LAMINAR FLOW OF AN ELECTRICALLY CONDUCTING INCOMPRESSIBLE FLUID BETWEEN TWO WAVY WALLS IN THE PRESENCE OF A TRANSVERSE MAGNETIC FIELD

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Magnetohydrodynamic laminar flow of an electrically conducting, viscous, incompressible fluid between two wavy walls, whose equations are taken in the form of Fourier Series, has been investigated under the assumptions that ε, the coefficient of roughness and R, the Reynold's number of the flow, are small. The stream function has been determined to the first order in ε. For numerical investigation, the walls are considered to have the sinusoidal deformation. The velocity profiles for the longitudinal and the transverse velocities are drawn at various cross-sections for $M=0, 1, 2$, $\varepsilon=0 \cdot 5$, $R=0 \cdot 1$ and $P_{c,x} \; R^2 = 0 \cdot 1$.

The solution to the problem of determining the steady state velocity distribution for a viscous incompressible fluid between two parallel plates is easily obtained from the equations of motion and is well known[1]. The slow viscous flow between rotating concentric infinite cylinders with axial roughness has been discussed by Citron[2] and the problem of flow of non-Newtonian fluids and heat transfer in them between wavy walls and wavy cylinders has been extensively studied by Bhatnagar and his collaborators[3].

The pioneer work in the study of steady magnetohydrodynamic channel flow of a conducting fluid under a uniform magnetic field transverse to an electrically insulated channel wall has been done by Hartmann[4]. In the present paper we have discussed the flow of an electrically conducting viscous incompressible fluid between two wavy walls which are situated symmetrical about a mid-plane in the presence of a transverse magnetic field.

The following assumptions have been made ;

(*a*) the roughness of the walls is small,

(*b*) the walls are insulated,

(*c*) electrical conductivity σ_e of the fluid is sufficiently large so as to ignore the displacement currents,

(*d*) no electric field is present, and

(*e*) the induced magnetic field produced by the motion of the electrically conducting fluid is negligible.

A method of Fourier series is employed and an expression for the stream function correct to the first order of the roughness is determined. The numerical work for the boundaries having the sinusoidal deformation is carried out. The longitudinal and transverse velocity profiles are shown graphically for various values of Hartmann number.

BASIC EQUATIONS

The equation of motion for laminar flow of an incompressible, electrically conducting fluid in the presence of a transverse magnetic field is, in the usual rotation

$$\rho \frac{d\underline{V}}{dt} = - \text{grad}\, p + \mu \triangle^2 \underline{V} + (\underline{J} \times \underline{B}) , \qquad (A)$$

where $\underline{J}$ and $\underline{B}$ are given by Maxewll's equation and Ohm's law, namely

$$\left.\begin{array}{ll} \text{curl}\, \underline{H} = 4\pi \underline{J}, & \text{div}\, \underline{B} = 0, \\ \text{curl}\, \underline{E} = 0, & \text{div}\, \underline{E} = 0, \end{array}\right\} \qquad (B)$$

and

$$\underline{J} = \sigma [\underline{E} + \underline{V} \times \underline{B}] \; . \qquad (C)$$

The equation of continuity is

$$\text{div}\, \underline{V} = 0 \qquad (D)$$

When a conducting fluid moves through the magnetic lines of force, $\underline{B}_0$, the positive and negative charges are each accelerated in such a way that their average motion gives rise to an electric current $\underline{j} = \sigma \underline{V} \times \underline{B}$, where $B = \underline{B}_0 + \underline{b}$. The quantity $\underline{b}$ is the magnetic induction resulting from the electric current $\underline{j}$ in the fluid. In this analysis $\underline{b}$ will be considered as a perturbation on the basic field strength and negligible in comparison with $\underline{B}_0$ *i.e.*, $\underline{B}_0 >> \underline{b}$.

The fluid is assumed to be ionized, and hence an electrical conductor. However, within any small but finite volume the number of particles with positive and negative charges are nearly equal. The total excess charge density θ and imposed electric field intensity $\underline{E}$ are assumed to be zero.

Let us consider the laminar flow of incompressible, viscous and electrically conducting fluid between two wavy walls situated symmetrically about a mid-plane in the presence of a constant transverse magnetic field H_0. Let x-axis be along the length of the walls in the mid-plane and z-axis perpendicular to it.

For this configuration the velocity components are

$$u = u\,(x, z), \quad v = 0 \quad , \quad w = w\,(x, z) \qquad (1)$$

With these assumptions, (A) to (C) are reduced simply to

$$\rho \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) - \sigma_e B_0^2 u \;, \qquad (2)$$

$$\rho \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) = - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) \;, \qquad (3)$$

and (D), the equation of continuity, is reduced to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

where ρ is the density of the fluid, μ the coefficient of viscosity, p the pressure, σ_e the electrical conductivity and $B_0 = \mu_e H_0$ the electromagnetic induction, μ_e being the magnetic permeability.

The boundary conditions for this flow are

$$u = 0,\ w = 0 \ \text{ on } z = z_1 \text{ and } z = z_2 \ ,$$

where

$$z_1 = h + \epsilon h \sum_{n=1}^{\infty} \left(a_n \cos \frac{nx}{h} + b_n \sin \frac{nx}{h} \right) ,$$

and

$$z_2 = -h - \epsilon h \sum_{n=1}^{\infty} \left(a'_n \cos \frac{nx}{h} + b'_n \sin \frac{nx}{h} \right) ,$$

ϵ being a small quantity.

To make the equations dimensionless, we make the following assumptions—

$$\bar{x} = \frac{x}{h} , \quad \bar{z} = \frac{z}{h} , \quad \bar{u} = \frac{u}{U} , \quad \bar{w} = \frac{w}{U} , \quad \bar{p} = \frac{p}{\rho U^2} ;$$

$$R \text{ (Reynolds number)} = \frac{hU}{\nu} ,$$

$$M \text{ (Hartmann number)} = \frac{ha}{(\eta\nu)^{\frac{1}{2}}} ,$$

$$a \text{ (Alfven velocity)} = \left(\frac{\mu_e H^2_0}{\rho} \right)^{\frac{1}{2}} , \quad \nu \text{ (Kinematic viscosity)} = \frac{1}{\mu_e \sigma_e} ,$$

where U is the characteristic velocity at the mouth of the channel.

Equations (2) to (4) reduce to (dropping the bars) :

$$\left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = - \frac{\partial p}{\partial x} + \frac{1}{R} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) - \frac{M^2}{R} u , \tag{5}$$

$$\left(u \frac{\partial \omega}{\partial x} + w \frac{\partial \omega}{\partial z} \right) = - \frac{\partial p}{\partial z} + \frac{1}{R} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right) , \tag{6}$$

and

$$\frac{u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{7}$$

with the boundary conditions

$$\left.\begin{aligned} & u = 0,\ w = 0 \text{ on } z = z_1 \text{ and } z = z_2 \\ & \text{where} \\ & z_1 = 1 + \epsilon \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) , \\ & \text{and} \\ & z_2 = -1 - \epsilon \sum_{n=1}^{\infty} \left(a'_n \cos nx + b'_n \sin nx \right) . \end{aligned}\right\} \tag{8}$$

By properly chosing ϵ, we can make the amplitude of the deformation of the boundaries as small as we like. Since we have assumed ϵ to be small, we expand the physical quantities in the powers of ϵ only and retain upto the first power of ϵ only ;

$$u(x, z) = u_0(x, z) + \epsilon u_1(x, z) ,$$

$$w(x, z) = w_0(x, z) + \epsilon w_1(x, z) ,$$

and

$$p = p_0 + \epsilon p_1 .$$

We can easily show that the zeroth and the first order velocity components satisfy the following boundary conditions :

$$u_0(x, \pm 1) = 0, \; w_0(x, \pm 1) = 0 , \tag{9}$$

$$\left.\begin{aligned} u_1(x, 1) &= -\left(\frac{\partial u_0}{\partial z}\right)_{z=1} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) , \\ w_1(x, 1) &= -\left(\frac{\partial w_0}{\partial z}\right)_{z=1} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) , \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} u_1(x, -1) &= \left(\frac{\partial u_0}{\partial z}\right)_{z=-1} \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx) , \\ w_1(x, -1) &= \left(\frac{\partial w_0}{\partial z}\right)_{z=-1} \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx) . \end{aligned}\right\} \tag{11}$$

We now introduce the stream function ψ, such that

$$u = \frac{\partial \psi}{\partial z} , \; w = -\frac{\partial \psi}{\partial x} . \tag{12}$$

Let

$$\psi = \psi_0 + \epsilon \psi_1 , \tag{13}$$

then from (5), (6) and (12), we have

$$\left[\psi_{0,z}\,\psi_{0,xz} - \psi_{0,x}\,\psi_{0,zz}\right] = -p_{0,x} - \frac{1}{R}\left[\psi_{0,xxz} + \psi_{0,zzz}\right] + \frac{M^2}{R}\psi_{0,z} , \tag{14}$$

$$\left[\psi_{0,x}\,\psi_{0,zx} - \psi_{0,z}\,\psi_{0,xx}\right] = -p_{0,z} + \frac{1}{R}\left[\psi_{0,xxx} + \psi_{0,zzx}\right] , \tag{15}$$

and

$$\left[\psi_{1,z}\,\psi_{0,xz} + \psi_{0,z}\,\psi_{1,xz} - \psi_{1,x}\,\psi_{0,zz} - \psi_{0,x}\,\psi_{1,zz}\right] = -p_{1,x} - \frac{1}{R}\left[\psi_{1,xxz} + \psi_{1,zzz}\right] + \frac{M^2}{R}\psi_{1,z} , \tag{16}$$

$$\left[\psi_{1,x}\,\psi_{0,zx} + \psi_{0,x}\,\psi_{1,zx} - \psi_{1,z}\,\psi_{0,xx} - \psi_{0,z}\,\psi_{1,xx}\right] = -p_{1,z} + \frac{1}{R}\left[\psi_{1,xxx} + \psi_{1,zzx}\right] , \tag{17}$$

where the suffix after comma denotes differentiation with respect to that suffix.

The equations (14) to (17) have to be solved under the boundary conditions :

$$\left.\begin{aligned}
&\psi_{0,z}(x, \pm 1) = 0, \ \psi_{0,x}(x, \pm 1) = 0, \\
&\psi_{1,z}(x, 1) = -\psi_{0,zz}(x, 1) \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \\
&\psi_{1,x}(x, 1) = -\psi_{0,zx}(x, 1) \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \\
&\psi_{1,z}(x, -1) = \psi_{0,zz}(x, -1) \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx), \\
&\psi_{1,x}(x, -1) = \psi_{0,zx}(x, -1) \sum_{n=1}^{\infty} (a'_n \cos nx + b'_n \sin nx).
\end{aligned}\right\} \quad (18)$$

SOLUTION OF ZERO-ORDER EQUATIONS

The zero-order flow is simply a Hartmann flow between two plane parallel walls under transverse magnetic field for which $w_0 = 0$ and u_0 is a function of z only. Thus, setting $\psi_{0,x} = 0$ in (14) and (15), we have

$$p_{0,x} = -\frac{1}{R}\,\psi_{0,zzz} + \frac{M^2}{R}\,\psi_{0,z}, \quad (19)$$

$$p_{0,z} = 0. \quad (20)$$

From (20) p_0 is independent of z and in (19) p_0 is a function of z only thereby showing that

$$p_{0,x} = \text{constant.}$$

The solution of (19) is

$$\psi_0 = -\frac{p_{0,x} R}{M^2}\left[\frac{\sinh Mz}{M \cosh M} - z\right]. \quad (21)$$

Therefore

$$u_0 = \frac{p_{0,x} R}{M^2}\left[1 - \frac{\cosh Mz}{\cosh M}\right]. \quad (22)$$

SOLUTION OF FIRST ORDER EQUATIONS

From (16), (17) and (21), we have

$$\left[\psi_{0,z}\,\psi_{1,xz} - \psi_{1,x}\,\psi_{0,zz}\right] = -p_{1,x} - \frac{1}{R}\left[\psi_{1,xxz} + \psi_{1,zzz}\right] + \frac{M^2}{R}\,\psi_{1,z} \quad (23)$$

and

$$\psi_{0,z}\,\psi_{1,xx} = p_{1,z} - \frac{1}{R}\left[\psi_{1,xxx} + \psi_{1,zzx}\right]. \quad (24)$$

Considering the boundary conditions (18) we choose $\psi_1(x, z)$ in the form :

$$\psi_1(x, z) = -p_{0,x} R \sum_{n=1}^{\infty} \left\{ A_n(z) \cos nx + B_n(z) \sin nx \right\}. \quad (25)$$

From (18) and (25), we have

$$\left.\begin{aligned} &A_n(\pm 1) = 0 ; B_n(\pm 1) = 0 ; \\ &A'_n(1) = -\frac{\tanh M}{M} a_n, \quad B'_n(1) = -\frac{\tanh M}{M} b_n ; \\ &A'_n(-1) = -\frac{\tanh M}{M} a'_n, B'_n(-1) = -\frac{\tanh M}{M} b'_n . \end{aligned}\right\} \tag{26}$$

Eliminating p_1 from (23) and (24) and equating the coefficients of cos nx and sin nx after using (21) and (25), we have

$$\frac{p_{0,x} R^2}{M^2}\left(1-\frac{\cosh Mz}{\cosh M}\right) n'' B_n(z) + \overline{p_{0,x}} R^2 \frac{\cosh Mz}{\cosh M} n B_n(z) - \frac{p_{0,x} R^2}{M^2}$$
$$\left(1-\frac{\cosh Mz}{\cosh M}\right) n^3 B_n(z) = -\left[n^4 A_n(z) + \overset{iv}{A}_n(z) - 2n^2 A''_n(z)\right] + M^2 A''_n(z), \tag{27}$$

and

$$-\frac{p_{0,x} R^2}{M^2}\left(1-\frac{\cosh Mz}{\cosh M}\right) n A''_n(z) - p_{0,x} R^2 \frac{\cosh Mz}{\cosh M} n A_n(z) + \frac{p_{0,x} R^2}{M^2}$$
$$\left(1-\frac{\cosh Mz}{\cosh M}\right) n^3 A_n(z) = -\left[n^4 B_n(z) + \overset{iv}{B}_n(z) - 2n^2 B''_n(z)\right] + M^2 B_n''(z), \tag{28}$$

where dashes denote differentiation with respect to z.

To avoid cumbersome mathematical analysis we now make another assumption, namely $p_{0,x} R^2$ is small and set

$$\left.\begin{aligned} &A_n(z) = \bar{A}_n(z) + p_{0,x} R^2 \bar{\bar{A}}_n(z), \\ \text{and} \\ &B_n(z) = \bar{B}_n(z) + p_{0,x} R^2 \bar{\bar{B}}_n(z), \end{aligned}\right\} \tag{29}$$

so that (25) becomes

$$\psi_1(x, z) = -p_{0,x} R \sum_{n=1}^{\infty} \left[(\bar{A}_n + p_{0,x} R^2 \bar{\bar{A}}_n) \cos nx + (\bar{B}_n + p_{0,x} R^2 \bar{\bar{B}}_n) \sin nx\right]. \tag{30}$$

From (27) to (29), we have

$$[n^4 \bar{A}_n(z) + \overset{iv}{\bar{A}}_n(z) - 2n^2 \bar{A}''_n(z)] - M^2 \bar{A}''_n(z) = 0, \tag{31}$$

$$[n^4 \bar{B}_n(z) + \overset{iv}{\bar{B}}_n(z) - 2n^2 \bar{B}''_n(z)] - M^2 \bar{B}''_n(z) = 0, \tag{32}$$

$$[n^4 \bar{\bar{A}}_n(z) + \overset{iv}{\bar{\bar{A}}}_n(z) - 2n^2 \bar{\bar{A}}''_n(z)] - M^2 \bar{\bar{A}}''_n(z)$$
$$= -\left[\frac{n}{M^2}\left(1-\frac{\cosh Mz}{\cosh M}\right) \bar{B}''_n(z) + \frac{\cosh Mz}{\cosh M} n \bar{B}_n(z) - \frac{n^3}{M^2}\left(1-\frac{\cosh Mz}{\cosh M}\right) \bar{B}_n(z)\right], \tag{33}$$

and

$$[n^4 \bar{\bar{B}}_n(z) + \bar{\bar{B}}_n^{iv}(z) - 2n^2 \bar{\bar{B}}''_n(z)] - M^2 \bar{\bar{B}}''_n(z)$$
$$= \left[\frac{n}{M^2}\left(1 - \frac{\cosh Mz}{\cosh M}\right) \bar{A}''_n(z) + \frac{\cosh Mz}{\cosh M}\, n \bar{A}_n(z) - \frac{n^3}{M^2} \left(1 - \frac{\cosh Mz}{\cosh M}\right) \bar{A}_n(z)\right], \tag{34}$$

with the boundary conditions

$$\left.\begin{aligned}
&\bar{A}_n(\pm 1) = 0,\ \bar{B}_n(\pm 1) = 0,\ \bar{\bar{A}}_n(\pm 1) = 0,\ \bar{\bar{B}}_n(\pm 1) = 0,\\
&\bar{A}'_n(1) = -\frac{\tanh M}{M}\, a_n,\ \bar{\bar{A}}'_n(1) = 0,\\
&\bar{B}'_n(1) = -\frac{\tanh M}{M}\, b_n,\ \bar{\bar{B}}'_n(1) = 0,\\
&\bar{A}'_n(-1) = -\frac{\tanh M}{M}\, a'_n,\ \bar{\bar{A}}'_n(-1) = 0,\\
&\bar{B}'_n(-1) = -\frac{\tanh M}{M}\, b'_n,\ \bar{\bar{B}}'_n(-1) = 0.
\end{aligned}\right\} \tag{35}$$

From (31) and (32), we have

$$\bar{A}_n(z) = C_1 e^{\alpha_{1n} z} + C_2 e^{\alpha_{2n} z} + C_3 e^{-\alpha_{1n} z} + C_4 e^{-\alpha_{2n} z}, \tag{36}$$

$$\bar{B}_n(z) = D_1 e^{\alpha_{1n} z} + D_2 e^{\alpha_{2n} z} + D_3 e^{-\alpha_{1n} z} + D_4 e^{-\alpha_{2n} z}, \tag{37}$$

where

$$\left.\begin{aligned}
\alpha_{1n} &= \left[\frac{(2n^2 + M^2) + M(M^2 + 4n^2)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}},\\
\alpha_{2n} &= \left[\frac{(2n^2 + M^2) - M(M^2 + 4n^2)^{\frac{1}{2}}}{2}\right]^{\frac{1}{2}}.
\end{aligned}\right\} \tag{38}$$

From (35) and (36), for the determination of C_1, C_2, C_3 and C_4, we have

$$\left.\begin{aligned}
&C_1 e^{\alpha_{1n}} + C_2 e^{\alpha_{2n}} + C_3 e^{-\alpha_{1n}} + C_4 e^{-\alpha_{2n}} = 0,\\
&C_1 e^{-\alpha_{1n}} + C_2 e^{-\alpha_{2n}} + C_3 e^{\alpha_{1n}} + C_4 e^{\alpha_{2n}} = 0,\\
&C_1 \alpha_{1n} e^{\alpha_{1n}} + C_2 \alpha_{2n} e^{\alpha_{2n}} - C_3 \alpha_{1n} e^{-\alpha_{1n}} - C_4 \alpha_{2n} e^{-\alpha_{2n}} = -\frac{\tanh M}{M}\, a_n,\\
&C_1 \alpha_{1n} e^{-\alpha_{1n}} + C_2 \alpha_{2n} e^{-\alpha_{2n}} - C_3 \alpha_{1n} e^{\alpha_{1n}} - C_4 \alpha_{2n} e^{\alpha_{2n}} = -\frac{\tanh M}{M}\, a'_n.
\end{aligned}\right\} \tag{39}$$

From (35) and (37), for the determination of D_1, D_2, D_3 and D_4 we have

$$\left.\begin{aligned}
&D_1 e^{\alpha_{1n}} + D_2 e^{\alpha_{2n}} + D_3 e^{-\alpha_{1n}} + D_4 e^{-\alpha_{2n}} = 0 ,\\
&D_1 e^{-\alpha_{1n}} + D_2 e^{-\alpha_{2n}} + D_3 e^{\alpha_{1n}} + D_4 e^{\alpha_{2n}} = 0 ,\\
&D_1 \alpha_{1n} e^{\alpha_{1n}} + D_2 \alpha_{2n} e^{\alpha_{2n}} - D_3 \alpha_{1n} e^{-\alpha_{1n}} - D_4 \alpha_{2n} e^{-\alpha_{2n}} = - \frac{\tanh M}{M} b_n ,\\
&D_1 \alpha_{1n} e^{-\alpha_{1n}} + D_2 \alpha_{2n} e^{-\alpha_{2n}} - D_3 \alpha_{1n} e^{\alpha_{1n}} - D_4 \alpha_{2n} e^{\alpha_{2n}} = - \frac{\tanh M}{M} b'_n .
\end{aligned}\right\} \quad (40)$$

From (33) and (34), (35), we have

$$\bar{\bar{A}}_n (z) = G_1 e^{\alpha_{1n} z} + G_2 e^{\alpha_{2n} z} + G_3 e^{-\alpha_{1n} z} + G_4 e^{-\alpha_{2n} z} + F_1 (z) , \quad (41)$$

and

$$\bar{\bar{B}}_n (z) = E_1 e^{\alpha_{1n} z} + E_2 e^{\alpha_{2n} z} + E_3 e^{-\alpha_{1n} z} + E_4 e^{-\alpha_{2n} z} + F_2 (z) , \quad (42)$$

where

$$F_1 (z) = - \frac{n}{2M^2\beta_1\beta_2} \Bigg[\left(D_1\alpha_{1n} e^{\alpha_{1n} z} - D_2\alpha_{2n} e^{\alpha_{2n} z} - D_3\alpha_{1n} e^{-\alpha_{1n} z} + D_4\alpha_{2n} e^{-\alpha_{2n} z}\right) z -$$

$$\left(\frac{D_1 e^{\alpha_{1n} z}}{\alpha_{1n}} - \frac{D_2 e^{\alpha_{2n} z}}{\alpha_{2n}} - \frac{D_3 e^{-\alpha_{1n} z}}{\alpha_{1n}} + \frac{D_4 e^{-\alpha_{2n} z}}{\alpha_{2n}} \right) n^2 z \Bigg] + \frac{n}{2M^2 \cosh M}$$

$$\cdot \Bigg[\frac{D_1\alpha^2_{1n} e^{(M + \alpha_{1n})z}}{(M + 2\alpha_{1n})(M + \beta_1)(M + \beta_2)} + \frac{D_2\alpha^2_{2n} e^{(M + \alpha_{2n})z}}{(M + 2\alpha_{2n})(M + \beta_1)(M - \beta_2)} +$$

$$\frac{D_3\alpha^2_{1n} e^{(M - \alpha_{1n})z}}{(M - 2\alpha_{1n})(M - \beta_1)(M - \beta_2)} + \frac{D_4\alpha^2_{2n} e^{(M - \alpha_{2n})z}}{(M - 2\alpha_{2n}) (M - \beta_1) (M + \beta_2)} +$$

$$\frac{D_1\alpha^2_{1n} e^{-(M - \alpha_{1n})z}}{(M - 2\alpha_{1n})(M - \beta_1)(M - \beta_2)} + \frac{D_2\alpha^2_{2n} e^{-(M - \alpha_{2n})z}}{(M - 2\alpha_{2n}) (M - \beta_1) (M + \beta_2)} +$$

$$\frac{D_3\alpha^2_{1n} e^{-(M + \alpha_{1n})z}}{(M + 2\alpha_{1n})(M + \beta_1)(M + \beta_2)} + \frac{D_4\alpha^2_{2n} e^{-(M + \alpha_{2n})z}}{(M + 2\alpha_{2n})(M + \beta_1)(M - \beta_2)} \Bigg] -$$

$$\frac{n(n^2 + M^2)}{2M^2 \cosh M} \times \Bigg[\frac{D_1 e^{(M + \alpha_{1n})z}}{(M + 2a_{1n})(M + \beta_1)(M + \beta_2)} + \frac{D_2 e^{(M + \alpha_{2n})z}}{(M + 2\alpha_{2n})(M + \beta_1)(M - \beta_2)} +$$

$$\frac{D_3 e^{(M - \alpha_{1n})z}}{(M - 2\alpha_{1n})(M - \beta_1)(M - \beta_2)} + \frac{D_4 e^{(M - \alpha_{2n})z}}{(M - 2\alpha_{2n})(M - \beta_1)(M + \beta_2)} +$$

$$\frac{D_1\, e^{-(M-\alpha_{1n})z}}{(M-2\alpha_{1n})(M-\beta_1)(M-\beta_2)} + \frac{D_2\, e^{-(M-\alpha_{2n})z}}{(M-2\alpha_{2n})(M-\beta_1)(M+\beta_2)} +$$

$$\left.\frac{D_3\, e^{-(M+\alpha_{1n})z}}{(M+2\alpha_{1n})(M+\beta_1)(M+\beta_2)} + \frac{D_4\, e^{-(M+\alpha_{2n})z}}{(M+2\alpha_{2n})(M+\beta_1)(M-\beta_2)}\right], \quad (43)$$

and

$$F_2(z) = \frac{nz}{2M^2\beta_1\beta_2}\Bigg[\left(C_1\alpha_{1n}\, e^{\alpha_{1n}z} - C_2\alpha_{2n}\, e^{\alpha_{2n}z} - C_3\alpha_{1n}\, e^{-\alpha_{1n}z} + C_4\alpha_{2n}\, e^{-\alpha_{2n}z}\right) - n^2$$

$$\left(\frac{C_1\, e^{\alpha_{1n}z}}{\alpha_{1n}} - \frac{C_2\, e^{\alpha_{2n}z}}{\alpha_{2n}} - \frac{C_3\, e^{-\alpha_{1n}z}}{\alpha_{1n}} + \frac{C_4\, e^{-\alpha_{2n}z}}{\alpha_{2n}}\right)\Bigg] - \frac{n}{2M^2\cosh M}$$

$$\left[\frac{C_1\alpha^2_{1n}\, e^{(M+\alpha_{1n})z}}{(M+2\alpha_{1n})(M+\beta_1)(M+\beta_2)} + \frac{C_2\alpha^2_{2n}\, e^{(M+\alpha_{2n})z}}{(M+2\alpha_{2n})(M+\beta_1)(M-\beta_2)} +\right.$$

$$\frac{C_3\alpha^2_{1n}\, e^{(M-\alpha_{1n})z}}{(M-2\alpha_{1n})(M-\beta_1)(M-\beta_2)} + \frac{C_4\alpha^2_{2n}\, e^{(M-\alpha_{2n})z}}{(M-2\alpha_{2n})(M-\beta_1)(M+\beta_2)} +$$

$$\frac{C_1\alpha^2_{1n}\, e^{-(M-\alpha_{1n})z}}{(M-2\alpha_{1n})(M-\beta_1)(M-\beta_2)} + \frac{C_2\alpha^2_{2n}\, e^{-(M-\alpha_{2n})z}}{(M-2\alpha_{2n})(M-\beta_1)(M+\beta_2)} +$$

$$\left.\frac{C_3\alpha^2_{1n}\, e^{-(M+\alpha_{1n})z}}{(M+2\alpha_{1n})(M+\beta_1)(M+\beta_2)} + \frac{C_4\alpha^2_{2n}\, e^{-(M+\alpha_{2n})z}}{(M+2\alpha_{2n})(M+\beta_1)(M-\beta_2)}\right] +$$

$$\frac{n(n^2+M^2)}{2M^2\cosh M}\left[\frac{C_1\, e^{(M+\alpha_{1n})z}}{(M+2\alpha_{1n})(M+\beta_1)(M+\beta_2)} + \frac{C_2\, e^{(M+\alpha_{2n})z}}{(M+2\alpha_{2n})(M+\beta_1)(M-\beta_2)} +\right.$$

$$\frac{C_3\, e^{(M-\alpha_{1n})z}}{(M-2\alpha_{1n})(M-\beta_1)(M-\beta_2)} + \frac{C_4\, e^{(M-\alpha_{2n})z}}{(M-2\alpha_{2n})(M-\beta_1)(M+\beta_2)} +$$

$$\frac{C_1\, e^{-(M-\alpha_{1n})z}}{(M-2\alpha_{1n})(M-\beta_1)(M-\beta_2)} + \frac{C_2\, e^{-(M-\alpha_{2n})z}}{(M-2\alpha_{2n})(M-\beta_1)(M+\beta_2)} +$$

$$\left.\frac{C_3\, e^{-(M+\alpha_{1n})z}}{(M+2\alpha_{1n})(M+\beta_1)(M+\beta_2)} + \frac{C_4\, e^{-(M+\alpha_{2n})z}}{(M+2\alpha_{2n})(M+\beta_1)(M-\beta_2)}\right], \quad (44)$$

with

$$\beta_1 = \alpha_{1n} + \alpha_{2n} \quad ,$$

$$\beta_2 = \alpha_{1n} - \alpha_{2n} \; .$$

From (35) and (41), for the determination of G_1, G_2, G_3 and G_4, we have

$$\left.\begin{aligned}
&G_1 e^{\alpha_{1n}} + G_2 e^{\alpha_{2n}} + G_3 e^{-\alpha_{1n}} + G_4 e^{-\alpha_{2n}} + F_1(1) = 0 \; , \\
&G_1 e^{-\alpha_{1n}} + G_2 e^{-\alpha_{2n}} + G_3 e^{\alpha_{1n}} + G_4 e^{\alpha_{2n}} + F_1(-1) = 0 \; , \\
&G_1 \alpha_{1n} e^{\alpha_{1n}} + G_2 \alpha_{2n} e^{\alpha_{2n}} - G_3 \alpha_{1n} e^{-\alpha_{1n}} - G_4 \alpha_{2n} e^{-\alpha_{2n}} + F'_1(1) = 0 \; , \\
&G_1 \alpha_{1n} e^{-\alpha_{1n}} + G_2 \alpha_{2n} e^{-\alpha_{2n}} - G_3 \alpha_{1n} e^{\alpha_{1n}} - G_4 \alpha_{2n} e^{\alpha_{2n}} + F'_1(-1) = 0 \; .
\end{aligned}\right\} \qquad (45)$$

From (35) and (42), for the determination of E_1, E_2, E_3 and $E4$, we have

$$\left.\begin{aligned}
&E_1 e^{\alpha_{1n}} + E_2 e^{\alpha_{2n}} + E_3 e^{-\alpha_{1n}} + E_4 e^{-\alpha_{2n}} + F_2(1) = 0 \; , \\
&E_1 e^{-\alpha_{1n}} + E_2 e^{-\alpha_{2n}} + E_3 e^{\alpha_{1n}} + E_4 e^{\alpha_{2n}} + F_2(-1) = 0 \; , \\
&E_1 \alpha_{1n} e^{\alpha_{1n}} + E_2 \alpha_{2n} e^{\alpha_{2n}} - E_3 \alpha_{1n} e^{-\alpha_{1n}} - E_4 \alpha_{2n} e^{-\alpha_{2n}} + F'_2(1) = 0 \; , \\
&E_1 \alpha_{1n} e^{-\alpha_{1n}} + E_2 \alpha_{2n} e^{-\alpha_{2n}} - E_3 \alpha_{1n} e^{\alpha_{1n}} - E_4 \alpha_{2n} e^{\alpha_{2n}} + F'_2(-1) = 0 \; .
\end{aligned}\right\} \qquad (46)$$

From (21) and (30), we have

$$\psi = - \frac{p_{0,x} R}{M^2} \left[\frac{\sinh Mz}{M \cosh M} - z \right] - \epsilon p_{0,x} R \sum_{n=1}^{\infty} \left[(\bar{A}_n + p_{0,x} R^2 \bar{\bar{A}}_n) \cos nx + \right.$$

$$\left. (\bar{B}_n + p_{0,x} R^2 \bar{\bar{B}}_n) \sin nx \right] , \qquad (47)$$

where $\bar{A}_n$, $\bar{\bar{A}}_n$, $\bar{B}_n$ and $\bar{\bar{B}}_n$ are given by (36) to (46).

PARTICULAR CASE

Let the two boundaries have sinusoidal deformation given by

(i) $a_n = a'_n = 0$ for all n ,

(ii) $b_n = b'_n = 0$ for $n > 1$; $b_1 = b'_1 = 1$.

From (47), we have

$$\psi = - \frac{p_{0,x} R}{M^2} \left[\frac{\sinh Mz}{M \cosh M} - z \right] - \epsilon\, p_{0,x} R \left[p_{0,x} R^2 \bar{\bar{A}} \cos x + \bar{B}_1 \sin x \right] , \qquad (48)$$

where

$$\bar{\bar{A}}_1 = G_1 e^{\alpha_{11} z} + G_2 e^{\alpha_{21} z} + G_3 e^{-\alpha_{11} z} + G_4 e^{-\alpha_{21} z} + F(z)$$

and

$$\bar{B}_1 = D_1 e^{\alpha_{11} z} + D_2 e^{\alpha_{21} z} + D_3 e^{-\alpha_{11} z} + D_4 e^{-\alpha_{21} z}.$$

NUMERICAL DISCUSSION

The longitudinal velocity profile for particular values of $\epsilon = 0 \cdot 5$, $R = 0 \cdot 1$ and $p_{0,x} R^2 = 0 \cdot 1$ are shown in Fig. 1. The velocity profiles are drawn for various values of Hartmann number and at different cross-sections of the channel. As the Hartmann number increases, the velocity profiles are flattened and the effect of roughness is also appreciable. As the width of the channel increases the magnitude of the velocity decreases and vice versa.

The transverse velocity profiles with various values of Hartmann number and $\epsilon = 0 \cdot 5$, $R = 0 \cdot 1$, $p_{0,x} R^2 = 0 \cdot 1$ are shown in Fig. 2. It is interesting to note that these velocity profiles have the point of inflexion on the mid-plane. Thus the resulting effect of the longitudinal and transverse velocity profiles is that the direction of the flow is towards the walls if the width of the channel increases, and away from the walls if the width of the channel decreases. This effect of the deformation of the walls diminishes in the neighbourhood of the mid-plane. The transverse velocity which is developed due to the deformation in the wall is reduced to zero on the mid-plane. As the Hartmann number increases, the magnitude of the transverse velocity decreases.

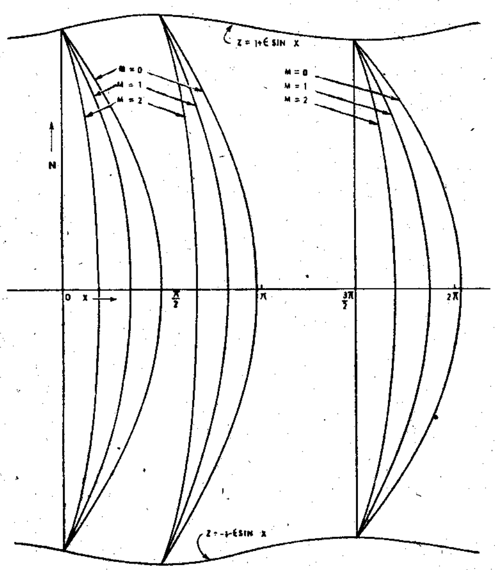

Fig. 1—Longitudinal velocity profiles for particular values ($\epsilon = 0 \cdot 5$, $R = 0 \cdot 1$, $p_{o,x} R^2 = 0 \cdot 1$)

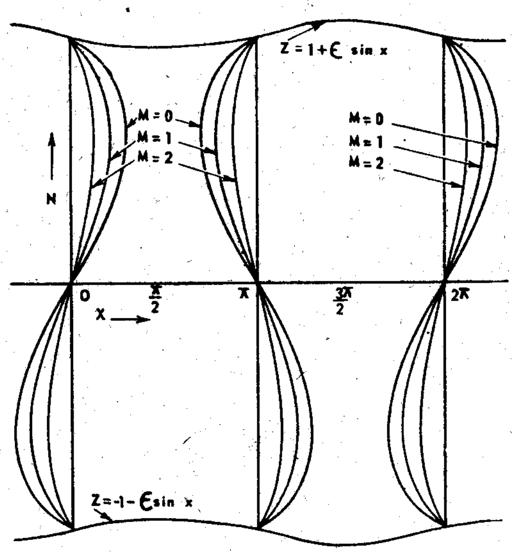

Fig. 2—Transverse velocity profiles for particular values ($\epsilon = 0 \cdot 5$, $R = 0 \cdot 1$, $p_{o,x} R^2 = 0 \cdot 1$)

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