STEADY RADIAL FLOW OF SECOND ORDER FLUIDS BETWEEN TWO INFINITE PARALLEL DISCS — ONE ROTATING AND THE OTHER AT REST

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The laminar radial flow of second order fluids between two infinite parallel discs (one rotating and the other at rest) has been discussed. A source of strength M is assumed to be present on the axis of rotation, which induces the radial flow. The solution is obtained in powers of the radius vector, for small values of the Reynolds number R ($= \rho\omega d^2/\phi_1$).

Livesay[1] discussed the incompressible radial flow between two parallel plates using the integral approach and the assumption of a parabolic velocity profile. Savage[2] pointed out that in the flow regime where intertia forces are important the flow must depart from a parabolic profile, even at radii sufficiently large for entry effects to be negligible. He presented the solution by expanding in terms of the downstream coordinate. For small values of the Reynolds number when the spacing to the radius ratio is small, the pressure distribution is found to agree well with experiment far away from the centre. But in his analysis no-slip condition on the plate $z=0$ is omitted. Radial flows have further been studied by several workers [3—7]. Kreith & Peube[6] have discussed incompressible radial flow between two parallel discs rotating in the same sense with equal velocities in the presence of a source at the axis. Their method is also that of a series expansion in powers of the radius vector. The solution so obtained satisfies no-slip condition at both the plates.

The radial flow of an incompressible non-Newtonian fluid between two parallel plates— one rotating and the other at rest has been investigated here. The constitutive equations of an incompressible second order fluid, as suggested by Coleman & Noll[8] are

$$\tau_{ij} = - p g_{ij} + \phi_1 A_{ij} + \phi_2 B_{ij} + \phi_3 A_i^{\ k} A_{kj} \ , \tag{1}$$

$$A_{ij} = v_{i,j} + v_{j,i} \ , \tag{2}$$

and

$$B_{ij} = a_{i,j} + a_{j,i} + v_{m,i} v^m{}_{,j} + v_{m,j} v^m{}_{,i} \tag{3}$$

where τ_{ij} is the stress tensor, g_{ij} is the metric tensor, v_i , a_i are the components of velocity and acceleration vectors, ϕ_1,ϕ_2,ϕ_3 are the fluid parameters, p is the pressure and comma denotes covariant differentiation. The solution of 6·8 percent poly-isobutylene in cetane at 30°C behaves as a second order fluid and the values of the constants ϕ_1, ϕ_2 and ϕ_3 have been determined experimentally by Markovitz & Brown[9], and Markovitz[10].

The method given by Kreith & Peube[6] has been adopted in the present investigation. The inlet condition is taken as a source flow of strength M along the axis. The functions in the series expansion have been determined for the small Reynolds number ($R=\rho\omega d^2/\phi_1$). Their variation against axial distance for different values of non-Newtonian fluid parameters have been shown in Fig. (1) to (6).

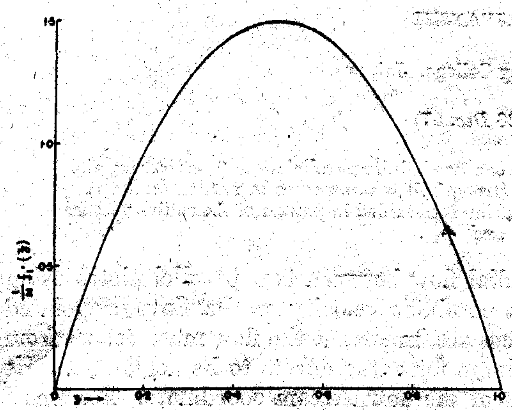

Fig. 1—Variation of $\frac{1}{M} f'_1(z)$ against z.
($R = 0{\cdot}5$, $K = -0{\cdot}1$, $S = 0{\cdot}3$)

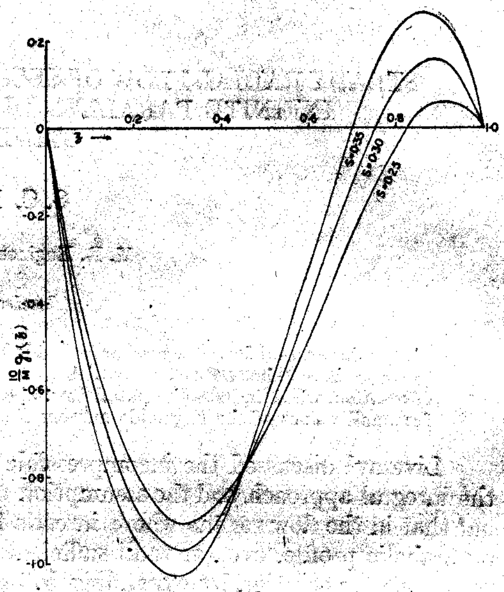

Fig. 2—Variation of $\frac{10}{M} g_1(z)$ against z.
($R = 0{\cdot}5$, $K = -0{\cdot}1$)

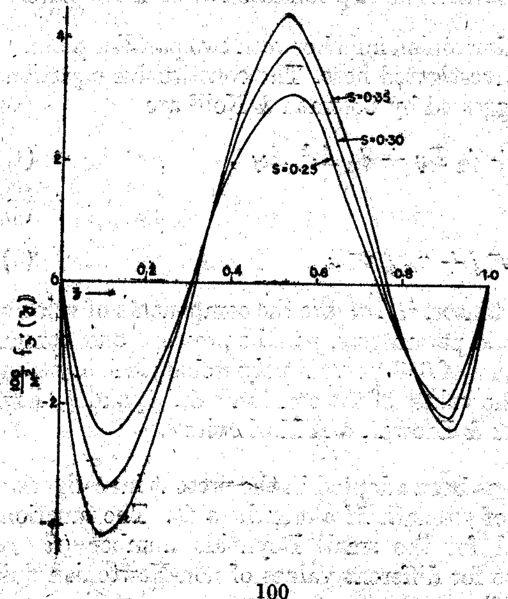

Fig. 3—Variation of $\frac{100}{M^2} f'_3(z)$ against z.
($R = 0{\cdot}5$, $K = -0{\cdot}1$)

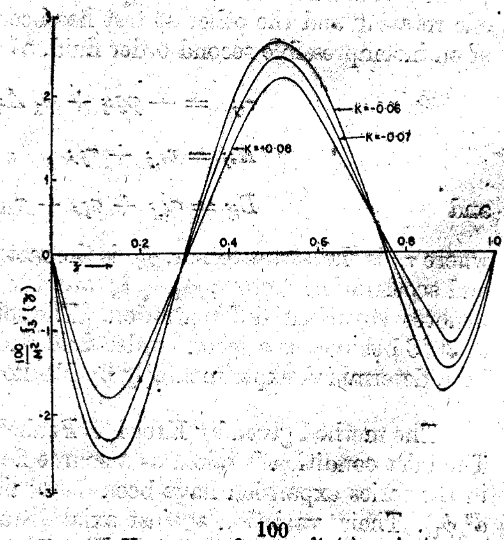

Fig. 4—Variation of $\frac{100}{M^2} f'_3(z)$ against z.
($R = 0{\cdot}5$, $S = 0{\cdot}2$)

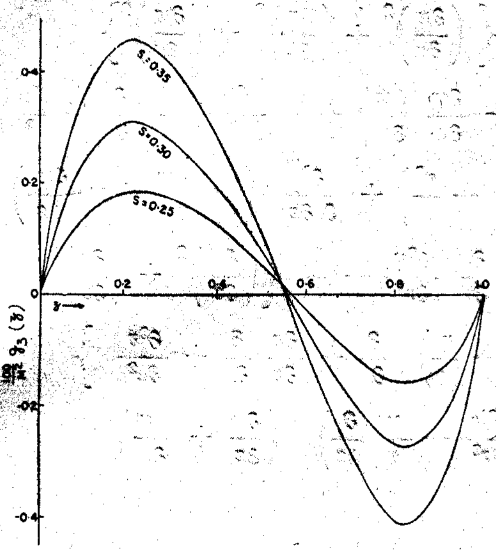

Fig. 5—Variation of $\frac{100}{M^2}\, g_3(z)$ against z.
$(R = 0{\cdot}5,\ K = -0{\cdot}1)$

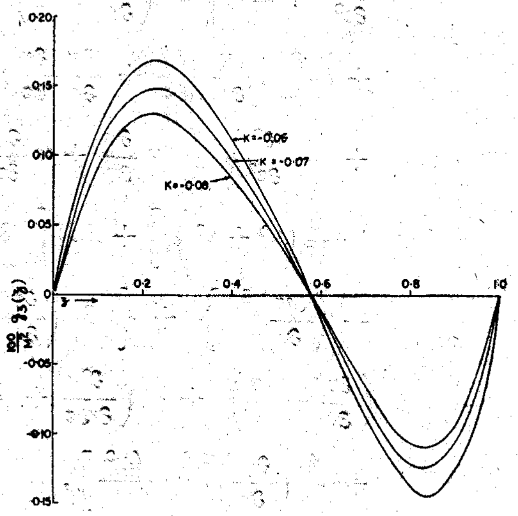

Fig. 6—Variation of $\frac{100}{M^2}\, g_3(z)$ against z.
$(R = 0{\cdot}5, S = 0{\cdot}2)$

EQUATIONS OF MOTION

The momentum equation for the incompressible flow are

$$\left(\frac{\partial v_i}{\partial t} + v^j\, v_{i,j}\right) = -\, p_{,i} + \tau^{j}_{i,j} \tag{4}$$

and the continuity equation is

$$v^i{}_{,j} = 0 \tag{5}$$

We shall use polar cylindrical coordinates (r', θ', z') in the following analysis. Let the velocity components in these directions be u', v' and w' respectively. The flow takes place between two parallel discs $z'=0$ and $z'=d$. The disc $z'=0$ rotates with angular velocity ω and the disc $z=d$ is assumed to be at rest. The source of strength M is taken on the axis of rotation $r'=0$. The flow is assumed to be axially symmetric about the z=axis, so that $\frac{\partial}{\partial \theta}\left(\ \ \right) = 0.$

We introduce the following non-dimensional quantities,

$$\begin{aligned} u &= (u'/\omega d), & v &= (v'/\omega d), & w &= (w'/\omega d), \\ p &= (p'/\phi_1 \omega), & r &= (r'/d), & z &= (z'/d), \\ R &= (\rho d^2 \omega/\phi_1), & K &= (\phi_2/\rho d^2), & S &= (\phi_3/\rho d^2). \end{aligned} \tag{6}$$

The equations (1) to (5) in non-dimensional form are

$$R\left(u\,\frac{\partial u}{\partial r} + w\,\frac{\partial u}{\partial z} - \frac{v^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} - \frac{u}{r^2} +$$

$$\frac{\partial^2 u}{\partial z^2} + K R \Big[2 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left\{ \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 + \left(\frac{\partial w}{\partial r} \right)^2 \right\}$$

$$+ 2 \frac{\partial}{\partial z} \left(\frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right) + 2 \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right.$$

$$\left. + \frac{1}{2} \frac{\partial^2}{\partial z^2} \right) \left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) + \frac{\partial^2}{\partial r \partial z} \left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right)$$

$$- \frac{2}{r^3} \left(u^2 + v^2 \right) \Big] + S R \Big[8 \frac{\partial u}{\partial r} \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \left\{ \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 \right.$$

$$\left. - \left(\frac{\partial w}{\partial r} \right)^2 \right\} + \left\{ 2 \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \frac{\partial}{\partial r} + \frac{\partial v}{\partial z} \frac{\partial}{\partial z} + \frac{\partial^2 v}{\partial z^2} \right\}$$

$$\left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) + 2 \left(\frac{\partial^2 u}{\partial r \partial z} + \frac{\partial^2 w}{\partial r^2} - \frac{u}{r} \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right)$$

$$+ \frac{4}{r} \left(\frac{\partial u}{\partial r} \right)^2 - \frac{4u^2}{r^3} \Big], \tag{7}$$

$$R \left(u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{uv}{r} \right) = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}$$

$$+ K R \Big[\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) \left(u \frac{\partial v}{\partial r} - w \frac{\partial v}{\partial z} + \frac{uv}{r} \right)$$

$$+ \frac{2}{r} \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \left(u \frac{\partial v}{\partial r} - v \frac{\partial u}{\partial r} \right) - \frac{2}{r} \frac{\partial}{\partial z} \left(v \frac{\partial u}{\partial z} - u \frac{\partial v}{\partial z} \right) \Big]$$

$$+ S R \Big[- 2 \left(\frac{\partial w}{\partial z} \frac{\partial}{\partial r} + \frac{\partial^2 w}{\partial r \partial z} + \frac{2}{r} \frac{\partial w}{\partial z} \right) \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)$$

$$+ \left\{ 2 \frac{\partial^2 v}{\partial r \partial z} + \frac{\partial v}{\partial z} \frac{\partial}{\partial r} + \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) \frac{\partial}{\partial z} + \frac{1}{r} \frac{\partial v}{\partial z} \right\}$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) - 2 \frac{\partial^2 u}{\partial r \partial z} \frac{\partial v}{\partial z} - 2 \frac{\partial u}{\partial r} \frac{\partial^2 v}{\partial z^2} \Big], \tag{8}$$

$$R \left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}$$

$$+ K R \Big[2 \frac{\partial}{\partial z} \left\{ \left(\frac{\partial u}{\partial z} \right)^2 + \left(\frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right\} + 2 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right)$$

$$\left(\frac{\partial u}{\partial r} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial r} \frac{\partial w}{\partial z} \right) + \left(\frac{\partial^2}{\partial r \partial z} + \frac{1}{r} \frac{\partial}{\partial z} \right)$$

$$\left(u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} \right) + \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + 2 \frac{\partial^2}{\partial z^2} \right)$$

$$\left(u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} \right) \Big] + S R \Big[8 \frac{\partial w}{\partial z} \frac{\partial^2 w}{\partial z^2}$$

$$+ \frac{\partial v}{\partial z}\left(2\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial r^2}\right) + \frac{\partial^2 v}{\partial r \partial z}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

$$+ 2\left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial r \partial z} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r}\frac{\partial}{\partial r}\right)$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)\Bigg], \tag{9}$$

and

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \tag{10}$$

We define a stream function ψ in the form

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{11}$$

which satisfies (10).

On considerations of symmetry of the flow, the stream function ψ, azimuthal velocity v and pressure p at sufficient distance from the source (so as to neglect entry effects), are assumed in the following form

$$\psi = r^2 f_1(z) + f_{-1}(z) + r^{-2} f_3(z) + \ldots\ldots\ldots\ldots + r^{-n+1} f_n(z), \tag{12}$$

$$v = r g_{-1}(z) + r^{-1} g_1(z) + r^{-3} g_3(z) + \ldots\ldots\ldots\ldots + r^{-n} g_n(z), \tag{13}$$

and

$$p = r^2 h_{-2}(z) + h_0(z) + h(z)\ln r + r^{-2} h_2(z) + \ldots\ldots\ldots\ldots$$
$$+ r^{-2n} h_{2n}(z), \tag{14}$$

where n is odd.

The boundary conditions are

$$f_s(0) = f_s(1) = 0, \qquad for\ s = -1, 3, 5, \ldots\ldots\ldots\ldots \tag{15}$$

$$f'_s(0) = f'_s = 0, \qquad for\ s = -1, 1, 3, 5, \ldots\ldots\ldots\ldots, \tag{16}$$

$$f_1(1) - f_0(0) = M, \tag{17}$$

such that $M = (Q/2\pi d^3)$, where Q is the volume flux of the flow, and prime denotes differentiation with respect to z.

$$g_s(0) = g_s(1) = 0, \quad \text{for } s = 1, 3, 5, \ldots\ldots\ldots\ldots \tag{18}$$

$$g_{-1}(0) = 1 \text{ and } g_{-1}(1) = 0. \tag{19}$$

Substituting (12) to (14) in equations (7) to (9) and equating the coefficients of equal powers of r on both sides, we get the following set of equations

$$h'_{-2} = KR\,(4f''_{-1}\, f'''_{-1} + 4g'_{-1}\, g''_{-1}) + SR\,(2g'_{-1}\, g''_{-1} + 2f''_{-1}\, f'''_{-1}), \tag{20}$$

$$f'''_{-1} = 2h_{-2} + R\left[(f'_{-1})^2 - 2f_{-1}\, f''_{-1} - (g_{-1})^2\right] + KR\left[2f_{-1}\, f^{iv}_{-1} - 2(f''_{-1})^2\right] + SR\left[2f'_{-1}\, f'''_{-1} + (g'_{-1})^2 - (f''_{-1})^2\right], \tag{21}$$

$$g''_{-1} = R\,(2f'_{-1}\, g_{-1} - 2f_{-1}\, g'_{-1}) + KR\,(2f_{-1}\, g'''_{-1} - 2f''_{-1}\, g'_{-1}) + SR\,(2f'_{-1}\, g''_{-1} - 2f''_{-1}\, g'_{-1}), \tag{22}$$

$$h'_0 = -2f''_{-1} - 4Rf_{-1}\, f'_{-1} + KR\,(4f'''_{-1}\, f''_1 + 4f''_{-1}\, f'''_1 + 4g''_{-1}\, g'_1 + 4g'_{-1}\, g''_1 + 44f'_{-1}\, f''_{-1} + 4f_{-1}\, f'''_{-1}) + SR\,(2g''_{-1}\, g'_1 + 2g'_{-1}\, g''_1 + 28f'_{-1}\, f''_{-1} + 2f''_1\, f'''_{-1} + 2f''_{-1}\, f'''_1), \tag{23}$$

$$h' = 0, \tag{24}$$

$$f'''_1 = h + R\,(-2f_{-1}\, f''_1 - 2g_{-1}\, g_1) + KR\,(2f'''_{-1}\, f'_1 + 2f'_{-1}\, f'''_1 + 4g''_{-1}\, g_1 + 2f''_{-1}\, f''_1 + 2f_{-1}\, f_1^{iv} + 4g'_{-1}\, g'_1) + SR\,(4g'_{-1}\, g'_1 + 2g''_{-1}\, g_1 + 2f'_{-1}\, f'''_1 + 2f'''_{-1}\, f'_1 + 2f''_{-1}\, f''_1), \tag{25}$$

$$g''_1 = 2R\,(f'_1\, g_{-1} - f_{-1}\, g'_1) + 2KR\,(f''_{-1}\, g'_1 + f'_{-1}\, g''_1 + f_{-1}\, g'''_1 + f'''_{-1}\, g_1 - 2f''_1\, g'_{-1} - 2f'_1\, g''_{-1}) + 2SR\,(f''_{-1}\, g'_1 + f'''_{-1}\, g_1 + f'_{-1}\, g''_1 - f'_1\, g''_{-1} - 2f''_1\, g'_{-1}), \tag{26}$$

$$h'_2 = 4KR\,(f''_1\, f'''_1 + f'''_{-1}\, f''_3 + f''_{-1}\, f'''_3 + g'_1\, g''_1 + g''_{-1}\, g'_3 + g'_{-1}\, g''_3) + 2SR\,(g''_{-1}\, g'_3 + g'_1\, g''_1 + g'_{-1}\, g''_3 + f'''_{-1}\, f''_3 + f''_{-1}\, f'''_3 + f''_1\, f'''_1), \tag{27}$$

$$f'''_3 = -2h_2 + R\left[-(f'_1)^2 - 2f'_{-1}\, f'_3 - 2f_{-1}\, f''_3 + 2f''_{-1}\, f_3 - (g_1)^2 - 2g_{-1}\, g_3\right] + 2KR\left[2f'''_{-1}\, f'_3 + 4f''_{-1}\, f''_3 + 2(f''_1)^2 + 2f'_1\, f'''_1 + 2f'_{-1}\, f'''_3 + 4g''_{-1}\, g_3 + 4g'_{-1}\, g'_3 + 2(g'_1)^2 + 2g_1\, g''_1 + f_{-1}\, f^{iv}_3 - f^{iv}_{-1}\, f_3\right] + SR\left[6g'_{-1}\, g'_3 + 3(g'_1)^2 + 2g_1\, g''_1 + 4g''_{-1}\, g_3 + 6f''_{-1}\, f''_3 + 3(f''_1)^2 + 2f'_{-1}\, f'''_3 + 2f'_1\, f'''_1 + 2f'''_{-1}\, f'_3\right], \tag{28}$$

$$g''_3 = 2R(f_3' g_{-1} - f'_{-1} g_3 - f_{-1} g'_3 + f_3 g'_{-1}) + 2KR(-3f''_3 g'_{-1} - 4f'_3 g''_{-1} - f_3 g'''_{-1} + 3f''_{-1} g'_3 + 2f'_{-1} g''_3 + f_{-1} g'''_3 + 2f'''_{-1} g_3 - f'_1 g''_1 + f'''_1 g_1) + 2SR(3f''_{-1} g'_3 + f'''_1 g_1 + 2f'''_{-1} g_3 - 3f''_3 g'_{-1} + f'_{-1} g''_3 + 2f'''_{-1} g_3 - f'_1 g''_1 - 3f'_3 g''_{-1}), \tag{29}$$

Equations (15) to (29) will determine the flow completely.

SOLUTION OF EQUATIONS

Elimination of h_{-2}, h and h_2 between the pair of equations [(20), (21)]; [(24), (25)]; [(27), (28)] respectively gives

$$f'''_{-1} = 2\alpha + R[(f_{-1})^2 - 2f_{-1}f''_{-1} - (g_{-1})^2] + KR[2(f''_{-1})^2 + 2f_{-1}f^{iv}_{-1} + 4(g'_{-1})^2] + SR[3(g'_{-1})^2 + (f''_{-1})^2 + 2f'_{-1}f'''_{-1}], \tag{30}$$

$$f'''_1 = \beta + 2R(-f_{-1}f''_1 - g_{-1}g_1) + 2KR(f'''_{-1}f'_1 + f'_{-1}f'''_1 + 2g''_{-1}g_1 + f''_{-1}f''_1 + f_{-1}f^{iv}_1 + 2g'_{-1}g'_1) + 2SR(2g'_{-1}g'_1 + g''_{-1}g_1 + f'_{-1}f'''_1 + f'''_{-1}f'_1 + f''_{-1}f''_1), \tag{31}$$

$$f'''_3 = -2\gamma - R[(f'_{-1})^2 + 2f'_{-1}f'_3 + 2f_{-1}f''_3 - 2f''_{-1}f_3 + (g_1)^2 + 2g_{-1}g_3] + 2KR(2f'''_{-1}f'_3 + 2f'_{-1}f'''_3 + 2f'_1f'''_1 + 4g''_{-1}g_3 + 2g_1g''_1 + f_{-1}f^{iv}_3 - f^{iv}_{-1}f_3) + SR[(g'_1)^2 + 2g'_{-1}g'_3 + 2f''_{-1}f''_3 + (f''_1)^2 + 2g_1g''_1 + 4g''_{-1}g_3 + 2f'_{-1}f'''_3 + 2f'_1f'''_1 + 2f'''_{-1}f'_3], \tag{32}$$

where α, β and γ are arbitrary constants to be determined.

The above non-linear ordinary differential equations have been solved for small Reynolds number R. A series solution in powers of R is assumed for f, g, α, β and γ in the form

$$f_s(z) = \sum_{m=0}^{\infty} R^m f_s^{(m)}(z), \qquad g_s(z) = \sum_{m=0}^{\infty} R^m g_s^{(m)}(z),$$

$$\alpha = \sum_{m=0}^{\infty} R^m \alpha^{(m)}, \qquad \beta = \sum_{m=0}^{\infty} R^m \beta^{(m)}, \qquad \gamma = \sum_{m=0}^{\infty} R^m \gamma^{(m)}, \tag{33}$$

In view of (28) the boundary conditions (15) to (19) are

$$f_s^{(m)}(0) = f_s^{(m)}(1) = 0, \qquad \text{for } s = -1, 3, 5, \ldots\ldots \tag{34}$$

$$\overset{(m)'}{f_s}(0) = \overset{(m)'}{f_s}(1) = 0, \qquad \text{for } s = -1, 1, 3, 5, \ldots \ldots \tag{35}$$

$$\overset{(0)}{f_1}(1) - \overset{(0)}{f_1}(0) = M,$$

$$\overset{(m)}{f_1}(1) - \overset{(m)}{f_1}(0) = 0, \qquad (m \neq 0), \tag{36}$$

$$\overset{(m)}{g_s}(0) = \overset{(m)}{g_s}(1) = 0, \qquad (m \neq 0), \qquad \text{for } s = 1, 3, 5, \ldots \ldots \tag{37}$$

$$\overset{(0)}{g_{-1}}(0) = 1 \text{ and } \overset{(0)}{g_{-1}}(1) = 0. \tag{38}$$

From equations (22), (26) and (29) to (33), we obtain a set of differential equations in $\overset{(m)}{f_s}$ and $\overset{(m)}{g_s}$, where $s = -1, 1, 3$. Using equations (34) to (38), these functions have been determined to the third order approximation.

$$\begin{aligned}
f_{-1}(z) = {} & R(0{\cdot}05z^2 - 0{\cdot}116667z^3 + 0{\cdot}083333z^4 - 0{\cdot}016666z^5) + R^3[-0{\cdot}000080z^2 \\
& - 0{\cdot}000102z^3 + 0{\cdot}000357z^4 - 0{\cdot}000143z^5 - 0{\cdot}000361z^6 + 0{\cdot}000520z^7 - 0{\cdot}000327z^3 \\
& + 0{\cdot}000118z^9 - 0{\cdot}000024z^{10} + 0{\cdot}000002z^{11} + K(-0{\cdot}006348z^3 + 0{\cdot}012582z^3 \\
& - 0{\cdot}011667z^4 + 0{\cdot}009667z^5 - 0{\cdot}006667z^6 + 0{\cdot}003175z^7 - 0{\cdot}000794z^8 + 0{\cdot}000088z^9) \\
& + S(-0{\cdot}002075z^2 + 0{\cdot}007216z^3 - 0{\cdot}011667z^4 + 0{\cdot}013z^5 - 0{\cdot}010556z^6 + 0{\cdot}005317z^7 \\
& - 0{\cdot}001389z^8 + 0{\cdot}000154z^9) + K^2(0{\cdot}012065z^2 - 0{\cdot}000636z^3 - 0{\cdot}066667z^4 \\
& + 0{\cdot}093333z^5 - 0{\cdot}044444z^6 + 0{\cdot}006349z^7) + S^2(0{\cdot}009047z^2 - 0{\cdot}000476z^3 \\
& - 0{\cdot}05z^4 + 0{\cdot}07z^5 - 0{\cdot}033333z^6 + 0{\cdot}004762z^7) + KS(0{\cdot}021115z^2 \\
& - 0{\cdot}001114z^8 - 0{\cdot}116667z^4 + 0{\cdot}163333z^5 - 0{\cdot}077778z^6 + 0{\cdot}011111z^7)],
\end{aligned} \tag{39}$$

$$\begin{aligned}
g_{-1}(z) = {} & (1 - z) + R^2[-0{\cdot}004286z + 0{\cdot}033333z^3 - 0{\cdot}066667z^4 \\
& + 0{\cdot}056667z^5 - 0{\cdot}022222z^6 + 0{\cdot}003175z^7 + (K+S)(0{\cdot}1z^2 - 0{\cdot}233333z^3 \\
& + 0{\cdot}166666z^4 - 0{\cdot}033333z^5)],
\end{aligned} \tag{40}$$

$$\begin{aligned}
(1/M)f_1(z) = {} & (3z^2 - 2z^3) + R^2[-0{\cdot}000118z^2 - 0{\cdot}020873z^3 + 0{\cdot}05z^4 \\
& - 0{\cdot}03z^5 - 0{\cdot}011667z^6 + 0{\cdot}02z^7 - 0{\cdot}008929z^8 + 0{\cdot}001587z^9 \\
& + (K+S)(-0{\cdot}227619z^2 + 0{\cdot}596190z^3 - 0{\cdot}566667z^4 + 0{\cdot}226667z^5 \\
& - 0{\cdot}028571z^7) + (K+S)^2(0{\cdot}8z^2 + 3{\cdot}2z^3 - 4z^4 + 1{\cdot}6z^5)],
\end{aligned} \tag{41}$$

$$\begin{aligned}
(1/M)g_1(z) = {} & R[-0{\cdot}6z + 2z^3 - 2z^4 + 0{\cdot}6z^5 + (K+S)(-4z + 12z^2 - 8z^3)] \\
& + R^3[0{\cdot}000999z - 0{\cdot}000079z^3 - 0{\cdot}009683z^4 + 0{\cdot}021833z^5 - 0{\cdot}026667z^6 \\
& + 0{\cdot}027142z^7 - 0{\cdot}022619z^8 + 0{\cdot}011997z^9 - 0{\cdot}003227z^{10} + 0{\cdot}000303z^{11} \\
& + K(-0{\cdot}046434z - 0{\cdot}009048z^2 + 0{\cdot}010476z^3 + 0{\cdot}107302z^4 - 0{\cdot}525524z^5 \\
& + 0{\cdot}845556z^6 - 0{\cdot}573333z^7 + 0{\cdot}180952z^8 + 0{\cdot}010053z^9) + S(-0{\cdot}075878z \\
& - 0{\cdot}009048z^2 + 0{\cdot}010476z^3 + 0{\cdot}207540z^4 - 0{\cdot}692191z^5 + 1{\cdot}058889z^6 \\
& - 0{\cdot}687619z^7 + 0{\cdot}216667z^8 - 0{\cdot}028836z^9) + K^2(2{\cdot}337905z - 3{\cdot}710476z^2 \\
& + 4{\cdot}518095z^3 - 4{\cdot}2z^4 + 1{\cdot}664z^5 - 0{\cdot}533333z^6 - 0{\cdot}076191z^7) \\
& + KS(2{\cdot}798095z - 7{\cdot}420952z^2 + 9{\cdot}436191z^3 - 9{\cdot}6z^4 + 6{\cdot}653333z^5 \\
& - 1{\cdot}866667z^6) + S^2(3{\cdot}276190z - 3{\cdot}710476z^2 + 4{\cdot}918095z^3 - 5{\cdot}4z^4 \\
& + 2{\cdot}173333z^5 - 1{\cdot}333333z^6 + 0{\cdot}076191z^7) + (K^3 + S^3)(-3{\cdot}2z^2 + 12{\cdot}8z^3 \\
& - 16z^4 + 6{\cdot}4z^5) + (K^2S + KS^2)(-9{\cdot}6z^2 + 38{\cdot}4z^3 - 48z^4 + 19{\cdot}2z^5)],
\end{aligned} \tag{42}$$

$$\begin{aligned}
(1/M^2)f_3(z) = {} & R[-0{\cdot}085714z^2 + 0{\cdot}257142z^3 - 0{\cdot}6z^5 + 0{\cdot}6z^6 \\
& - 0{\cdot}171428z^7 + (K+S)(-2{\cdot}4z^2 + 9{\cdot}6z^3 - 12z^4 + 4{\cdot}8z^5)] \\
& + R^3[-0{\cdot}001266z^2 + 0{\cdot}002942z^3 - 0{\cdot}000040z^4 - 0{\cdot}005365z^5 + 0{\cdot}004714z^6
\end{aligned}$$

$$-0{\cdot}003320z^7 + 0{\cdot}011293z^8 - 0{\cdot}020828z^9 + 0{\cdot}019444z^{10} - 0{\cdot}009870z^{11} + 0{\cdot}002551z^{12} - 0{\cdot}000255z^{13} + K(-2{\cdot}577527z^2 + 3{\cdot}286584z^3 - 0{\cdot}089524z^4 + 0{\cdot}059429z^5 - 0{\cdot}558476z^6 + 0{\cdot}884463z^7 - 1{\cdot}820952z^8 + 1{\cdot}119365z^9 - 0{\cdot}346190z^{10} + 0{\cdot}042828z^{11}) + S(-1{\cdot}294256z^2 + 1{\cdot}678213z^3 - 0{\cdot}089524z^4 + 0{\cdot}065143z^5 - 0{\cdot}468952z^6 + 0{\cdot}916707z^7 - 1{\cdot}456667z^8 + 0{\cdot}893174z^9 - 0{\cdot}28z^{10} + 0{\cdot}036162z^{11}) + K^2(-6{\cdot}572628z^2 + 8{\cdot}894224z^3 + 3{\cdot}340952z^4 - 14{\cdot}509714z^5 + 21{\cdot}52z^6 - 18{\cdot}834285z^7 + 7{\cdot}085714z^8 - 0{\cdot}924263z^9) + S^2(-3{\cdot}588001z^2 + 4{\cdot}685969z^3 + 3{\cdot}209524z^4 - 12{\cdot}104z^5 + 17{\cdot}92z^6 - 15{\cdot}074286z^7 - 5{\cdot}757143z^8 - 0{\cdot}806349z^9) + KS(-10{\cdot}160627z^2 + 13{\cdot}580191z^3 + 6{\cdot}550476z^4 - 26{\cdot}613714z^5 + 39{\cdot}44z^6 - 33{\cdot}908571z^7 + 12{\cdot}842857z^8 - 1{\cdot}730612z^9) + K^3(-1{\cdot}462858z^2 + 3{\cdot}108572z^3 + 6{\cdot}4z^4 - 21{\cdot}76z^5 + 19{\cdot}2z^6 - 5{\cdot}485714z^7) + S^3(-0{\cdot}297142z^2 - 0{\cdot}868572z^3 + 8{\cdot}8z^4 - 17{\cdot}92z^5 + 14{\cdot}4z^6 - 4{\cdot}114286z^7) + K^2S(-3{\cdot}222858z^2 + 5{\cdot}348572z^3 + 21{\cdot}6z^4 - 61{\cdot}44z^5 + 52{\cdot}8z^6 - 15{\cdot}085714z^7) + KS^2(-2{\cdot}057142z^2 + 1{\cdot}371428z^3 + 24z^4 - 57{\cdot}6z^5 + 48z^6 - 13{\cdot}714286z^7)] \quad (43)$$

$$(1/M^2)g_3(z) = R^2 [0{\cdot}000476z - 0{\cdot}057143z^3 + 0{\cdot}171429z^4 - 0{\cdot}102857z^5 - 0{\cdot}2z^6 + 0{\cdot}342857z^7 - 0{\cdot}192857z^8 + 0{\cdot}038095z^9 + (K+S)(0{\cdot}125715z - 0{\cdot}514286z^2 + 2{\cdot}342857z^3 - 6z^4 + 6{\cdot}96z^5 - 3{\cdot}6z^6 + 0{\cdot}685714z^7) + (K+S)^2(3{\cdot}2z - 14{\cdot}4z^2 + 25{\cdot}6z^3 - 24z^4 + 9{\cdot}6z^5)], \quad (44)$$

$$\alpha = R(0{\cdot}15 - 2K - 1{\cdot}5S) + R^3[-0{\cdot}000306 + 0{\cdot}010602K + 0{\cdot}003792S - 0{\cdot}001908K^2 - 0{\cdot}001428S^2 - 0{\cdot}003342KS], \quad (45)$$

$$(1/M)\beta = -12 + R^2[-0{\cdot}125238 - 0{\cdot}022857(K+S) + 3{\cdot}2(K+S)^2], \quad (46)$$

$$(1/M^2)\gamma = -R(0{\cdot}771429 + 28{\cdot}8K + 10{\cdot}8S) - R^3(0{\cdot}008826 + 9{\cdot}859752K + 4{\cdot}873686S + 26{\cdot}682672K^2 + 14{\cdot}995050S^2 + 41{\cdot}677716KS + 9{\cdot}325716K^3 + 2{\cdot}194284S^3 + 20{\cdot}845716K^2S + 13{\cdot}714284KS^2). \quad (47)$$

Equations (39) and (40) provide the solution of laminar flow between two parallel discs, the disc $z = 0$ rotating and the disc $z = 1$ at rest, in the absence of source. These functions have been determined by Goel[11], and our results agree with him: Further, with $K = 0$, these functions are comparable with Bhatnagar[12]. The non-dimensional transverse shearing stress on the rotating disc is

$$(\tau_{\theta z})_{z=0} = -r(1 + 0{\cdot}004286R^2) + (M/r)[-R(0{\cdot}6 + 4K + 4S] + R^3(0{\cdot}000999 - 0{\cdot}046434K - 0{\cdot}075878S + 2{\cdot}337905K^2 + 2{\cdot}798095KS + 3{\cdot}276191S^2)] + (M^2R^2/r^3)(0{\cdot}000476 + 0{\cdot}125714K + 0{\cdot}125714S + 3{\cdot}2K^2 + 3{\cdot}2S^2 + 6{\cdot}4KS). \quad (48)$$

The non-dimensional normal stress on the stationary disc is

$$(\tau_{zz})_{z=1} = -p + Rr^2(2K+S)(1 + 0{\cdot}001905R^2) + MR^2(2K+S)[-1{\cdot}6 + 8(K+S)] + (1/r^2)M^2R(2K+S)[36 + R^2\{1{\cdot}076191 + 1{\cdot}325714(K+S) - 9{\cdot}6(K+S)^2\}]. \quad (49)$$

DISCUSSION

For the numerical work we have assumed $R = 0{\cdot}5$. Fig. 1 shows the variation of $f_1'(z)$ against the axial distance. The small variation of non-Newtonian fluid parameters K and S have no effect on $f_1'(z)$ and the profile is parabolic as in the Newtonian case. Fig. 2 shows the effect of K and S on $g_1(z)$. Having assigned a fixed value to $K(=-0{\cdot}1)$, the value of $g_1(z)$ changes sign as we go from the rotating disc to the stationary disc and its magnitude increases with the increase in S. The profiles of $f'_3(z)$ which depicts the effects of the variation of S and K are shown in Fig. 3 and 4 respectively. The general character of profiles in the figures is similar. With increase in K and S respectively, the magnitude of $f'_3(z)$ increases. However, the magnitude near the rotating disc is always greater than the magnitude near the stationary disc. Fig. 5 and 6 present the variation of $g_3(z)$ against z for fixed K and S respectively. With the increase in K and S there is corresponding increase in the magnitude of $g_3(z)$.

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