)} - \sqrt{R} \right] e^{-\lambda^2 t} \qquad (26)$$

CONCLUSION

From equations (21) and (22) it is seen that both the fluid and dust particles which are nearer to the axis of the cylinder, move with greater velocity in the first case. Since τ, λ^2 are positive, the velocity of dust particles is more than that of fluid particles. When the dust is very fine, the relaxation time of dust particles decreases and ultimately as $\tau \rightarrow 0$ the velocity of dusty fluid becomes that of clean fluid in both the cases.

If the masses of the dust particles are small, their influence and the fluid flow is reduced, and in the limit as $m \rightarrow 0$ the fluid becomes ordinary viscous, and we get the solution of the laminar flow of a viscous liquid through circular cylinder under the influence of exponential pressure gradient.

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