A NOTE ON THE DERIVATION OF THE BEST ESTIMATE OF RELIABILITY IN THE EXPONENTIAL CASE

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In this paper, the best estimate of reliability in the exponential case is obtained by deriving the distribution of an unbiased estimate conditional on a sufficient statistic. Pugh has also given this derivation. The derivation is interesting in itself and is obtained in this paper by the use of inversion theorem.

If a sample of size n is available from the exponential distribution

$$\frac{1}{\theta} e^{-t/\theta} \, dt \qquad 0 < t < \infty$$

the minimum variance unbiased estimate of $e^{-T/\theta}$, where T is a fixed quantity, is $\left[1 - \frac{T}{u}\right]^{n-1}$ where $t_1, t_2, \ldots\ldots\ldots, t_n$ are the sample values and $u = \sum_1^n t_i$. The result was first obtained by Pugh[1] and later by Enns[2] by a different method. Pugh used the Rao-Blackwell theorem[3] to obtain the estimate. It is well known that $u = \Sigma t_i$ is a sufficient statistic for θ and hence the minimum variance unbiased estimate of $e^{-T/\theta}$ will be obtained by taking the expectation of any unbiased estimate of $e^{-T/\theta}$, conditioned on u. Pugh has remarked that the method of obtaining the conditional expectation is a tedious one and he, therefore, verifies that $(1-T/u)^{n-1}$ is unbiased for $e^{-T/\theta}$ and concludes that it is the required estimate as it is a function of u alone. However, the derivation of the conditional expectation is interesting in itself and in this note a derivation of this result based on the inversion theorem in statistics is presented first. This has then been verified by obtaining the conditional distribution itself and its expectation. We have no way of knowing what method of derivation Pugh had in mind but ours appears to be a simple one.

This result is useful in reliability theory as $e^{-T/\theta}$ measures as the probability that an equipment following an exponential distribution does not fail in time T.

If r of the n value $t_1, t_2, \ldots\ldots\ldots, t_n$ are greater than or equal to T, it is obvious that r/n is an unbiased estimate of $e^{-T/\theta}$ as it represents the sample proportion of equipments not failing before time T.

CONDITIONAL EXPECTATION OF r/n, GIVEN u (THE SUFFICIENT STATISTIC)

To be specific let us suppose that $t_1, t_2, \ldots\ldots, t_{n-r}$ are the values which are less than T and $t_{n-r+1}, \ldots\ldots, t_n$ all exceed T. Consequently the joint distribution of r and t_i $(i = 1, 2, \ldots\ldots, n)$ is

$$f(r, t_1, t_2, \ldots\ldots t_n)\, dt_1 \ldots\ldots dt_n$$

$$= {}^nC_r \prod_{i=1}^{n} \left\{ \frac{1}{\theta} e^{-t_i/\theta}\, dt_i \right\}$$

$$0 \leqslant t_1, t_2 \ldots\ldots, t_{n-r} \leqslant T$$
$$T < t_{n-r+1} \ldots\ldots, t_n < \infty \qquad (1)$$
$$r = 0, 1, \ldots\ldots, n$$

The marginal distribution of r is obviously

$$p(r) = {}^nC_r \left\{ e^{-T/\theta} \right\}^r \left\{ 1 - e^{-T/\theta} \right\}^{n-r} \qquad (2)$$

$$r = 0, 1, \ldots\ldots, n.$$

Hence the characteristic function of u, conditional on r is

$$E\left(e^{iu\phi/r} \right) = \frac{{}^nC_r}{p(r)} \prod_{s=1}^{n-r} \left\{ \int_0^T e^{i\phi t_s} \cdot \frac{1}{\theta} e^{-t_s/\theta}\, dt_s \right\}$$

$$\cdot \prod_{s=n-r+1}^{n} \left\{ \int_T^{\infty} e^{i\phi t_s} \cdot \frac{1}{\theta} e^{-t_s/\theta}\, dt_s \right\}$$

$$= {}^nC_r \, \frac{A^r (1 - A)^{n-r}}{(1 - i\theta\phi)^n} \qquad (3)$$

where $\qquad A = \text{Exp} \{ -T(1 - i\theta\phi)/\theta \}$

Hence, applying the inversion theorem to $E(e^{iu\phi/r})$ and multiplying by $p(r)$, the joint distribution of u and r is

$$g(r, u)\, du = \frac{{}^nC_r}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{e^{-iu\phi} A^r (1-A)^{n-r}}{(1 - i\theta\phi)^n} d\phi \right\} du$$

$$r = 0, 1, \ldots\ldots, n \qquad (4)$$

But the marginal distribution of u is the well known distribution (see Pugh)

$$k(u)\, du = \frac{U^{n-1} e^{-u/\theta}\, d\theta}{\theta^n (n-1)} \qquad 0 < u < \infty \qquad (5)$$

Consequently the required conditional expectation is

$$E\left(\frac{r}{n}/u\right) = \sum_{r=0}^{n} \frac{r}{n} \cdot \frac{{}^nC_r}{2\pi k(u)} \int_{-\infty}^{\infty} \frac{e^{-iu\phi} A^r (-A)^{n-r}\, d\phi}{(1 - i\theta\phi)^n}$$

$$= \sum_{r=1}^{n} \frac{{}^{n-1}C_{r-1}}{2\pi k(u)} \int_{-\infty}^{\infty} \frac{e^{-iu\phi} A^r (1-A)^{n-r} d\phi}{(1-i\theta\phi)^n}$$

$$= \frac{1}{2\pi k(u)} \int_{-\infty}^{\infty} \frac{e^{-iu\psi} A \; d\phi}{(1-i\theta\phi)^n}$$

$$= \frac{e^{-T/\theta}}{2\pi k(u)} \int_{-\infty}^{\infty} \frac{e^{-i\varphi(u-T)}}{(1-i\theta\phi)} d\phi$$

$$= (1-T/u)^{n-1}, \text{ if } u > T \text{ and } = 0 \text{ otherwise.} \tag{6}$$

by a well known result in complex integration.

CONDITIONAL DISTRIBUTION OF r, GIVEN u

In the previous case we avoided the actual derivation of the conditional distribution of r, given u as we were interested only in the mean of this distribution. However, for the sake of completeness, we derive here the actual conditional distribution of r, given u. From (4) and (5), it is

$$h(r/u) = \frac{{}^{n}C_r}{k(u)} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-iu\phi} A^r (1-A)^{n-r} d\phi}{(1-i\theta\phi)^n} \tag{7}$$

Expanding $(1-A)^{n-r}$, we obtain

$$h(r/u) = \frac{{}^{n}C_r}{k(u)} \sum_{s=0}^{n-r} {}^{n-r}C_s (-1)^s \frac{e^{-(r+s)T/\theta}}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\varphi[u-(r+s)T]}}{(1-i\theta\phi)^n} d\phi$$

$$= {}^{n}C_r \sum_{s=0}^{n-r} {}^{n-r}C_s (-1)^s \left[1 - \frac{(r+s)T}{u}\right]^{n-1} \text{ if } u > nT$$

and $= 0$ otherwise , (8)

$r = 0, 1, \ldots\ldots, n$

As a check we calculate $E\left(\frac{r}{n}/u\right)$ from this also.

$$E\left(\frac{r}{n}\Big/u\right) = \sum_{r=0}^{n} \frac{r}{n} {}^{n}C_r \sum_{s=0}^{n-r} {}^{n-r}C_s (-1)^s \left[1 - \frac{(r+s)T}{u}\right]^{n-1}$$

$$= \sum_{k=0}^{n-1} {}^{n-1}C_k \left[1 - \frac{(k+1)T}{u}\right]^{n-1} \sum_{s=0}^{k} {}^{k}C_s (-1)^s , \tag{9}$$

where $k = r + s - 1$

However the last summation in (9) is $(1-1)^k$ and hence is zero unless k itself is zero. In that case, it is one and hence

$$E\left(\frac{r}{n}\Big/ u\right)=\left(1-\frac{T}{u}\right)^{n-1}$$

as it should be.

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