OPTIMUM ANGLE OF ATTACK FOR A RECTANGULAR TARGET

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The optimum angles of attack for a rectangular target along with the necessary conditions are discussed.

Coverage problems play an important role in the defence of a country. They help in planning and providing the necessary sorties and ammunition for destroying a target. These studies also help the Air Force pilots as guide lines for achieving success in their bombing missions.

Earlier papers considered either point targets or only circular targets. A useful review of the coverage problems was given by Guenther *et al*[1], which includes published and unpublished material, in respect of point targets. A good deal of work on circular targets was also done by Jarnagin & DiDontato[2], and others. It is felt that in practice, rectangular targets like bridges and railway tracks at strategic points, occur equally if not more frequently than point and circular targets. In this paper, the optimum angles of attack for a rectangular target along with the necessary conditions are derived.

RECTANGULAR TARGET

Consider a rectangular target of sides $2a$ and $2b$ as shown in Fig. 1. Let O, the centre of the rectangular target which is also the origin of the coordinate system, be the aiming point of the target. Let the line of flight BC (passing through O) make an angle θ with the x-axis. The coordinates of a random point A, *i.e.* impact point of the bomb due to errors, be (ξ, η) with respect to BC and its perpendicular, the origin remaining the same. ξ and η are $N(O, \sigma_1)$ and $N(O, \sigma_2)$ respectively and independent. Now the target is the rectangular contained between the 4 lines (1·1), (1·2), (1·3), and (1·4) whose equations with respect to Ox and Oy are :

$$\left.\begin{array}{ll} x = a & (1\cdot1) \\ y = b & (1\cdot2) \\ x = -a & (1\cdot3) \\ y = -b & (1\cdot4) \end{array}\right\} \qquad (1)$$

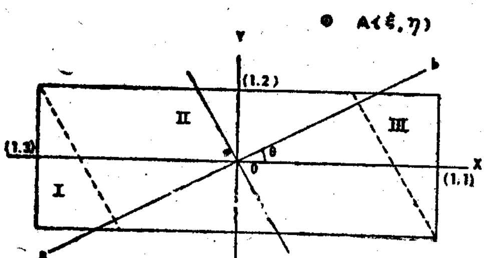

Fig. 1

Transforming them in terms of the coordinate axis BC and its perpendicular we get them as

$$\left.\begin{array}{l} \xi \cos\theta - \eta \sin\theta = a \\ \xi \sin\theta + \eta \cos\theta = b \\ \xi \cos\theta - \eta \sin\theta = -a \\ \xi \sin\theta + \eta \cos\theta = -b \end{array}\right\} \qquad (2)$$

Dividing the rectangle into 3 parts I, II and III (as shown in the Fig 1) and integrating the function

$$\frac{1}{2\pi\sigma_1\sigma_2} \exp - \frac{1}{2}\left\{ \frac{\xi^2}{\sigma_1^2} + \frac{\eta^2}{\sigma_2^2} \right\}$$

over these divisions, we get the probability of covering the rectangle. The limits of η and ξ in the 3 divisions are given below :

	Limits of η .	*Limits of* ξ
I	$(-\xi \tan\theta - b\sec\theta)$ to $(\xi\cot\theta + a\,\mathrm{cosec}\,\theta)$	$(-a\cos\theta - b\sin\theta)$ to $(-a\cos\theta + b\sin\theta)$
II	$(-\xi\tan\theta - b\sec\theta)$ to $(-\xi\tan\theta + b\sec\theta)$	$(-a\cos\theta + b\sin\theta)$ to $(a\cos\theta - b\sin\theta)$
III	$(\xi\cot\theta - a\,\mathrm{cosec}\,\theta)$ to $(-\xi\tan\theta + b\sec\theta)$	$(a\cos\theta - b\sin\theta)$ to $(a\cos\theta + b\sin\theta)$

$$\begin{aligned}\therefore P = &\int_{-a\cos\theta - b\sin\theta}^{-a\cos\theta + b\sin\theta} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\xi^2/\sigma_1^2} \int_{-\xi\tan\theta - b\sec\theta}^{\xi\cot\theta + a\,\mathrm{cosec}\,\theta} \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\eta^2/\sigma_2^2}\, d\eta\, d\xi \\ &+ \int_{-a\cos\theta + b\sin\theta}^{a\cos\theta - b\sin\theta} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\xi^2/\sigma_1^2} \int_{-\xi\tan\theta - b\sec\theta}^{-\xi\tan\theta + b\sec\theta} \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\eta^2/\sigma_2^2}\, d\eta\, d\xi \\ &+ \int_{a\cos\theta - b\sin\theta}^{a\cos\theta + b\sin\theta} \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\xi^2/\sigma_1^2} \int_{\xi\cot\theta - a\,\mathrm{cosec}\,\theta}^{-\xi\tan\theta + b\sec\theta} \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\eta^2/\sigma_2^2}\, d\eta\, d\xi \end{aligned} \tag{3}$$

Instead of evaluating this integral and differentiating with respect to θ to find the optimum angle, we can straight away differentiate (3) with respect to θ.

Differentiating (3) with respect to θ and making the necessary algebraic manipulations, we get

$$\begin{aligned}\frac{\partial P}{\partial \theta} = &\int_{-a\cos\theta - b\sin\theta}^{a\cos\theta + b\sin\theta} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\xi^2/2\sigma_1^2} (\xi\sec^2\theta + b\sec\theta\tan\theta)\, e^{-\frac{1}{2\sigma_2^2}\left\{\xi\tan\theta + b\sec\theta\right\}^2} d\xi \\ &+ \int_{-a\cos\theta + b\sin\theta}^{a\cos\theta + b\sin\theta} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\xi^2/2\sigma_1^2} (-\xi\sec^2\theta + b\sec\theta\tan\theta)\, e^{-\frac{1}{2\sigma_2^2}\left\{-\xi\tan\theta + b\sec\theta\right\}^2} d\xi\end{aligned}$$

$$-2\int_{-a\cos\theta-b\sin\theta}^{-a\cos\theta+b\sin\theta} \frac{1}{2\pi\sigma_1\sigma_2}\, e^{-\xi^2/2\sigma_1^2} (\xi\operatorname{cosec}^2\theta + a\cot\theta\operatorname{cosec}\theta)\, e^{-\frac{1}{2\sigma_2^2}\left\{\xi\cot\theta + a\operatorname{cosec}\theta\right\}^2} d\xi \tag{4}$$

$$\frac{\partial P}{\partial \theta} = 0 \qquad \text{at } \theta = 0° \text{ and } \pi/2$$

i.e. the attack should be along either of the sides only.

Conditions under which the attack along with the length and breadth are optimum

If a $> b$ and $\sigma_1 > \sigma_2$, it is clear that

$$e^{-a^2/2\sigma_1^2 - b^2/2\sigma_2^2} > e^{-a^2/2\sigma_2^2 - b^2/2\sigma_1^2}$$

Integrating both sides with respect to a and b in the limits 0 to a and 0 to b respectively, we get

$$\int_0^a e^{-x^2/2\sigma_1^2} dx \int_0^b e^{-y^2/2\sigma_2^2} dy > \int_0^a e^{-x^2/2\sigma_2^2} dx \int_0^b e^{-y^2/2\sigma_1^2} dy \tag{5}$$

The left-hand side of (5) represents the probability that a bomb dropped, when the attack is at 0°, will fall in the rectangle, but for the constant term. Similarly, the right-hand side gives the probability when the attack is at 90°. This shows that the probability is optimum when the larger variance is associated with the larger side of the rectangle.

TABLE 1

EXPECTED NUMBER OF HITS FALLING INSIDE THE RECTANGULAR TARGET

$a = 20 \qquad \sigma_1 = 20$

		b					
σ_2	θ	5	10	15	20	25	30
5	0°	18·24 (24)	26·06 (30)	27·23 (31)	27·30 (31)	27·30 (31)	27·30 (31)
	90°	7·89 (11)	15·31 (19)	21·86 (25)	27·30 (31)	31·54 (32)	34·54 (35)
10	0°	10·46 (9)	18·64 (18)	23·66 (25)	26·06 (28)	26·97 (29)	27·23 (29)
	90°	7·54 (11)	14·62 (22)	20·87 (22)	26·06 (28)	30·11 (31)	33·08 (33)
15	0°	7·13 (4)	13·52 (11)	18·64 (14)	22·32 (19)	24·69 (20)	26·06 (22)
	90°	6·45 (6)	12·52 (12)	17·88 (16)	22·32 (19)	25·79 (21)	28·33 (23)
25	0°	4·33 (4)	8·49 (8)	12·33 (13)	15·74 (18)	18·64 (22)	21·02 (24)
	90°	4·55 (10)	8·83 (14)	12·60 (17)	15·74 (19)	18·18 (21)	19·97 (22)
30	0°	3·61 (2)	7·13 (4)	10·46 (7)	13·52 (10)	16·27 (12)	18·67 (17)
	90°	3·91 (5)	7·58 (8)	10·82 (8)	13·52 (10)	15·62 (11)	17·15 (14)

NUMERICAL ILLUSTRATION

The expected number of hits, falling inside the rectangular target, for $a = 20$ and $b = 5, 10, 15, 20, 25, 30$ and for $\sigma_1 = 20$ and $\sigma_2 = 5, 10, 15, 25, 30$, corresponding to the angles $\theta = 0°$ and $90°$ were calculated on the basis of the probabilities in (5) and given in Table 1. Independent random samples of size 40 each were drawn from random normal deviates (1·3) with standard deviations 5, 10, 15, 20, 25, and 30 and they were plotted in various rectangular targets with sides a and b given above. The number of points lying within the targets for $\theta = 0°$ and $90°$ are counted and given in brackets along with the expected values.

From Table 1, we observe that the number of hits obtained from random samples show the same trend as reflected by the expected number of hits in various sizes of targets and errots for $\theta = 0°$ and $\theta = 90°$ but for some sampling fluctuations. This confirms our earlier conclusions.

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