MINIMAL TRANSFER TRAJECTORIES WITH OPTIMUM CORRECTIONAL MANOEUVRE PROGRAMME

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Inter-orbital transfer trajectories with correctional manoeuvres are investigated with the object to reduce the energy consumption to the minimum in the transfer operation. Various cases arising out of the required missions have been analysed and it is demonstrated that cotangential Hohmann type trajectories are not minimal in transfers with correctional manoeuvres contrary to the case of inter-orbital transfers without correctional manoeuvres. Transfers with no corrective thrusts have also been discussed and laws for optimum heading angles of the space vehicle are derived.

NOMENCLATURE

V = velocity

γ = heading angle, that is, the angle between V and perpendicular to the radius vector at the point

r = radius vector

θ = vectorial angle

h = twice the aerial velocity

E = eccentricity

α = angle of inclination of the major axis of the transfer trajectory measured with respect to the reference line, that is, the line passing through the force centre (focus) and the pericentre of the launch orbit

K = gravitational constant times mass of the attracting body

T = total transfer time from the launch to the destination point

t = time

$\triangle V$ = impulsive velocity change

Subscripts

1—denotes values at the launch point

2—denotes values at the destination point

l—relates to values corresponding to space vehicle at the launch point

d—relates to values corresponding to space vehicle at the destination point

p—relating to pericentre

a—relating to apocentre

INTRODUCTION

Because of unavoidable operational imprecision at the time of launch, the path of a space vehicle is observed to diverge from its precomputed trajectory and correctional thrusts are therefore applied during flight to annul the vehicle divergence. An optimal programme for the correctional manoeuvres so as to consume the least propellant has been investigated[1] by minimizing the mean characteristic velocity of the correctional manoeuvres $^1\overline{W}$. It is shown that

$$W_{min} \doteqdot ke \log \left(\frac{m}{k} \frac{\tau_0}{s} \right) \tag{I}$$

where m is the statistical mean value of the error in the impulsive velocity change brought about at the time of launch, k the statistical mean value of the subsequent correctional manoeuvres, τ_0 the total time of flight of the vehicle from the given launch to the destination point, s (a preassigned value) represents the time interval between final correctional manoeuvre and arrival of the vehicle at the target, and e is the base of the natural logarithms. Prior to launching, m and k are assumed to have been determined.

If, however, $m > ke$, then

$$\overline{W}_{min} \doteqdot \frac{\tau_0}{\tau_1} m + ke \log \left(\frac{\tau_1}{s} \right) \tag{II}$$

where τ_1 represents the time interval between the first correctional manoeuvre and arrival of the vehicle at the target and is a predetermined quantity. Since in this case the first correction takes place as soon after launching as is practicable, τ_1 can be taken practically equal to τ_0 in relation (II) which can be thus written as

$$\overline{W}_{min} \doteqdot m + ke \log \left(\frac{\tau_0}{s} \right) \tag{III}$$

Since m is governed by the operational error at the launch instant and k by the imprecision in the divergence detection and correctional manoeuvre instruments, it is logical to assume as a consequence of results (I) and (III) that for similar launch operation procedure and same set of detection and correction instruments with some preassigned acceptable value to s for all ballistic transfer trajectories between two specified launch and destination points, the total transfer time τ_0 along a trajectory will serve as a measure of the corresponding W_{min} along that trajectory.

In the transfer operation, energy will have to be spent in imparting impulsive velocity changes at the launch, the destination, in correctional manoeuvres during transfer phase. Our aim will be to find out the transfer trajectory which shall be most optimum in the sense of least expenditure of energy in the entire transfer operation. According to the mission requirements three cases can arise :

(A) "Quasi-minimal transfer trajectory of the first type" defined as that along which least energy is spent in launching operation and correctional manoeuvres.

(B) "Quasi-minimal transfer trajectory of the second type" defined as that along which least energy is spent in correctional manoeuvres and in the attainment of the orbital velocity at the destination by the space vehicle.

(C) "Minimal transfer trajectory" is that which requires the least energy consumption at the launch, in the correctional manoeuvres and at the destination.

Lastly, the problem of minimal transfer trajectories with no correctional manoeuvres arises as degenerate case of A), (B) and (C). The above mentioned problems have been investigated in the present paper.

QUASI-MINIMAL TRANSFER TRAJECTORY OF THE FIRST TYPE

If equation of the transfer trajectory be

$$h^2 u = K\,[1 + E \cos(\theta - \alpha)] \tag{1}$$

where $u = 1/r$

then by conservation of momentum principle

$$u^2 h \delta t = \delta\theta \tag{2}$$

Equations (1) and (2) yield

$$T = t_2 - t_1 = \frac{1}{K^2}\left(\frac{h}{\sqrt{1-E^2}}\right)^3 \left[2\tan^{-1}\left\{\left(\frac{1-E}{1+E}\right)^{\frac{1}{2}} \tan\left(\frac{\theta_2-\alpha}{2}\right)\right\} - 2\tan^{-1}\left\{\left(\frac{1-E}{1+E}\right)^{\frac{1}{2}} \tan\left(\frac{\theta_1-\alpha}{2}\right)\right\} - E\left(1-E^2\right)^{\frac{1}{2}} \left\{\frac{\sin(\theta_2-\alpha)}{1+E\cos(\theta_2-\alpha)} - \frac{\sin(\theta_1-\alpha)}{1+E\cos(\theta_1-\alpha)}\right\}\right] \tag{3}$$

The transverse and the radial components of the launch velocity can be expressed as

$$V_l \cos\gamma_l = \dot{\theta}_1/u_1 = h u_1 = \frac{K}{h}\left[1 + E\cos(\theta_1 - \alpha)\right] \tag{4}$$

$$V_l \sin\gamma_l = -\dot{u}_1/(u_1)^2 = -h\left(\frac{\partial u}{\partial\theta}\right)_{\theta=\theta_1} = \frac{K E u_1 \sin(\theta_1-\alpha)}{V_l \cos\gamma_l} \tag{5}$$

From equations (4) and (5), we have

$$\alpha = \theta_1 - \tan^{-1}\left[\frac{V_l^2 \sin\gamma_l \cos\gamma_l}{(V_l\cos\gamma_l)^2 - u_1 K}\right] \tag{6}$$

$$E = \left[\left(\frac{V_l^2}{u_1 K} - 1\right)^2 \cos^2\gamma_l + \sin^2\gamma_l\right]^{\frac{1}{2}} \tag{7}$$

The velocity change required for the space vehicle to adopt the transfer path will be

$$\triangle V_1 = [V_1^2 + V_l^2 - 2\,V_1 V_l \cos(\gamma_l - \gamma_1)]^{\frac{1}{2}} \tag{8}$$

The relationship between V_l and γ_l is given[2] by

$$V_l^2 = \frac{K u_1\,[\,1 - \cos(\theta_2 - \theta_1)]\sec^2\gamma_l}{[R_1 + \sin(\theta_2-\theta_1)\tan\gamma_l - \cos(\theta_2-\theta_1)]} \tag{9}$$

where $|\gamma_l| \leqslant \frac{\pi}{2}$ and $u_2/u_1 = R_1$

Substituting the value of V_l from equation (9) in equations (4), (6) and (7), we have

$$h = \left[\frac{K \{ 1 - \cos (\theta_2 - \theta_1)\}}{u_1 \{R_1 + \sin (\theta_2 - \theta_1) \tan \gamma_l - \cos (\theta_2 - \theta_1)\}} \right]^{\frac{1}{2}} \tag{10}$$

$$\alpha = \theta_1 - \tan^{-1} \left[\frac{1 - \cos (\theta_2 - \theta_1)}{(1 - R_1) \cot \gamma_l - \sin (\theta_2 - \theta_1)} \right] \tag{11}$$

$$E = [\{ 1 - \cos (\theta_2 - \theta_1)\}^2 \sec^2 \gamma_l - 2A \{ 1 - \cos (\theta_2 - \theta_1)\} + A^2]^{\frac{1}{2}} / A \tag{12}$$

where $$A = R_1 + \sin (\theta_2 - \theta_1) \tan \gamma_l - \cos (\theta_2 - \theta_1)$$

For the minimal transfer path characterised by the minimum energy consumption at launch and in correctional manoeuvres $(\overline{W}_{min} + \triangle V_1)$ should be minimum. Hence taking into account the reasons given in the Introduction we have,

$$ke \frac{\partial (\log T)}{\partial \gamma_l} + \frac{\partial (\triangle V_1)}{\partial \gamma_l} = 0 \tag{13}$$

Using equation (3) and (8), equation (13) gives

$$\phi_1 (\gamma_l) = \frac{1}{K^2} \left(\frac{h}{\sqrt{1 - E^2}} \right)^3 \left[F (\theta)_{\theta = \theta_1} - F (\theta)_{\theta = \theta_2} + G (\theta)_{\theta = \theta_2} - G (\theta)_{\theta = \theta_1} \right]$$
$$+ 3T \left[\frac{1}{h} \frac{\partial h}{\partial \gamma_l} + \frac{E}{(1 - E^2)} \frac{\partial E}{\partial \gamma_l} \right] + \frac{T}{ke \triangle V_1} \left[V_l \frac{\partial V_l}{\partial \gamma_l} \right.$$
$$\left. - V_1 \left\{ \frac{\partial V_l}{\partial \gamma_l} \cos (\gamma_l - \gamma_1) - V_l \sin (\gamma_l - \gamma_1) \right\} \right] = 0 \tag{14}$$

where

$$F (\theta) = \frac{1 + E}{1 + E + (1 - E) \tan^2 \left(\frac{\theta - \alpha}{2} \right)} \left[\frac{\partial \alpha}{\partial \gamma_l} \sec^2 \left(\frac{\theta - \alpha}{2} \right) \left(\frac{1 - E}{1 + E} \right)^{\frac{1}{2}} \right.$$
$$\left. + 2 \frac{\partial E}{\partial \gamma_l} \tan \left(\frac{\theta - \alpha}{2} \right) \frac{1}{(1 - E)^{\frac{1}{2}} (1 + E)^{3/2}} \right] \tag{15}$$

$$G (\theta) = \frac{E (1 - E^2)^{\frac{1}{2}}}{[1 + E \cos (\theta - \alpha)]^2} \left[\frac{\partial \alpha}{\partial \gamma_l} \left\{ E + \cos (\theta - \alpha) \right\} + \frac{\partial E}{\partial \gamma_l} \sin (\theta - \alpha) \right.$$
$$\left. \left\{ \cos (\theta - \alpha) - \left(1 + E \cos (\theta - \alpha) \right) \frac{1 - 2E^2}{E (1 - E^2)} \right\} \right] \tag{16}$$

From equations (9) to (12), we obtain

$$\frac{\partial V_l}{\partial \gamma_l} = \frac{K u_1}{V_l A^2} \left\{ 1 - \cos (\theta_2 - \theta_1) \right\} \sec^2 \gamma_l \left[A \tan \gamma_l - \frac{1}{2} \sin (\theta_2 - \theta_1) \sec^2 \gamma_l \right] \tag{17}$$

$$\frac{\partial h}{\partial \gamma_l} = \frac{h \sin(\theta_1 - \theta_2) \sec^2 \gamma_l}{2A} \tag{18}$$

$$\frac{\partial \alpha}{\partial \gamma_l} = \frac{[\{1 - \cos(\theta_2 - \theta_1)\}(R_1 - 1)\operatorname{cosec}^2 \gamma_l]}{[(1 - R_1)^2 \cot^2 \gamma_l - 2\{(1 - R_1)\cot \gamma_l \sin(\theta_2 - \theta_1) + \cos(\theta_2 - \theta_1) - 1\}]} \tag{19}$$

$$\frac{\partial E}{\partial \gamma_l} = \frac{\sec^2 \gamma_l}{A}\left[\frac{\{1 - \cos(\theta_2 - \theta_1)\}}{AE}\left\{\left(1 - \cos(\theta_2 - \theta_1)\right)\tan \gamma_l - \sin(\theta_2 - \theta_1)\right\} + \sin(\theta_2 - \theta_1)\left(\frac{1}{E} - E\right)\right] \tag{20}$$

Substitution from equations (3), (8), (9 to 12) and (17 to 20) into equation (14) will reduce equation (14) into an equation in the unknown γ_l which can be numerically solved to yield the optimum value of γ_l. Having known optimum γ_l, elements (h, α and E) and other parameters of the quasi-minimal transfer trajectory of the first type can be obtained.

If the launch point be the pericentre of the launch orbit and apocentre of the destination orbit the target (taking launch and destination orbits coplanar with major axes aligned),

$$u_1 = u_{1,p} \quad ; \quad \theta_1 = 0 \tag{21}$$

$$u_2 = u_{2,a} \quad ; \quad \theta_2 = \pi \tag{22}$$

Evidently then $\gamma_1 = 0$. Now if cotangential Hohmann type trajectory be the minimal transfer trajectory, $\gamma_l = 0$ must satisfy equation (14) obtained after substituting in it equations (21), (22) and $\gamma_1 = 0$. Putting equations (21), (22) and $\gamma_1 = \gamma_l = 0$ in the above respective equations we obtain

$$\left.\begin{aligned}
&V_l = \sqrt{\frac{2Ku_{1,p}}{R_{1,p}+1}} \quad \text{where } R_{1,p} = \frac{u_{2,a}}{u_{1,p}} && (i)\\
&h = \left[\frac{2K}{u_{1,p}(R_{1,p}+1)}\right]^{\frac{1}{2}}, \quad \alpha = 0 && (ii)\\
&E = \frac{1 - R_{1,p}}{1 + R_{1,p}}, \quad T = \frac{\pi}{K^2}\left[\frac{K(R_{1,p}+1)}{2R_{1,p}u_{1,p}}\right]^{3/2} && (iii)\\
&\triangle V_1 = \sqrt{\frac{2Ku_{1,p}}{R_{1,p}+1}} - V_1, \quad \frac{\partial \alpha}{\partial \gamma_l} = \frac{2}{R_{1,p}-1} && (iv)\\
&\frac{\partial h}{\partial \gamma_l} = \frac{\partial V_l}{\partial \gamma_l} = \frac{\partial E}{\partial \gamma_l} = 0 && (v)
\end{aligned}\right\} \tag{23}$$

Substituting equations (23) in equations (15) and (16)

$$\left.\begin{aligned}
&F(\theta)_{\theta=0} = \frac{2(R_{1,p})^{\frac{1}{2}}}{R_{1,p}-1}, \quad F(\theta)_{\theta=\pi} = \frac{2}{(R_{1,p})^{\frac{1}{2}}(R_{1,p}-1)} && (i)\\
&(R_{1,p})^{-1}\,G(\theta)_{\theta=0} = -G(\theta)_{\theta=\pi} = -\frac{2}{(R_{1,p})^{\frac{1}{2}}(R_{1,p}+1)} && (ii)
\end{aligned}\right\} \tag{24}$$

Substitution of equations (23) and (24) in equation (14) gives

$$\phi_1(0) = \left[\frac{R_{1,p}+1}{2K\,R_{1,p}} \right]^{\frac{1}{2}} \frac{2(R_{1,p}+1)}{(R_{1,p}\,u_{1,p})^{\frac{3}{2}}} \neq 0$$

Hence it is proved that cotangential Hohmann type trajectory is not the minimal transfer trajectory unlike the case of ballistic transfer without correctional manoeuvres.

(B) QUASI—MINIMAL TRANSFER TRAJECTORY OF THE SECOND TYPE

In this case the velocity change required for the space vehicle to attain the orbital velocity V_2 of the destination orbit at the target will be

$$\triangle V_2 = [\,V_2^2 + V_d^2 - 2\,V_2 V_d\,\cos(\gamma_d - \gamma_2)]^{\frac{1}{2}} \tag{25}$$

We can show that

$$\tan\gamma_d = \frac{KE}{u_2 h^2}\sin(\theta_2-\alpha) = \frac{\sin(\theta_2-\alpha)}{R_1\sin(\theta_1-\alpha)}\tan\gamma_l \tag{26}$$

Also from dynamical relationship

$$V_d^2 = V_l^2 + 2K(u_2 - u_1) \tag{27}$$

Combining equations (25) to (27), we have

$$\triangle V_2 = \Big[V_2^2 + V_l^2 + 2K(u_2-u_1) - 2V_2\{V_l^2 + 2K(u_2-u_1)\}^{\frac{1}{2}} \cos\left\{\tan^{-1}\left(\frac{\sin(\theta_2-\alpha)}{R_1\sin(\theta_1-\alpha)}\tan\gamma_l\right) - \gamma_2\right\}\Big]^{\frac{1}{2}} \tag{28}$$

For minimal transfer trajectory

$$ke\frac{\partial(\log T)}{\partial\gamma_l} + \frac{\partial(\triangle V_2)}{\partial\gamma_l} = 0 \tag{29}$$

which by equations (3), (25) and (27) is transformed into

$$\phi_2(\gamma_l) = C(\gamma_l) + \frac{T}{ke\triangle V_2}\Big[V_l\frac{\partial V_l}{\partial\gamma_l}\left(\frac{1 - V_2\cos(\gamma_d-\gamma_2)}{\{V_l^2+2K(u_2-u_1)\}^{\frac{1}{2}}}\right) + \left\{V_l^2 + 2K(u_2-u_1)\right\}^{\frac{1}{2}}\cdot V_2\frac{\partial\gamma_d}{\partial\gamma_l}\sin(\gamma_d-\gamma_2)\Big] = 0 \tag{30}$$

where

$$C(\gamma_l) = \frac{1}{K^2}\left(\frac{h}{(1-E^2)^{\frac{1}{2}}}\right)^3\left[F(\theta)_{\theta=\theta_1} - F(\theta)_{\theta=\theta_2} + G(\theta)_{\theta=\theta_2} - G(\theta)_{\theta=\theta_1}\right] + 3T\left[\frac{\partial h}{h\partial\gamma_l} + \frac{E}{(1-E^2)}\frac{\partial E}{\partial\gamma_l}\right]$$

Substituting from equations (9), (17), (26) and (28) in equation (30), the latter is an equation in the unknown γ_l which after numerical solution will give the optimum heading angle γ_l and hence elements and other parameters of the transfer trajectory can be evaluated. Proceeding in the manner as illustrated in case (i), it can be shown that in this case also cotangential Hohmann type trajectory is not the minimal transfer trajectory.

(C) MINIMAL TRANSFER TRAJECTORY

In order that the transfer trajectory be minimal

$$ke \frac{\partial (\log T)}{\partial \gamma_l} + \frac{\partial (\triangle V_1)}{\partial \gamma_l} + \frac{\partial (\triangle V_2)}{\partial \gamma_l} = 0 \tag{31}$$

Using equations (3), (8), (25) and (27), equation (31) yields

$$\phi_3 (\gamma_l) = C (\gamma_l) + \frac{T}{ke} \frac{\partial V_l}{\partial \gamma_l} \left[\frac{1}{\triangle V_1} \left\{ V_l - V_1 \cos (\gamma_l - \gamma_1) \right\} + \frac{1}{\triangle V_2} \right.$$
$$\left. \left\{ \frac{V_l - V_2 V_l \cos (\gamma_d - \gamma_2)}{\{V_l^2 + 2K (u_2 - u_1)\}^{\frac{1}{2}}} \right\} \right] + \frac{T}{ke} \left[\frac{V_1 V_l \sin (\gamma_l - \gamma_1)}{\triangle V_1} \right.$$
$$\left. + \frac{\partial \gamma_d}{\partial \gamma_l} \cdot \frac{V_2}{\triangle V_2} \left\{ V_l^2 + 2K (u_2 - u_1) \right\}^{\frac{1}{2}} \sin (\gamma_d - \gamma_2) \right] = 0 \tag{32}$$

from which optimum heading angle can be evaluated and hence elements of the minimal transfer trajectory and optimum launch velocity become known. From equation (26) we obtain,

$$\frac{\partial \gamma_d}{\partial \gamma_l} = \left(\frac{\cos \gamma_d}{u_2 h^2} \right)^2 \left[u_2 h^2 K \left\{ \sin (\theta_2 - \alpha) \frac{\partial E}{\partial \gamma_l} - E \cos (\theta_2 - \alpha) \frac{\partial \alpha}{\partial \gamma_l} \right\} \right.$$
$$\left. - 2KEu_2 h \sin (\theta_2 - \alpha) \frac{\partial h}{\partial \gamma_l} \right] \tag{33}$$

Substituting equations (21), (22) and

$$\gamma_1 = \gamma_2 = \gamma_l = 0 \tag{34}$$

in equations (26), (28), (33) and using equations (23) we obtain

$$\left.\begin{aligned} &\gamma_d = 0 \,, \ \triangle V_2 = V_2 - \left[2K (u_{2,a} - u_{1,p}) + \frac{2K u_{1,p}}{R_{1,p} + 1} \right]^{\frac{1}{2}} && (i) \\ &\text{and} \\ &\frac{\partial \gamma_d}{\partial \gamma_l} = - \frac{1}{R_{1,p}} && (ii) \end{aligned}\right\} \tag{35}$$

Hence by equations (23), (24), (34) and (35), equation (32) yields

$$\phi_3 (0) = \phi_1 (0) \neq 0$$

Therefore, again cotangential Hohmann type trajectory does not represent the minimal transfer trajectory.

MINIMAL TRAJECTORIES WITH NO CORRECTIONAL MANOEUVRES

The problem treated in ref. (2) with additional knowledge of the elements of the minimal transfer trajectory, comes out as a particular case of (A) when in equation (13) we drop the first term, and thus using equation (12), we have

$$V_l \frac{\partial V_l}{\partial \gamma_l} - V_1 \left[\frac{\partial V_l}{\partial \gamma_l} \cos(\gamma_l - \gamma_1) - V_l \sin(\gamma_l - \gamma_1) \right] = 0 \qquad (36)$$

Using equation (9), equation (36) will be transformed into a quartic equation in tangent γ_l which can be numerically solved for γ_l and therefrom elements and other parameters of the optimum transfer trajectory can be obtained.

Putting $\theta_2 - \theta_1 = \pi$ in equations (9) to (12), we have

$$\left.\begin{aligned} V_l &= L \sec \gamma_l & (i) \\ h &= L/u_1 & (ii) \\ \alpha &= \theta_1 - \tan^{-1}\left[\frac{2 \tan \gamma_l}{(1 - R_1)}\right] & (iii) \\ E &= \frac{[4 \tan^2 \gamma_l + (1 - R_1)^2]^{\frac{1}{2}}}{(R_1 + 1)} & (iv) \end{aligned}\right\} \qquad (37)$$

where

$$L = \left(\frac{2 K u_1}{R_1 + 1} \right)^{\frac{1}{2}} \qquad (38)$$

Using equations (36) and [37(i)], we have

$$\tan \gamma_l = \left(\frac{V_1 \sin \gamma_1}{L} \right) \qquad (39)$$

Having known γ_l, equations (37) then give the required elements of the minimal trajectory characterized by least exit energy at the launch point. Here L has a significant interpretation. Equation (38) suggests that L is the orbital velocity at the pericentre (Hohmann velocity) corresponding to the orbit whose pericentre is the launch point (u_1, θ_1) and apocentre the destination point $(u_2, \theta_1 + \pi)$. Hence from equation (39) we conclude that when $\theta_2 - \theta_1 = \pi$, then : Tangent of the optimum heading angle of the space vehicle at the launch point times the corresponding Hohmann velocity = Radial velocity corresponding to the launch orbit at the launch point.

Further, it is evident from the equation [37 (ii)] that the aerial velocity of the transfer trajectory is independent of the launch angle and is equal to half the launch radius vector times corresponding Hohmann velocity.

If $V_1 = 0$, equation (39) yields $\gamma_l = 0$. Hence by equation [37(i)], optimum V_l when $\theta_2 - \theta_1 = \pi$ and $V_1 = 0$ is given by L, a result derived in ref. (2).

Letting $\tan\beta = 2\tan\gamma_l/(1 - R_1)$ and combining equations (26) and [37(*iii*)], we have

$$\tan\gamma_d = -\frac{\tan\gamma_l}{R_1} \tag{40}$$

Hence when launch and destination points are separated by an angle π, tangent of the optimum heading angle of the space vehicle at the destination point = — (tangent of the optimum heading angle at the launch point times the ratio of the length of destination radius vector to that of the launch radius vector).

When $\theta_2 - \theta_1 = \pi$, taking the destination point as the launch point and vice versa (consequently replacing suffixes l and 1 by d and 2 respectively in the corresponding equations; R_2 evidently would mean $\frac{1}{R_1}$) equations (9) to (12) yield

$$\left.\begin{aligned}
V_d &= \left[\frac{2Ku_2}{1/R_1 + 1}\right]^{\frac{1}{2}} \sec\gamma_d = R_1\, L \sec\gamma_d && (i)\\
h &= L/u_1 && (ii)\\
\alpha &= \pi + \theta_1 - \tan^{-1}\left[\frac{2R_1\tan\gamma_d}{R_1 - 1}\right] && (iii)\\
E &= \frac{(2R_1\tan\gamma_d)^2 + (R_1 - 1)^2]^{\frac{1}{2}}}{R_1 + 1} && (iv)
\end{aligned}\right\} \tag{41}$$

Combining equation [41(*i*)] with equation (36) (obtained after changing the suffixes as mentioned earlier),

$$\tan\gamma_d = \frac{V_2\sin\gamma_2}{R_1\, L} \tag{42}$$

Having known γ_d, equations (41) determine the elements of the minimal trajectory requiring least entry energy at the destination point. A comparison of equations (37) with (41) leads to the interesting result that although in the two cases the minimal transfer paths differ but their aerial velocity remains the same.

Putting $V_2 = 0$, equation (42) gives $\gamma_d = 0$ which when substituted in equation [41 (*i*)] yields optimum $V_d = R_1 L$.

Let $\tan\delta = 2R_1\tan\gamma_d/(R_1 - 1)$. Substituting equation [41(*iii*)] in equation (26) and remembering the reverse picture of this case, we have

$$\tan\gamma_l = -R_1\tan\gamma_d \tag{43}$$

where now in equation (43), $\tan\gamma_d$ is given by equation (42). Equations (40) and (43) lead the conclusion that for both the cases the same relation holds good between the optimum heading angles of the space vehicle at the launch and destination points.

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