GRAVITATIONAL EFFECT ON THE SHAPE OF A BUBBLE FORMED BY AN UNDER-WATER EXPLOSION

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The shape of a gas bubble initially formed by an explosion or by some other means in a liquid just before it starts to move upwards has been determined by considering surface tension and the pressure on the interface. The external pressure is taken to be non-uniform due to the presence of gravity and the internal pressure is assumed to be constant. Two cases have been considered. In the first case of a two-dimensional bubble an exact solution has been obtained whereas in the case of a three-dimensional bubble only an approximate solution could be found.

There are various theories to describe and explain the different phases of an explosion under water, such as detonation process, shock wave propagation and oscillations of the gas bubble. The gas bubble after a few milliseconds of detonation or formed otherwise starts migrating upwards or towards a rigid boundary in the vicinity. Generally the shape of this migrating gas bubble is determined by ignoring the force due to gravity or in other words assuming the pressure field outside the gas bubble to be uniform, i.e. taking the shape to be spherical. But due to the presence of gravity the outside pressure field is non-uniform and is responsible for the distortion of the shape of the bubble. Experimental observations of Davidson & Schuler[1] also show that it is incorrect to assume the bubbles as they form to be spherical or circular. In this paper we have studied the resulting distortion and obtained the expressions for the shape of the bubble by taking into account surface tension and gravity.

Walters & Davidson[2] have examined the subsequent distortions in the case of a two-dimensional bubble formed in an inviscid liquid by assuming the shape to be circular. Since the forces due to surface tension are considered to be negligible, their theory would be applicable to the study of the motion of a large rising bubble under the force of bouyancy.

In his attempt to determine the initial shape of the bubble, Moore[3] observed that due to the non-uniformity of the outside pressure it is very difficult to solve the equation

$$p_{01} - p_{02} = TJ \tag{1}$$

where J is the total curvature of the bubble surface, T the surface tension, p_{01} the constant internal pressure, and p_{02} the external non-uniform pressure. Therefore without solving (1) he assumed the bubble to be an ellipsoid of revolution. The assumption is not justified as the condition (1) is not satisfied at every point of the bubble surface. However, this has been made to satisfy by Moore at those points of the surface where the total curvature is maximum or minimum.

Recently, Rosenthal[4] has studied the shape of a bubble at the axis of a rotating fluid by considering the surface tension and the constant pressure difference at the interface, i.e. without taking into account the effect of gravity. The shape is assumed to be spherical when the angular velocity of the liquid is zero.

In this paper, the shape is determined by considering together the forces due to surface tension and gravity. This amounts to the solution of equation (1). The external pressure field is non-uniform due to gravity, whereas the internal pressure is assumed to be constant. The problem has been examined for two cases when the bubble is in the static equilibrium. In Section 1, the case of a two-dimensional bubble due to a line explosion is considered and an exact solution giving the bubble shape has been found out. The size and shape depends upon the parameter involving the internal pressure, the depth of the origin from the free surface and the surface tension. In Section 2, an approximate shape of a bubble formed by a point explosion has been obtained. This approximate solution clearly shows the effect of gravity on the shape of the bubble. In both the cases the force due to gravity manifests itself by flattening the lower portion of the bubble.

SECTION 1

Two-dimensional bubble

A horizontal line source in water generates a cylinderical bubble having vertical circular section in the absence of gravity. But due to gravity the pressure field outside the bubble is non-uniform. Thus in the presence of gravity the vertical section of the bubble surface will be a two-dimensional closed curve. Let the line source be along the x-axis, z be the vertical axis and y be the axis normal to the xz plane. Let p_{01} be the uniform pressure inside the bubble. The pressure outside the bubble can be written as

$$g\rho_2 (h - z) \quad \text{or} \quad p_{02} - g\rho_2 z$$

where g is the acceleration due to gravity, ρ_2 the density of the outer liquid and h is the depth of the line source from free surface.

The shape of the interface is expressed by balancing the forces due to surface tension and the pressure discontinuity at every point of the surface. If T is the interfacial surface tension and $\dfrac{z''}{(1 + z'^2)^{3/2}}$ is the curvature for a section of the bubble, we have

$$T \frac{z''}{(1 + z'^2)^{3/2}} = (p_{01} - p_{02}) + g\rho_2 z \tag{2}$$

where dashes denote the derivatives with respect to f.
Equation (2) can be rewritten as

$$-\frac{d}{dz}\left[1 + z'^2\right]^{-\frac{1}{2}} = \frac{p_{01} - p_{02}}{T} + \frac{g\rho_2 z}{T} \tag{3}$$

Integrating we have

$$-\left[1 + z'^2\right]^{-\frac{1}{2}} = \frac{(p_{01} - p_{02})\, z}{T} + \frac{g\rho_2 z^2}{2T} \tag{4}$$

The constant of integration is zero as when

$$z \to 0, \frac{dz}{df} \to \infty$$

Squaring (4) we get

$$[1 + z'^2]^{-1} = (Az + Bz^2)^2 \tag{5}$$

where $A = \frac{p_{01} - p_{02}}{T}$ and $B = \frac{g\rho_2}{2T}$

Equation (5) can be reduced to the elliptic integrals of the first and second kind by making the substitution

$$Az + Bz^2 = \cos 2\phi \tag{6}$$

$$f = \pm \left[\frac{\sqrt{A^2 + 4B}}{2B} \int_{\pi/4}^{\phi} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}}\, d\phi - \frac{A^2}{2B\sqrt{A^2 + 4B}} \int_{\pi/4}^{\phi} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}}\, d\phi \right] + C \tag{7}$$

where $k^2 = \frac{8B}{A^2 + 4B}$ and C is the constant of integration. When $\phi = 0$, $f = 0$, therefore

$$C = \pm \left[\frac{\sqrt{A^2 + 4B}}{2B} \int_{0}^{\pi/4} (1 - k^2 \sin^2 \phi)^{\frac{1}{2}}\, d\phi - \frac{A^2}{2B\sqrt{A^2 + 4B}} \int_{0}^{\pi/4} (1 - k^2 \sin^2 \phi)^{-\frac{1}{2}}\, d\phi \right] \tag{8}$$

Equation (7) can be written in more abridged form as

$$f = \pm \frac{1}{k} \sqrt{\frac{2}{B}} \left[E(\phi, k) - \frac{1}{2} (2 - k^2) F(\phi, k) \right] \tag{9}$$

where $F(\phi, k)$ and $E(\phi, k)$ are elliptic integrals of first and second kind respectively. Writing z from (6) in terms of k we have

$$z = \frac{1}{k\sqrt{B}} \left[\sqrt{2 - 2k^2 \sin^2 \phi} - \sqrt{2 - k^2} \right] \tag{10}$$

f and z can be made dimensionless by introducing

$$\eta = \sqrt{B} f \text{ and } \xi = \sqrt{B}\, z$$

$$\eta = \pm \frac{\sqrt{2}}{k} \left[E(\phi, k) - \frac{1}{2}(2 - k^2) F(\phi, k) \right] \tag{11}$$

$$\xi = \frac{1}{k}\left[\sqrt{2 - 2k^2 \sin^2 \phi} - \sqrt{2 - k^2}\right] \quad (12)$$

Since there is no other external force except gravity, the bubble will be symmetrical about the vertical ξ-axis. This is also evident from the values of η and ξ given by (11) and (12) by the fact that for a particular value of ξ there corresponds two values of η, which are equal in magnitude but opposite in sign.

We can get different shapes of the bubble by giving various values to the parameter $k \left(= \frac{8}{A^2/B + 4}\right)$. These will correspond to the different values of A only as B is a physical constant depending upon gravity and surface tension. It is also observed that the size of the bubble is inversely proportional to the square of the pressure difference on the boundary, and in the limiting case as $k \rightarrow 0$ or $A^2/B \rightarrow \infty$ in (11) and (12), the resulting bubble reduces to a point only, which is an ideal situation. This is otherwise obvious from equation (2) also that a point bubble being of infinite curvature will be able to withhold infinite pressure differences. This can be verified from Fig. 1 that the flattening of the lower portion increases as the ratio A^2/B decreases (or as k increases) and for larger values of A^2/B the shape of the bubble becomes circular. We get from (7) that for f to be real, k has to be less than unity always or the ratio A^2/B will always be greater than 4.

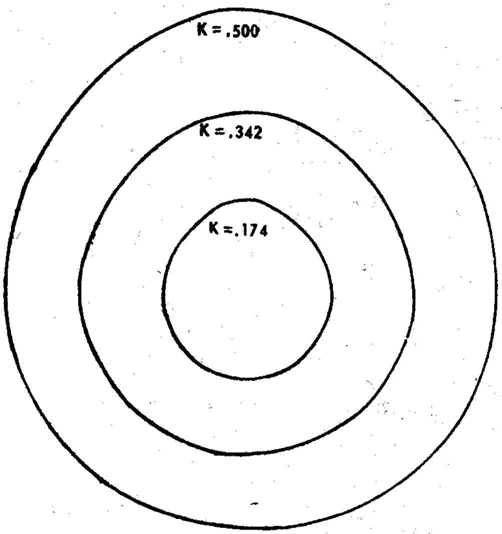


Fig. 1—Shapes of bubble for various values of k.

The distortion in the shape of the bubble will be appreciable when A, which is directly proportional to the pressure difference i.e. $p_{01} - g\rho_2 h$, is small. This will correspond to various physical situations of the bubble, For instance, when it is near the free surface and inside pressure is also quite small, i.e. when the size of the bubble is quite large. This is otherwise also quite clear as in such a case the variation in hydrostatic pressure over the bubble boundary is quite significant.

SECTION II

Here the shape of the bubble will obviously be a surface of revolution. Keeping in view the non-uniformity of the pressure distribution outside the bubble the governing equation of the bubble surface will be

$$\frac{d}{df}\left[\frac{f}{(1 + f'^2)^{\frac{1}{2}}}\right] = \frac{(p_{01} - p_{02})f}{T} + \frac{g\rho_2 z f}{T} \quad (13)$$

where $\frac{1}{f}\frac{d}{df}\left[\frac{f}{(1+f'^2)^{\frac{1}{2}}}\right]$ is the total curvature for the surface of revolution $r=f(z)$

and the notations are the same as those in the first case.

Equation (13) can be made dimensionless by introducing the following set of variables

$$\eta = \frac{f}{R},\ \xi = \frac{z}{R} \text{ where } \frac{p_{01}-p_{02}}{T} = \frac{2}{R} \text{ and } \frac{g\rho_2}{2T} = \frac{1}{b_0^2} \tag{14}$$

where $\frac{2}{R}$ is the total curvature at $z = 0$ when gravitational force is ignored.

Equation (13) in the non-dimensional form can be written as

$$\frac{1}{\eta}\frac{d}{d\eta}\left[\frac{\eta}{(1+\eta'^2)^{\frac{1}{2}}}\right] = 2 + 2\epsilon\xi \tag{15}$$

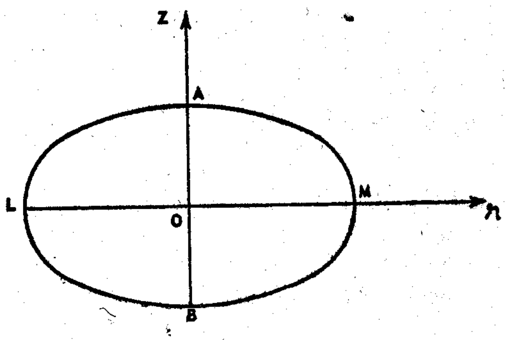
where $$\epsilon = \frac{2g\rho_2 T}{(p_{01}-p_{02})^2} = \frac{R^2}{b_0^2} \text{ and } \eta' = \frac{d\eta}{d\xi} \tag{16}$$

We have found it rather difficult to get the exact expression for ξ as a function of η by solving (15). This difficulty is due to the presence of ξ in the right hand side of (15).

However, a solution which satisfies (15) approximately has been found out.

Since the interface is a surface of revolution, let it meet the axis of the bubble (z-axis) at A and B.

The lower portion of the bubble surface will be flattened due to gravity or in other words the curvature at A will be greater than the curvature at B, as shown in the Figure.



Thus the shapes of bubble near the points A and B can be assumed to be ellipsoids of revolution as :

$$\frac{\eta^2}{k^2} + \frac{\xi^2}{\alpha^2} = 1 \tag{17}$$

$$\frac{\eta^2}{l^2} + \frac{\xi^2}{\beta^2} = 1 \tag{18}$$

where $| OA | = \alpha$ and $| OB | = \beta$ and k is the radius of the horizontal circular section through the origin. When the gravity effect is neglected, (17) and (18) coincide with a sphere of unit radius i.e. $k = \alpha = \beta = 1$ and the subsequent effect of gravity is manifested in the changed values of these parameters.

The justification in assuming the forms (17) and (18) lies in the fact that the curvatures of these ellipses at $\eta = 0$ are of the same order as those of the section through the axis of the bubble surface (15). The values of these curvatures are $1 + \epsilon\alpha$ and $1 - \epsilon\beta$ at A and B respectively. The values of α and β can be determined by comparing (17) and (18) with equation (15) in the neighbourhood of $\eta = 0$. Thus we find that

$$\alpha = \frac{k^2}{1 - \epsilon k^2} \text{ and } \beta = \frac{k^2}{1 + \epsilon k^2} \tag{19}$$

It may also be added that at $\xi = 0$, $\frac{d\eta}{d\xi}$ comes out to be the same when derived from (17) and (18). But the curvatures of these ellipses at $\xi = 0$ are not the same. So by using (15), η has been expanded in the neighbourhood of $\xi = 0$, in the ascending powers of ξ.

This gives :

$$\eta = k + \frac{\xi^2}{2!}\left(\frac{1}{k} - 2\right) - 2\epsilon \frac{\xi^3}{3!} + \frac{\xi^4}{4!}\left(\frac{1}{k^3} - \frac{12}{k^2} + \frac{32}{k} - 24\right) - \ldots \tag{20}$$

where $\eta\,(0) = k$

Since k depends upon ϵ only, its value can be approximated by applying perturbation method to its initial value unity. In this way the value of k comes out to be

$$k = 1 - \frac{2\epsilon}{3} \tag{21}$$

where the powers of ϵ higher than the first have been neglected.

Considering this value of k, (20) can be rewritten as :

$$\eta = \left(1 - \frac{2\epsilon}{3}\right) - \frac{\xi^2}{2!}\left(1 - \frac{2\epsilon}{3}\right) - \frac{\xi^3}{3!}\,2\epsilon - \frac{\xi^4}{4!}\left(3 - \frac{22\epsilon}{3}\right) - \ldots \tag{22}$$

The values of α and β can also be expressed in terms of ϵ by substituting (21) in (19).

It is observed that in the neighbourhood of $\xi = 0$ values of η obtained from (22) approximately coincide with those given by (17) and (18). Thus these ellipsoids of revolution satisfy equation (15) not only in the neighbourhood of $\eta = 0$ but also are fairly good approximations throughout the contour i.e. for $0 < \eta \leq k$.

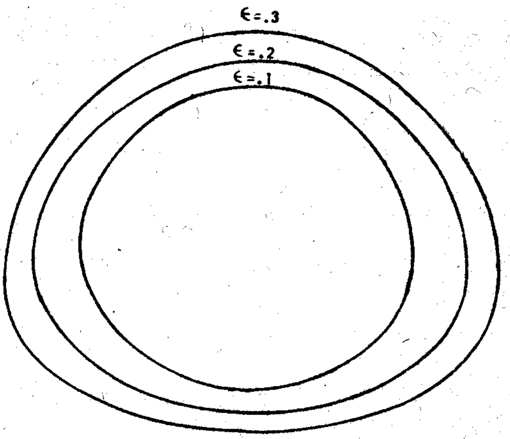


Fig.2—Shapes of bubble for various values of ϵ.

It is apparent from equation (15) that the curvature of the bubble surface decreases when ξ assumes negative values. This is borne out of the fact that the force of gravity will tend to flatten the lower portion of the bubble. In order that (15) should represent a closed contour, the curvature should not change its sign and this is possible only when $\epsilon\xi < 1$ where ξ is negative for the lower portion. As $\epsilon \rightarrow 0$ i.e. when the effect of gravitational field is neglected, (15) represents a spherical bubble. The effect of gravity becomes more significant at certain situations depending on the size and location of the bubble. These situations can be studied by giving various values to ϵ.

It can be verified from Fig. (2) that as the value of ϵ is increased from ·1 to ·3, the flattening of the lower portion also, increases. Also, it is observed from Table 1 that with the increase of ϵ, which is a consequence of the fall of pressure difference at the interface, the volume of the bubble also increases. The maximum effect of the gravitational field is realised in that physical situation where the size of the bubble is quite significant and it is situated near the free surface where the variations in the gravitational force over the bubble boundary are maximum.

TABLE 1

NON = DIMENSIONALISED VALUES OF α, β, k & V FOR VARIOUS VALUES OF ϵ

ϵ	$\frac{R}{b_0}\alpha$	$\frac{R}{b_0}\beta$	$\frac{R}{b_0}k$	$\frac{R^2}{b_0^3}V$
·1	·30567	·24243	·29513	·10210
·2	·41730	·23851	·38750	·20420
·3	·49294	·16431	·43817	·26153

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