OPTIMUM EXIT TRAJECTORIES WITH SPECIFIED TRANSFER ANGLE

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Optimum trajectory with specified transfer angle between two terminals in an inverse square gravitational field is investigated, criterion of optimization being minimum velocity increment at the initial terminal. Results are numerically illustrated for two orbits in Earth's gravitational field.

Stark[1] has discussed the optimum trajectory between two terminals in an inverse square gravitational field, with the velocity being given only at the initial terminal. Stark defines an optimum trajectory as one which minimizes the single velocity increment which is applied at the initial terminal. Consider the initial terminal to be an arbitrary point on a given elliptic orbit in an inverse square gravitational field and the final terminal a point on another given elliptic orbit in the same field separated from the initial terminal by a specificed, transfer angle. The choice of two such terminals being infinite, out of the transfer trajectories between the two terminals, the problem of finding out the optimum transfer trajectory is of interest, the optimization criterion being the same as that of Stark. This problem is analysed in this paper and numerical results for the case of orbits in Earth's gravitational field are obtained.

NOMENCLATURE

r = radius vector

θ = vectorial angle

r_1 = radius vector corresponding to the initial terminal

r_2 = radius vector corresponding to the final terminal

θ_1 = vectorial angle corresponding to the initial terminal

e_1 = eccentricity of the initial orbit

e_2 = eceentricity of the final orbit

l_1 = semi-latus rectum of the initial orbit

l_2 = semi-latus rectum of the final orbit

α = angle between the major axes of the initial and final orbits

ϕ = transfer angle, that is, angle which the line joining the force centre (focus) to the initial terminal makes with the line joining the force centre to the final terminal

K = Gravitation constant times mass of the Earth

ANALYSIS

Let the elliptic orbits corresponding to the initial and final terminals be respectively

$$l_1 = r(1 + e_1 \cos \theta) \tag{1}$$

$$l_2 = r[1 + e_2 \cos (\theta - \alpha)] \tag{2}$$

force centre being the pole, and the line joining the force centre to the peri-apsis of the initial orbit as the initial line.

If (r_1, θ_1) and $(r_2, \theta_1, + \phi)$ be the initial and final terminals, by (1) and (2)

$$\rho = \frac{r_1}{r_2} = \frac{l_1}{l_2} \left[\frac{1 + e_2 \cos (\theta_1 + \phi - \alpha)}{1 + e_1 \cos \theta_1} \right] \tag{3}$$

If V_0 and V_1 be the rocket velocities just before and after the application of the impulse at the initial terminal, γ_0 and γ_1 the corresponding heading angles (angle between rocket velocity vector and the local horizontal), the velocity change is given by

$$\triangle V = [V_0^2 + V_1^2 - 2V_0V_1 \cos (\gamma_0 - \gamma_1)]^{\frac{1}{2}} \tag{4}$$

It can be shown that

$$\tan \gamma_0 = \frac{e_1 \sin \theta_1}{1 + e_1 \cos \theta_1} \tag{5}$$

and

$$V_0^2 = \frac{K}{l_1} (1 + e_1^2 + 2e_1 \cos \theta_1) \tag{6}$$

Now since the transfer trajectory passes through the final terminal ,

$$\frac{r_1}{r_2} = \frac{K}{r_1 V_1^2} \left(\frac{1 - \cos \phi}{\cos^2 \gamma_1} \right) + \frac{\cos (\gamma_1 + \phi)}{\cos \gamma_1}$$

$$\text{------------} \; | \gamma_1 | \leqslant \pi/2$$

which can be written as

$$V_1^2 = \frac{K(1 - \cos \phi) \sec^2 \gamma_1}{r_1 (\rho + \sin \phi \tan \gamma_1 - \cos \phi)} \tag{7}$$

By virtue of (3), (5), (6) and (7), $\triangle V$ is a function of two variables θ_1 and γ_1 for specified value of ϕ. The problem now reduces to the mathematical problem of finding out the values of θ_1 and γ_1 which make $| \triangle V |$ minimum. For this, taking the first derivative with respect to θ, and γ_1, we have from (4),

$$V_0 \left[\frac{\partial V_0}{\partial \theta_1} - V_1 \sin (\gamma_1 - \gamma_0) \frac{\partial \gamma_0}{\delta \theta_1} - \cos (\gamma_1 - \gamma_0) \frac{\partial V_1}{\partial \theta_1} \right] + V_1 \left[\frac{\partial V_1}{\delta \theta_1} - \cos (\gamma_1 - \gamma_0) \frac{\partial V_0}{\partial \theta_1} \right] = 0 \tag{8}$$

$$V_0 \left[\frac{\partial V_0}{\partial \gamma_1} + V_1 \sin (\gamma_1 - \gamma_0) \left(1 - \frac{\partial \gamma_0}{\partial \gamma_1} \right) - \cos (\gamma_1 - \gamma_0) \frac{\partial V_1}{\partial \gamma_1} \right] + V_1 \left[\frac{\partial V_1}{\partial \gamma_1} - \cos (\gamma_1 - \gamma_0) \frac{\partial V_0}{\partial \gamma_1} \right] = 0 \tag{9}$$

whereas from (5), (6) and (7)

$$\frac{\partial \gamma_0}{\partial \gamma_1} = \frac{\partial V_0}{\partial \gamma_1} = 0 \tag{10a}$$

$$\frac{\partial \gamma_0}{\partial \theta_1} = \frac{e_1(e_1 + \cos \theta_1)}{1 + 2e_1 \cos \theta_1 + e_1^2} \tag{10b}$$

$$\frac{\partial V_0}{\partial \theta_1} = - \frac{K e_1 \sin \theta_1}{V_0\, l_1} \tag{10c}$$

$$\frac{\partial V_1}{\partial \gamma_1} = \frac{K(1 - \cos \phi)(1 + e_1 \cos \theta_1)}{2 V_1\, l_1} \left[\frac{\sec^2 \gamma_1\, (2A \tan \gamma_1 - \sin \phi \sec^2 \gamma_1)}{A^2} \right] \tag{10d}$$

$$\frac{\partial V_1}{\partial \theta_1} = \frac{K(1 - \cos \phi) \sec^2 \gamma_1}{2 V_1\, l_1} \left[\frac{l_1 (e_1 e_2 \sin(\phi - \alpha) + e_2 \sin(\theta_1 + \phi - \alpha) - e_1 \sin \theta_1)}{A^2\, l_2 (1 + e_1 \cos \theta_1)} - \frac{e_1 \sin \theta_1}{A} \right] \tag{10e}$$

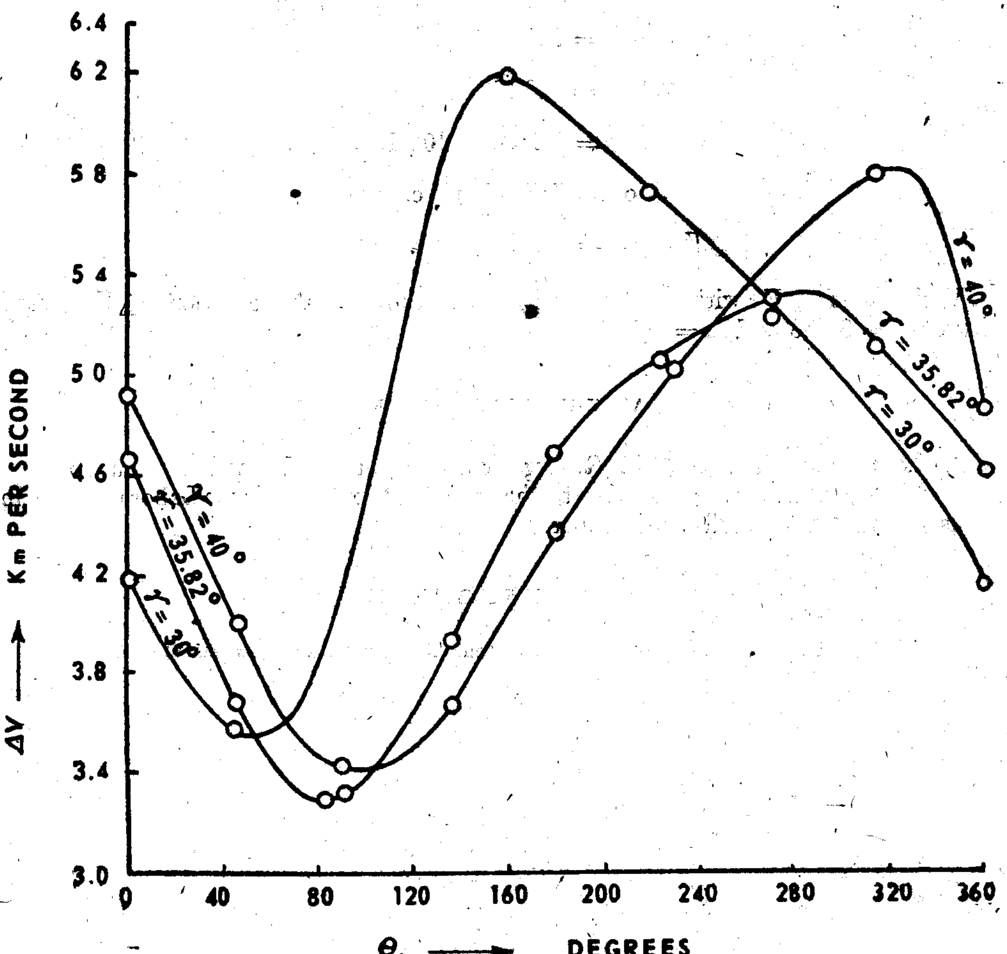

Fig. 1—Variation of ΔV with respect to vectorial angle θ_1.

and

$$A = \frac{l_1}{l_2}\left(\frac{1 + e_2 \cos(\theta_1 + \phi - \alpha)}{1 + e_1 \cos\theta_1}\right) + \sin\phi \tan\gamma_1 - \cos\phi$$

Equations (8) and (9) are two equations in two unknowns θ_1 and γ_1 and can be solved numerically for given values of e_1, e_2, l_1, l_2 and ϕ_1. θ_1 will give the point of application of the impulse on the initial orbit and γ_1 the heading angle along which the rocket should start just after the impulse for the optimum trajectory. Substitution of values of θ_1 and γ_1 in (5), (6), and (7) will give optimum V_1, V_0 and γ_0 and then from (4) $|\triangle V|_{min}$ can be evaluated.

Numerical illustration

Equations (8) and (9) have been solved for initial orbit ($e_1 = \cdot 2$, $l_1 = (10\cdot 56 \times 10^3)$ km. and final orbit $e_2 = \cdot 3$, $l_2 = (19\cdot 11 \times 10^3)$ km, $\alpha = 30°$ in the Earth's gravitational field. Taking $\phi = 40°$, the following results are obtained:

$$\theta_1 \text{ opt} = 83° \; 18'$$

$$\gamma_1 \text{ opt} = 35° \; 49'$$

$$r_1 \text{ opt} = (10\cdot 32 \times 10^3) \text{ km.}$$

$$V_1 \text{ opt} = 7\cdot 757 \text{ km/sec.}$$

$$|\triangle V|_{min} = 3\cdot 319 \text{ km/sec.}$$

Variation of $|\triangle V|$ with respect to θ_1 is shown in Fig. 1. It shows that $|\triangle V|_{min}$ really exists at $\theta_1 = 83°18'$ and $\gamma_1 = 35°49'$.

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