STEADY FLOW OF TWO CONDUCTING INCOMPRESSIBLE AND IMMISCIBLE FLUIDS BETWEEN TWO CONDUCTING PLATES UNDER CLOSED CIRCUIT CONDITIONS

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This paper discusses the flow of two conducting, viscous, incompressible and immiscible fluids between two conducting parallel plates in presence of an applied transverse magnetic field under general electrical loading conditions. It has been shown that the flow fluxes of the two fluids, the interface velocity and the skin friction on the two plates decrease with the increase of the total current.

NOMENCLATURE

B_0 — strength of the applied magnetic field

E_0 — electric field in z direction

h_1 — depth of fluid 1

h_2 — depth of fluid 2

$h^1{}_1$ — thickness of plate 1

$h^1{}_2$ — thickness of plate 2

I — total current

Jz — current density in z direction

p — pressure

Q_1 — flow flux of fluid 1

Q_2 — flow flux of fluid 2

u_0 — interface velocity

u_1 — velocity of fluid 1

u_2 — velocity of fluid 2

x, y, z — cartesian coordinates

μ_1 — viscosity of fluid 1

μ_2 — viscosity of fluid 2

σ_1 — conductivity of fluid 1

σ_2 — conductivity of fluid 2

$\sigma^1{}_1$ — conductivity of plate 1

$\sigma^1{}_2$ — conductivity of plate 2

τ_1 — stress on plate 1

τ_2 — stress on plate 2

The problem of simultaneous flow of immiscible fluids in channels may be of importance in industrial processes such as transportation of two or more fluids in the same pipe. Bird *et al*[1] discussed the flow of two viscous incompressible and immiscible fluids between two parallel plates and showed that when the heights of the two fluids are equal, the velocity maximum occurs in the less viscous fluid. Kapur & Shukla[2] generalised this discussion to the case of the flow of any (finite) number of immiscible fluids occupying different heights between two parallel plates and it showed that a unique velocity maximum always exists. Later on, Kapur & Shukla[3] again considered the unsteady flow of two immiscible fluids occupying equal heights between two parallel plates under a time dependent pressure gradient and showed that the velocity maximum can also occur in the more viscous fluid if its density is less than tnat of the less viscous fluid. Kapur & Shukla[4] discussed the flow of two viscous, conducting, incompressible and immiscible fluids between two parallel non-conducting plates in the presence of a transverse magnetic field neglecting induced fields. They have shown that the flow fluxes of the two fluids, the skin friction at the plates and the interface velocity decrease as the strength of the magnetic field increases. The same problem has been extended by Shukla & Prasad[5] taking induced fields into consideration and similar results were obtained.

Here we have considered the flow of two conducting incompressible and immiscible fluids between two parallel conducting plates in the presence of a uniform transverse magnetic field under closed circuit conditions. The main aim is to discuss the effects of the magnetic field, the total current and the conductivities of the plates on the interface velocity, the flow fluxes of the two fluids and the skin friction on the two plates.

BASIC EQUATIONS

Consider the flow of two conducting incompressible and immiscible fluids between two conducting parallel plates in the presence of a uniform transverse magnetic field under general electrical loading conditions. A constant pressure gradient is assumed to be applied to both the fluids. The physical situation of the problem is illustrated in Fig. 1.

The basic equations governing the flow are[6]

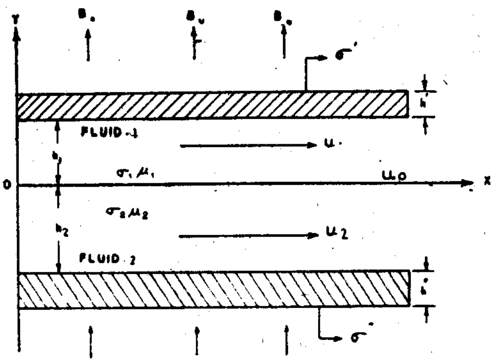

Fig. 1—Flow of two immiscible and conducting fluids.

$$\mu_i \frac{\partial^2 u_i}{\partial y^2} = J_z B_0 - P \qquad (1)$$

and

$$J_z = \sigma_i (E_0 + u_i B_0) \qquad (2)$$

$$i = 1, 2$$

where u_i is the velocity of the two fluids and $-\frac{dp}{dx} = P$. Since the tangential components of the velocity are continuous at the interface, the boundary conditions for the velocities in the two regions are as follows :

$$\left.\begin{array}{l}\left.\begin{array}{llll} u_1 = 0 & \text{at} & y = & h_1 \\ u_1 = u_0 & \text{at} & y = & 0 \end{array}\right\} \\ \left.\begin{array}{llll} u_2 = 0 & \text{at} & y = & -h_2 \\ u_2 = u_0 & \text{at} & y = & 0 \end{array}\right\}\end{array}\right\} \quad (3)$$

where u_0 is the velocity at the interface.

From (1), (2) and (3) one gets

$$u_1 = \frac{u_0}{\sinh M_1} \sinh M_1 \left(1 - \frac{y}{h}\right) + \left(\frac{E_0}{B_0} - \frac{P}{\sigma_1 B_0^2}\right) \left\{\frac{\sinh M_1 \left(1 - \frac{y}{h}\right) + \sinh \frac{M_1 y}{h_1}}{\sinh M_1} - 1\right\} \quad (4)$$

and

$$u_2 = \frac{u_0}{\sinh M_2} \sinh M_2 \left(1 + \frac{y}{h_2}\right) + \left(\frac{E_0}{B_0} - \frac{P}{\sigma_2 B_0^2}\right) \left\{\frac{\sinh M_2 \left(1 + \frac{y}{h_2}\right) - \sinh \frac{M_2 y}{h_2}}{\sinh M_2} - 1\right\} \quad (5)$$

where $M_1^2 = \frac{\sigma_1}{\mu_1} B_0^2 h_1^2$ and $M_2^2 = \frac{\sigma_2}{\mu_2} B_0\ h_2^2$.

It is convenient to use the following dimensionless variables for further analysis

$$\left.\begin{array}{lll} \bar{u}_1 = \frac{\mu_1 u_1}{P h_1^2} ; & \bar{u}_2 = \frac{\mu_2 u_2}{P h_2^2} ; & \bar{u}_0 = \frac{\mu_1 u_0}{P h_1^2} \\ \bar{E}_0 = \frac{\mu_1 E_0}{B_0 P h_1^2} ; & \bar{y} = \frac{y}{h_1} ; & \bar{h} = \frac{h_1}{h_2} \\ \mu^2 = \frac{\mu_2}{\mu_1} ; & \frac{\sigma_2}{\sigma_1} = \lambda^2 = \frac{\bar{h}^2 \mu^2 M_2^2}{M_1^2} & \end{array}\right\} \quad (6)$$

Equations (4) and (5) then take the forms:

$$\bar{u}_1 = \frac{\bar{u}_0 \sinh M_1 (1 - \bar{y})}{\sinh M_1} + \left(\bar{E}_0 - \frac{1}{M_1^2}\right) \left\{\frac{\sinh M_1 (1 - \bar{y}) + \sinh M_1 \bar{y}}{\sinh M_1} - 1\right\} \quad (7)$$

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$$\bar{v}_2 = \frac{\mu^2 \bar{u}_0 \bar{h}^2 \sinh M_2 (1 + \bar{h}\bar{y})}{\sinh M_2} + \left(\mu^2 \bar{h}^2 \bar{E}_0 - \frac{1}{M_2^2}\right) \left\{\frac{\sinh M_2 (1 + \bar{h}\bar{y}) - \sinh M_2 \bar{h}\bar{y}}{\sinh M_2} - 1\right\} \quad (8)$$

Since at the interface the shearing stress is constant

$$\bar{h}^{-2}\left(\frac{\partial \bar{u}_1}{\partial \bar{y}}\right)_{\bar{y}=0}=\left(\frac{\partial \bar{u}_2}{\partial \bar{y}}\right)_{\bar{y}=0} \quad (9)$$

From (7), (8) and (9) one gets

$$\bar{u}_0=\frac{\frac{1}{M_1}\tanh\frac{M_1}{2}+\frac{1}{M_2\bar{h}}\tanh\frac{M_2}{2}-\bar{E}_0 M_1\left(\tanh\frac{M_1}{2}+\lambda\,\mu\tanh\frac{M_2}{2}\right)}{M_1(\coth M_1+\lambda\,\mu\coth M_2)} \quad (10)$$

The total current I is given by[6]

$$I=\int_0^L\Bigg[\sigma_2^{1}\int_{-(h''+h_2)}^{-h_2}E_0\,dy+\sigma_2\int_{-h_2}^{0}(E_0+u_2B_0)\,dy$$

$$+\sigma_1\int_0^{h_1}(E_0+u_1B_0)\,dy+\sigma_1^{1}\int_{h_1}^{h_1+h'}E_0\,dy\Bigg]dx \quad (11)$$

which on using (4), (5) and (6) gives

$$\bar{I}=M_1\bar{E}_0\cdot\left[\frac{\bar{h}_1\lambda_1^2}{\bar{h}}+\frac{\bar{h}_2\lambda_2^2}{\bar{h}}+\frac{2}{M_1}\tanh\frac{M_1}{2}+\frac{2\lambda\mu}{M_1}\tanh\frac{M_2}{2}\right]$$

$$+\bar{u}_0\left[\tanh\frac{M_1}{2}+\lambda\,\mu\tanh\frac{M_2}{2}\right]-\frac{1}{M_1}\left[\left(\frac{2}{M_1}\tanh\frac{M_1}{2}-1\right)\right.$$

$$\left.+\frac{1}{\bar{h}}\left(\frac{2}{M_2}\tanh\frac{M_2}{2}-1\right)\right] \quad (12)$$

where $\bar{I}=\dfrac{I}{LPh_1^2}\sqrt{\dfrac{\mu_1}{\sigma_1}}$

$$\lambda_1^2=\frac{\sigma_1^1}{\sigma_1},\quad \lambda_2^2=\frac{\sigma_2^1}{\sigma_1},\quad \bar{h}_1=\frac{h_1^1}{h_2},\quad \bar{h}_2=\frac{h_2^1}{h_2}$$

which on using (10) gives $\bar{E}_0$ as a function of $\bar{I}$

$$M_1\bar{E}_0\left[\frac{\bar{h}_1\lambda_1^2}{\bar{h}}+\frac{\bar{h}_2\lambda_2^2}{\bar{h}}+\frac{2}{M_1}\left(\tanh\frac{M_1}{2}+\lambda\,\mu\tanh\frac{M_2}{2}\right)-\right.$$

$$\left.\frac{\left(\tanh\frac{M_1}{2}+\lambda\,\mu\tanh\frac{M_2}{2}\right)^2}{M_1(\coth M_1+\lambda\,\mu\coth M_2)}\right]=\bar{I}+\frac{1}{M_1}\left[\left(\frac{2}{M_1}\tanh\frac{M_1}{2}-1\right)\right.$$

$$\left.+\frac{1}{\bar{h}}\left(\frac{2}{M_2}\tanh\frac{M_2}{2}-1\right)\right]$$

$$- \left[\frac{\left(\tanh \frac{M_1}{2} + \lambda \mu \tanh \frac{M_2}{2} \right) \left(\tanh \frac{M_1}{2} + \frac{\mu}{\lambda} \tanh \frac{M_2}{2} \right)}{M^2_1 \, (\coth M_1 + \lambda \mu \coth M_2)} \right] \tag{13}$$

This equation determines the constant E_0 explicitly in terms of the total current I.

DETERMINATION OF SKIN FRICTION AND FLOW FLUX

If the skin frictions on the upper and the lower plates be τ_1 and τ_2 then, with the help of (7) and (8), one gets

$$\bar{\tau}_1 = \frac{\tau_1}{Ph_1} = - \left(\frac{d\bar{u}_1}{d\bar{y}} \right)_{\bar{y}=1}$$

$$= \frac{M_1 \bar{u}_0}{\sinh M_1} - M_1 \left(\bar{E}_0 - \frac{1}{M_1^2} \right) \tanh \frac{M_1}{2} \tag{14}$$

and

$$\bar{\tau}_2 = \frac{\tau_2}{Ph_2} = \frac{1}{\bar{h}} \left(\frac{d\bar{u}_2}{d\bar{y}} \right)_{\bar{y} = -\frac{1}{\bar{h}}}$$

$$= \frac{\mu^2 \bar{u}_0 \bar{h}^2 M_2}{\sinh M_2} - M_2 \left(\mu^2 \bar{h}^2 \bar{E}_0 - \frac{1}{M_2^2} \right) \tanh \frac{M_2}{2} \tag{15}$$

If Q_1 and Q_2 be the corresponding flow fluxes, one also gets

$$\bar{Q}_1 = \frac{\mu_1 Q_1}{Ph_1^3} = \int_0^1 \bar{u}_1 \, d\bar{y}$$

$$= \frac{\bar{u}_0 \tanh \frac{M_1}{2}}{M_1} - \left(\bar{E}_0 - \frac{1}{M_1^2} \right) \left(1 - \frac{2}{M_1} \tanh \frac{M_1}{2} \right) \tag{16}$$

and

$$\bar{Q}_2 = \frac{\mu_2 Q_2}{Ph_2^3} = \frac{1}{\bar{h}} \int_{-\frac{1}{\bar{h}}}^{0} \bar{u}_2 \, d\bar{y}$$

$$= \frac{\mu^2 \bar{u}_0}{M_2} \tanh \frac{M_2}{2} - \left(\mu^2 \bar{h} \bar{E}_0 - \frac{1}{M_2^2} \right) \left(\frac{1}{\bar{h}^2} - \frac{2 \tanh \frac{M_2}{2}}{M_2 \bar{h}^3} \right) \tag{17}$$

Now from equations (7), (8), (10), and (13) to (17) one can find the explicit expressions for u_1, u_2, u_0, τ_1, τ_2, Q_1 and Q_2 in terms of I.

Particular Case

Let us discuss the following particular case :

Consider the plates to be of the same material and of the same thickness and assume that the fluids occupy the same height between the plates. Then

$$\bar{h} = 1 \,; \qquad \frac{h_1^1}{h_2} = \bar{h}_1 = \bar{h}_2 = \frac{h_2^1}{h_2} = b \qquad \text{(say)}$$

$$\lambda_1 = \sqrt{\frac{\sigma_1^1}{\sigma_1}} = \lambda_2 = \sqrt{\frac{\sigma_2^1}{\sigma_1}} = K \qquad \text{(say)}$$

With the help of (10) and (12) the various expressions take the forms :

$$\bar{u}_0 = \frac{\dfrac{2}{M_1} + 2bK^2 \left(\dfrac{\dfrac{1}{M_1}\tanh\dfrac{M_1}{2} + \dfrac{1}{M_2}\tanh\dfrac{M_2}{2}}{\tanh\dfrac{M_1}{2} + \lambda\,\mu\tanh\dfrac{M_2}{2}} \right) - \bar{I}}{M_1 b K^2 \left(1 + \dfrac{\coth\dfrac{M_1}{2} + \lambda\,\mu\coth\dfrac{M_2}{2}}{\tanh\dfrac{M_1}{2} + \lambda\,\mu\tanh\dfrac{M_2}{2}} \right) + \left(\coth\dfrac{M_1}{2} + \lambda\,\mu\coth\dfrac{M_2}{2} \right)} \tag{18}$$

Equations (10) and (18) show that $\bar{u}_0$ is a linear function of $\bar{I}$ and decreases as the total current increases. This is also obvious from Fig. 2.

$$M_1 \bar{E}_0 \left\{ 2\,bK^2 + \frac{2}{M_1}\left(\tanh\frac{M_1}{2} + \lambda\,\mu\tanh\frac{M_2}{2}\right) - \frac{\left(\tanh\dfrac{M_1}{2} + \lambda\,\mu\tanh\dfrac{M_2}{2}\right)^2}{M_1\left(\coth M_1 + \lambda\,\mu\coth M_2\right)} \right\}$$

$$= \bar{I} + \frac{1}{M_1}\left[\frac{2}{M_1}\tanh\frac{M_1}{2} + \frac{2}{M_2}\tanh\frac{M_2}{2} - 2 \right]$$

$$- \frac{\left(\tanh\dfrac{M_1}{2} + \lambda\,\mu\tanh\dfrac{M_2}{2}\right)\left(\tanh\dfrac{M_1}{2} + \dfrac{\mu}{\lambda}\tanh\dfrac{M_2}{2}\right)}{M_1^2\left(\coth M_1 + \lambda\,\mu\coth M_2\right)} \tag{19}$$

$$\bar{\tau}_1 = \frac{M_1\,\bar{u}_0}{\sinh M_1} - M_1\left(\bar{E}_0 - \frac{1}{M_1^2}\right)\tanh\frac{M_1}{2} \tag{20}$$

$$\bar{\tau}_2 = \frac{\mu^2 M_2\,\bar{u}_0}{\sinh M_2} - M_2\left(\mu^2\bar{E}_0 - \frac{1}{M_2^2}\right)\tanh\frac{M_2}{2} \tag{21}$$

$$\bar{Q}_1 = \frac{\bar{u}_0}{M_1}\tanh\frac{M_1}{2} - \left(\bar{E}_0 - \frac{1}{M_1^2}\right)\left(1 - \frac{2}{M_1}\tanh\frac{M_1}{2}\right) \tag{22}$$

$$\bar{Q}_2 = \frac{\mu^2\,\bar{u}_0}{M_2}\tanh\frac{M_2}{2} - \left(\mu^2\bar{E}_0 - \frac{1}{M_2^2}\right)\left(1 - \frac{2}{M_2}\tanh\frac{M_2}{2}\right) \tag{23}$$

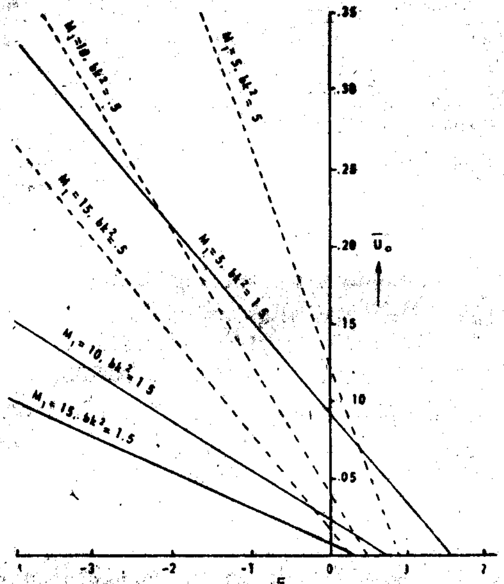

Fig. 2—Variation of $\overline{u_0}$ vs $\bar{I}$

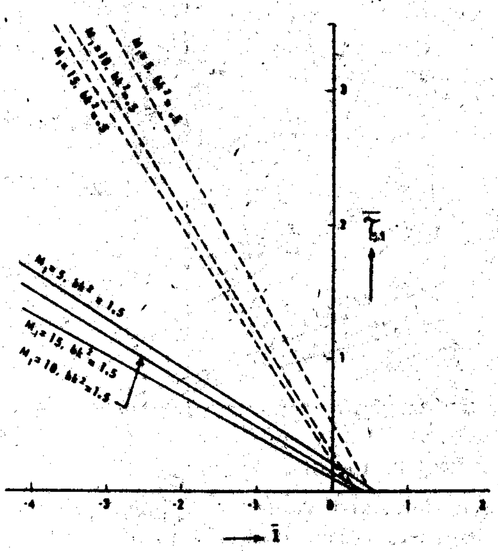

Fig. 3—Variation of $\bar{\tau}_1$ vs $\bar{I}$

Keeping the conductivities and the viscosities of the fluids fixed, the variations of $\overline{u_0}$, $\overline{\tau_1}$, $\overline{\tau_2}$, $\overline{Q_1}$ and $\overline{Q_2}$ versus $\bar{I}$ are shown in Fig. 2 to 6 respectively for various values of M. The following results are obtained :

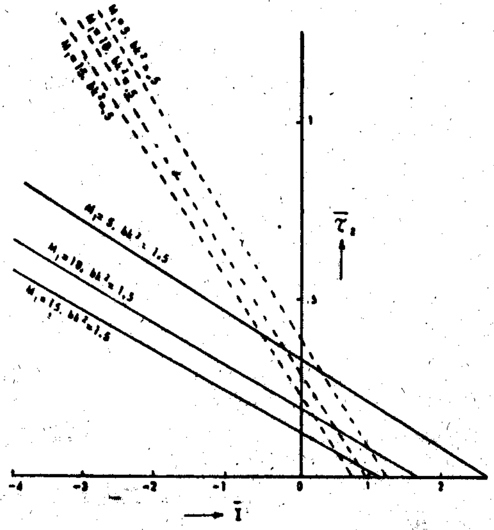

Fig. 4—Variation of $\overline{\tau_2}$ vs $\bar{I}$

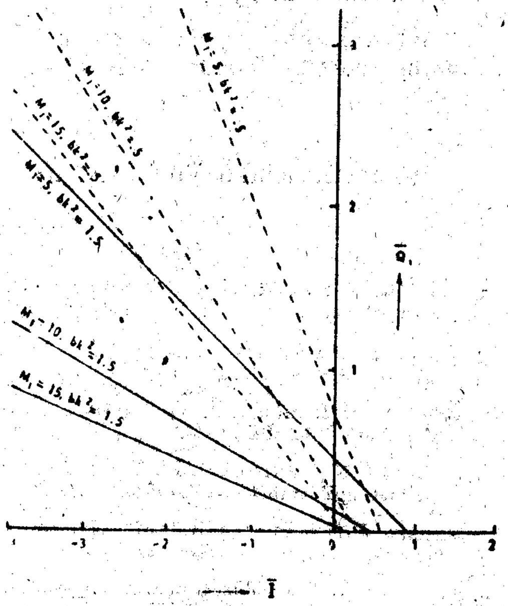

Fig. 5—Variation of $\overline{Q_1}$ vs $\bar{I}$

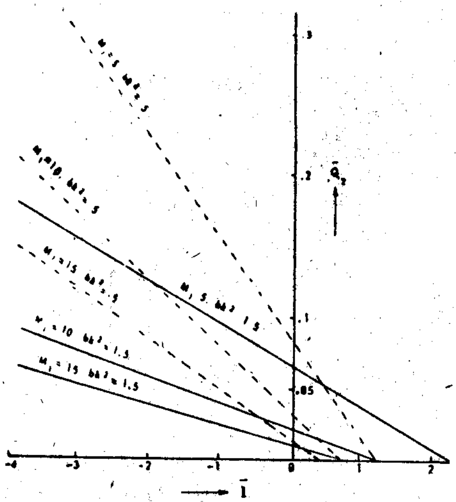

Fig. 6—Variation of $\overline{Q}_2$ vs. $\overline{I}$

Results obtained from Fig 2: (*i*) the interface velocity $\overline{u}_0$ decreases as the strength of the magnetic field increases for fixed values of $\overline{I}$ and bK^2; (*ii*) the interface velocity $\overline{u}_0$ decreases as bK^2 *i.e.*, the conductivity or the thickness of the plates increases for fixed negative values of $\overline{I}$ and the strength of the magnetic field.

Results obtained from Fig. 3 and 4: (*i*) for fixed bK^2 and negative values of $\overline{I}$, the skin friction at the plates increases as the strength of the magnetic field increases; (*ii*) for fixed values of the applied magnetic field and negative values of $\overline{I}$, the skin friction at the plates decreases as bK^2, *i.e.* the conductivity or the thickness of the plates increases; (*iii*) for fixed values of applied magneto field and bK^2, the skin friction at the plates decreases as the total current increases. Thus it is possible to minimise the skin friction at the plates by choosing suitable configuration of the external electric field.

Results obtained from Fig. 5 and 6: (*i*) for fixed bK^2 and negative values of $\overline{I}$, the flow fluxes of the fluids decrease as the strength of the magnetic field increases; (*ii*) for fixed values of the applied magnetic field and negative values of $\overline{I}$, the flow fluxes of the fluids decrease as bK^2, *i.e.*, the conductivity or the thickness of the plates increases; (*iii*) for fixed values of applied magnetic field and bK^2, the flow fluxes of the fluids decrease as the total current $\overline{I}$ increases.

When the total current is zero and the plates are non-conducting one gets all the results obtained by Shukla & Prasad[5].

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