TRANSIENT TEMPERATURE IN A GROWING SOLID

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Using two-parameter quadratic temperature profile in conjunction with the heat balance integral method, approximate solutions for the freezing of a semi-infinite slab and inward freezing of a circular cylinder are investigated, with the assumption that the freezing takes place at the surface of the solidifying medium according to Newton's law of cooling. Comparison with the existing solutions on the former problem shows good agreement.

Transient heat conduction problems accompanied by change of phase occur during aerodynamic heating of high-speed vehicles, nuclear reactor operations, food processing and ice formation, etc. Since such types of problems are non-linear because of the unspecified nature of the moving boundary, their closed analytical solutions present a considerable difficulty.

A few investigations have earlier been made on the transient temperature distribution in a melting solid when the melt is immediately removed on formation. Landau[1] and Masters[2] have studied sublimation problems for very large heat inputs at the exposed surface and have obtained the solutions of the differential equation by the method of numerical integration. Recently, Goodman[3] has developed a mathematical technique of heat balance integral and applied it to a number of phase change problems. The technique transforms the non-linear problem into an ordinary initial value problem whose solution can generally be expressed in closed analytical form. Ahuja & Kumar[4] have discussed the problem of melting thin cylindrical tubes by the application of this technique when the molten mass was not removed.

In this paper an attempt has been made to find out the transient temperature distribution in a growing solid. The solidification is effected by considering Newtonian cooling at the surface. The problem has been studied for two different configurations : (1) freezing of a semi-infinite plane slab, and (2) inward freezing of a circular cylinder. The results of study on plane slab were compared with solutions already known[5] and a noticeable agreement was observed. Finally, the results of these investigations were evaluated numerically and depicted graphically.

STATEMENT OF THE PROBLEM

For the purpose of this study, the liquid has been assumed to satisfy the following conditions :

(*i*) The whole liquid remains at the fusion temperature.

(*ii*) All the thermal properties of the material are uniform and constant.

(*iii*) No change of volume occurs during phase transformation, *i.e.* convection effects in the liquid have been ignored.

On the basis of these assumptions, the energy equation for the solidified phase can be written as

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$$\frac{\partial T}{\partial t}=\frac{K}{\rho C}\,\frac{1}{x^{\lambda}}\,\frac{\partial}{\partial x}\left(x^{\lambda}\frac{\partial T}{\partial x}\right) \tag{1}$$

where λ is the configuration parameter—zero for the plane boundary and unity for the cylinder; K, ρ and C are thermal conductivity, density and specific heat of the solid respectively.

Boundary conditions

At the solid-liquid interface, one of the boundary conditions to be satisfied is the equality of the two temperatures. If T_F is the fusion temperature* and $s(t)$ the position of the moving front at any instant t, condition 1 for (i) plane boundary slab with boundary $x = s(t)$ and (ii) cylinder with boundary $x = a - s(t)$ where a is the radius of the cylinder, is given by

$$T = T_F \tag{2}$$

A second boundary condition at the moving interface relates to the liberation of heat at this surface. This condition for the solidification of the plane medium and for the inward freezing of the cylinder is given respectively by

$$K\frac{\partial T}{\partial x} = L\rho\,\frac{ds}{dt}\ ;\quad x = s\,(t) \tag{3}$$

$$K\frac{\partial T}{\partial x} = -\,L\rho\,\frac{ds}{dt}\,;\ x = a - s\,(t) \tag{4}$$

The wall condition for the solidification of semi-infinite slab at the boundary $x = 0$ and for the freezing of the cylinder at the boundary $x = a$ is written respectively as

$$K\frac{\partial T}{\partial x} = H\,(\,T - Ts\,) \tag{5}$$

$$K\frac{\partial T}{\partial x} = -\,H\,(\,T - Ts\,) \tag{6}$$

where Ts is the temperature of the surrounding medium and H is the coefficient of the surface heat transfer.

Subject to the boundary conditions (2) through (6), it is required to find out the solution of the problem (1) for the two configurations under study.

SOLUTION OF THE PROBLEM†

Introducing the non-dimensional quantities :

$$\theta = \frac{T - T_F}{T_F - Ts}\,,\ \tau = \frac{(\,T_F - Ts\,)\,H^2}{L\rho K}\,t\,,\ \epsilon = \frac{H}{K}\,s\,(t) \tag{7}$$

$$\delta = \frac{(\,T - T_F\,)\,C}{L}\,,\ a = \frac{H}{K}$$

equation (1) for the two situations may be expressed as

$$\delta\,\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial r^2} \quad \text{(A)}$$

$$\delta\,(1 - r)\,\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial r}\left\{(\,1 - r\,)\,\frac{\partial\theta}{\partial r}\right\} \quad \text{(B)}$$

$$0 < r < \epsilon,\quad \tau > 0 \tag{8}$$

*The temperature scale is so chosen that solidification occurs at zero temperature.

†Henceforth capital letter A in the parenthesis will refer to the solidification of the plane slab and capital letter B to the freezing of the cylinder.

The corresponding boundary conditions in terms of the dimensionless space variable r are

$$\theta = 0 \quad ; \quad r = \epsilon \tag{9}$$

$$\frac{\partial \theta}{\partial r} = \frac{d\epsilon}{d\tau} \quad ; \quad r = \epsilon \tag{10}$$

$$\frac{\partial \theta}{\partial r} = 1 + \theta \quad ; \quad r = 0 \tag{11}$$

where r for the freezing of the semi-infinite slab and the inward freezing of the cylinder is given respectively by

$$\left.\begin{aligned} r &= \frac{H}{K}\, x \qquad &\text{(A)} \\ r &= \frac{K - Hx}{K} \qquad &\text{(B)} \end{aligned}\right\} \tag{12}$$

The method of heat balance integral is now used. Integrating equations (8) with respect to r between the limits $r = 0$ and $r = \epsilon$ and using the boundary conditions (10) and (11), we obtain

$$\left.\begin{aligned} \frac{d\epsilon}{d\tau} - (1 + \theta)\Big|_{r=0} &= \delta \int_0^{\epsilon} \frac{\partial \theta}{\partial \tau}\, dr \qquad &\text{(A)} \\ (1 - \epsilon)\frac{d\epsilon}{d\tau} - (1 + \theta)\Big|_{r=0} &= \delta \int_0^{\epsilon} (1 - r) \frac{\partial \theta}{\partial \tau}\, dr \qquad &\text{(B)} \end{aligned}\right\} \tag{13}$$

Again multiplying (8A) by $2\,\frac{\partial \theta}{\partial r}$ and (8B) by $2\,(1 - r)\frac{\partial \theta}{\partial r}$ and integrating with respect to r, we get

$$\left.\begin{aligned} \left\{ \left(\frac{\partial \theta}{\partial r} \right)^2 \right\}_0^{\epsilon} &= 2\delta \int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} \frac{\partial \theta}{\partial r}\, dr \qquad &\text{(A)} \\ \left\{ \left[(1 - r) \frac{\partial \theta}{\partial r} \right]^2 \right\}_0^{\epsilon} &= 2\delta \int_0^{\epsilon} (1 - r)^2 \frac{\partial \theta}{\partial \tau} \frac{\partial \theta}{\partial r}\, dr \qquad &\text{(B)} \end{aligned}\right\} \tag{14}$$

Now considering curves of constant temperature in the $r\tau$—plane, using relations (9), (10) and (11) equations (14) reduce to

$$\left.\begin{aligned} \frac{\partial \theta}{\partial \tau}\Big|_{r=\epsilon} + (1 + \theta)^2\Big|_{r=0} &= -\,2\delta \int_0^{\epsilon} \frac{\partial \theta}{\partial \tau} \frac{\partial \theta}{\partial r}\, dr \qquad &\text{(A)} \\ (1 - \epsilon)^2 \frac{\partial \theta}{\partial \tau}\Big|_{r=\epsilon} + (1 + \theta)^2\Big|_{r=0} &= -\,2\delta \int_0^{\epsilon} (1 - r)^2 \frac{\partial \theta}{\partial \tau} \frac{\partial \theta}{\partial r}\, dr \qquad &\text{(B)} \end{aligned}\right\} \tag{15}$$

To evaluate the integrals on the right-hand side of equations (13) and (15), we assume the following two-parameter quadratic temperature profile for θ :

$$\theta = -\frac{\epsilon}{1+\epsilon}\left(1-\frac{r}{\epsilon}\right)+g\left(\frac{1+r}{1+\epsilon}-\frac{r^2}{\epsilon^2}\right) \tag{16}$$

which clearly satisfies the boundary conditions (9) and (11). The initial conditions for g and ϵ are

$$\left.\begin{array}{l} \epsilon = 0\,, \quad \tau = 0 \\ \lim\limits_{\epsilon \to 0} \epsilon g = 0 \end{array}\right| \tag{17}$$

Here, ϵ and g are two unknown functions of time and are to be determined from equations (13), (15) and (16).

Substituting for θ from (16) in (13), we obtain the following relations after evaluating the integral expressions

$$\left.\begin{array}{lr} \dfrac{1+\epsilon}{1+g}\,\dfrac{3-2\delta g}{3}\,\dfrac{d\epsilon}{d\tau}+\dfrac{\delta(2+\epsilon)\epsilon}{2(1+\epsilon)}\,\dfrac{d\epsilon}{d\tau}-\dfrac{\delta(4+\epsilon)\epsilon}{6(1+g)}\,\dfrac{dg}{d\tau}=1 & \text{(A)} \\ (1+\epsilon)\dfrac{d\epsilon}{d\tau}-\dfrac{\delta\{(4-\epsilon-2\epsilon^2-\epsilon^3)g-2(3-\epsilon^2)\epsilon\}}{6(1+\epsilon)^2}\,\dfrac{d\epsilon}{d\tau}-\dfrac{\delta(8-\epsilon-\epsilon^2)\epsilon}{12(1+\epsilon)}\,\dfrac{dg}{d\tau}=\dfrac{1+g}{1+\epsilon} & \text{(B)} \end{array}\right\} \tag{18}$$

Integration of equations (18) with respect to τ with the initial condition (17) gives

$$\tau = \frac{\delta}{4}\left\{(1+\epsilon)^2-1-2\log(1+\epsilon)\right\}-\frac{\delta(4+\epsilon)\epsilon}{6}\log(1+g) + \int_0^{\epsilon}\left\{\frac{1+\epsilon}{1+g}\,\frac{3-2\delta g}{3}+\frac{\delta(2+\epsilon)}{3}\log(1+g)\right\}d\epsilon \tag{19A}$$

$$\tau = \delta\left\{\frac{2\epsilon}{3}+\frac{\epsilon^2}{6}-\frac{\epsilon^3}{9}-\frac{2}{3}\log(1+\epsilon)-\frac{8\epsilon-\epsilon^2-\epsilon^3}{12}\log(1+g)\right\} + \int\left\{\left[1-\epsilon-\frac{\delta(4-3\epsilon)}{6}g\right]\frac{1+\epsilon}{1+g}+\frac{\delta}{12}\left[(8-2\epsilon-3\epsilon^2)\log(1+g)\right]\right\}d\epsilon \tag{19B}$$

Again combining (15) and (16) and solving the integral expressions, we get

$$\left(\frac{2g}{\epsilon}-\frac{1+g}{1+\epsilon}\right)\frac{d\epsilon}{d\tau}+\left(\frac{1+g}{1+\epsilon}\right)^2 = -2\delta\left\{\frac{\epsilon-g}{(1+\epsilon)^2}+\frac{(3\epsilon-4g-g\epsilon)\epsilon}{6(1+\epsilon)^2}-\frac{2\epsilon-3g-g\epsilon}{6(1+\epsilon)}\right\}\frac{dg}{d\tau}+2\delta\left\{\frac{(\epsilon-g)(1+g)}{(1+\epsilon)\epsilon}+\frac{(1+g)(3\epsilon-4g-g\epsilon)\epsilon}{6(1+\epsilon)^3}-\frac{2(2\epsilon-3g-g\epsilon)g}{6(1+\epsilon)\epsilon}\right\}\frac{d\epsilon}{d\tau} \tag{20A}$$

$$(1-\epsilon)^2\left(\frac{2g}{\epsilon}-\frac{1+g}{1+\epsilon}\right)\frac{d\epsilon}{d\tau}-\left(\frac{1+g}{1+\epsilon}\right)^2=-\frac{\delta}{30(1+\epsilon)^2}\Big\{40\epsilon-20\epsilon^2-2\epsilon^3+3\epsilon^4$$

$$+(-30+32\epsilon+4\epsilon^2-4\epsilon^3-\epsilon^4)\,g\Big\}\frac{dg}{d\tau}$$

$$-\frac{\delta}{30(1+\epsilon)^3\epsilon}\Big\{-(60\epsilon^2-30\epsilon^3-20\epsilon^4+15\epsilon^5)+(100\epsilon-80\epsilon^2-66\epsilon^3+22\epsilon^4+18\epsilon^5)\,g$$

$$+(-60+16\epsilon+48\epsilon^2+12\epsilon^3-22\epsilon^4-7\epsilon^5)\,g^2\Big\}\frac{d\epsilon}{d\tau} \qquad (20B)$$

Eliminating τ from (18) and (20) the resulting equations in g and ϵ are

$$\Big\{4\epsilon+\epsilon^2-(6+4\epsilon+\epsilon^2)\,g\Big\}\,\epsilon(1+\epsilon)\frac{dg}{d\epsilon}=6\epsilon^2+2\epsilon^3-(16\epsilon+16\epsilon^2+6\epsilon^3)\,g$$

$$+(12+20\epsilon+14\epsilon^2+3\epsilon^3)\,g^2-\frac{12(1+\epsilon)^2}{\delta}\,g \qquad (21A)$$

with the condition that

$$\underset{\epsilon\to 0}{Lt}\ \epsilon g=0$$

$$\delta\Big\{-40\epsilon+35\epsilon^2-\epsilon^3-6\epsilon^4+(60-24\epsilon-13\epsilon^2+3\epsilon^3+2\epsilon^4)\,g\Big\}\,\epsilon(1+\epsilon)\frac{dg}{d\epsilon}=$$

$$60(1+\epsilon)(1-\epsilon^2)(\epsilon^2+2g-g\epsilon^2)$$

$$-2\delta(30\epsilon^2-30\epsilon^3-10\epsilon^4+15\epsilon^5)$$

$$+2\delta(80\epsilon-45\epsilon^2-56\epsilon^3+17\epsilon^4+18\epsilon^5)\,g$$

$$-2\delta(60+4\epsilon-53\epsilon^2-22\epsilon^3+17\epsilon^4+7\epsilon^5)\,g^2 \qquad (21B)$$

with the condition that $\underset{\epsilon\to 0}{Lt}\ \epsilon g=0$

Equations (19) and (21) together determine the temperature-time history of the solidifying medium. It may be remarked that inspite of the simplification achieved by the integral method, equations (21) still remain non-linear and their solutions in closed analytical forms are difficult to obtain. They have, however, been solved numerically by the method given by Fox & Goodwin[6], for different values of δ. For the numerical work the following initial values have been used

$$g(\epsilon)=g'(\epsilon)=0 \ ; \ \epsilon=0$$

where dash denotes differentiation of g with respect to ϵ.

The sets of values of g corresponding to the different values of ϵ for any particular value of δ have been used in equations (19) and the values of $\tau(\epsilon)$ have been obtained for these values of δ.

RESULTS AND DISCUSSION

Fig. 1 is the plot of ϵ, the dimensionless plane thickness solidified *vs* τ for different values of δ. The results of the present investigation have been compared

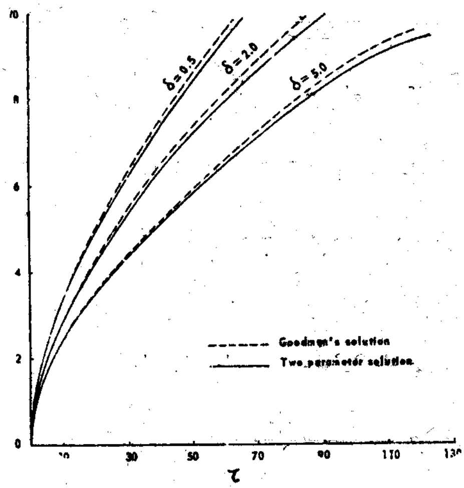

FIG. 1—Thickness of the plane solidified *vs* time for radiation boundary condition at the surface.

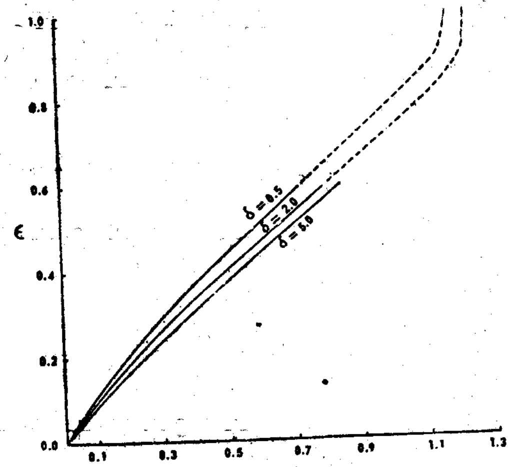

FIG. 2—Thickness of the cylinder solidified *vs* time for the radiation boundary condition and various values of δ.

graphically with those obtained by Goodman[3]. It is noticed that during the initial stages of solidification, both methods give almost similar results. It is also observed that for the case : $\delta = 0$ ($\delta = 0$ corresponds to $\beta = 1$ of Goodman[3]) equation (19A) gives

$$\tau = \epsilon + \tfrac{1}{2}\,\epsilon^2$$

This expression for τ is the same as that obtained by Goodman[3] in equation (48) for the case $\beta = 1$.

TABLE 1

TEMPERATURE DISTRIBUTION θ IN A CYLINDER FOR VARIOUS DEPTHS OF SOLIDIFICATION (ϵ) AND FOR $\delta = 0\cdot5$

	θ				
r	$\epsilon = 0\cdot3$	$\epsilon = 0\cdot5$	$\epsilon = 0\cdot8$	$\epsilon = 0\cdot9$	$\epsilon = 1\cdot0$
0·00	—0·25136	—0·38564	—0·55226	—0·58966	—0·60218
0·05	—0·21317	—0·35414	—0·52912	—0·56847	—9·58177
0·10	—0·17352	—0·32108	—0·50446	—0·54591	—0·56035
0·15	—0·13236	—0·28643	—0·47828	—0·52198	—0·53842
0·20	—0·08973	—0·25022	—0·45058	—0·49671	—0·51444
0·25	—0·04562	—0·21244	—0·42138	—0·47007	—0·48995
0·30	—0·0	—0·17309	—0·39065	—0·44208	—0·46444
0·35		—0·13217	—0·35841	—0·41272	—0·43790
0·40	—0·08969	—0·08969	—0·32465	—0·38200	—0·41035
0·45		—0·04562	—0·28937	—0·34992	—0·38177
0·50		0·0	—0·25259	—0·31647	—0·35218
0·55			—0·21428	—0·28168	—0·32156
0·60			—0·17445	—0·24552	—0·28991
0·65			—0·13313	—0·20800	—0·25725
0·70			—0·09026	—0·16912	—0·22357
0·75			—0·04589	—0·12888	—0·18886
0·80			0·0	—0·08728	—0·15313
0·85				—0·04432	—0·11638
0·90				0·0	—0·07861
0·95					—0·03982
1·00					0·θ

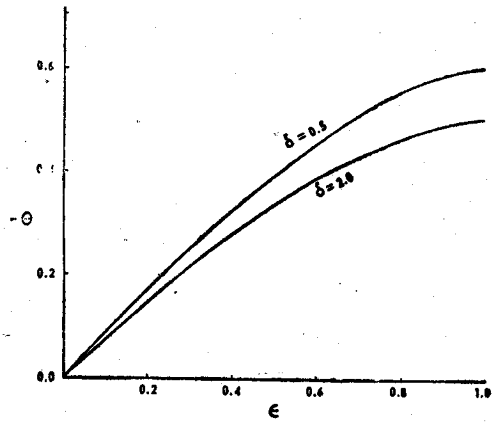

FIG. 3—Temperature at the surface of the cylinder *vs* time for the radiation boundary condition.

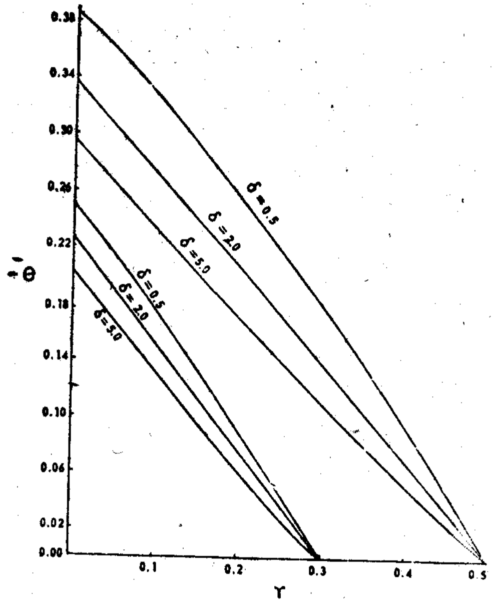

FIG. 4—Temperature distribution in two different solidified regions of the cylinder for radiation boundary condition at the surface and various values of δ.

Fig. 2 gives the dimensionless thickness of the frozen part of the cylinder *vs* τ for different values of δ. It has been observed that during the closing periods, the solidification is rather abrupt. The dimensionless time for complete solidification of the cylinder for two different values of δ ($\delta = 0{\cdot}5$ and $\delta = 2{\cdot}0$) has been observed to be $1{\cdot}18$ and $1{\cdot}235$ respectively. Fig. 3 shows temperature at the surface of the solidifying cylinder *vs* ϵ for two different values of δ, while Fig. 4 is the temperature distribution in two solidified regions of the cylinder which correspond to $\epsilon = 0{\cdot}3$ and $0{\cdot}5$.

Table 1 gives the temperature distribution in the solidified regions of the cylinder for various depths of solidification ϵ and for $\delta = 0{\cdot}5$. For this value of δ the temperature T at the surface of ice cylinder has also be calculated. It has been noticed that under the given conditions and with the use of the temperature profile given in equation (16) a temperature of —30·6°C should be maintained at the surface of the cylinder in order to freeze it upto half of its thickness, and for the complete solidification to occur the surface temperature should be —47·8°C.

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