LINEAR COMBINATIONS OF NON-CENTRAL *CHI*-SQUARE VARIATES

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In this paper the probability density functions for definite and indefinite quadratic forms of non-central normal variates have been derived by integrating the joint distribution of sum and difference of two weighted *chi*-squares.

INTRODUCTION

The distribution of a positive definite quadratic form in normal variates (central or non-central) can be applied to hit probability problems. Development of the distribution for an indefinite quadratic form in non-central normal variates was motivated by a classification problem. The distribution of quadratic forms in normal variates has been studied by Robbins & Pitman[1], Pachares[2], Gurland[3], Shah & Khatri[4], Shah[5] and others. Kabe[6] has obtained the distribution of sum of gamma variates. The probability density functions for definite and indefinite quadratic forms in non-central normal variates was obtained recently by Press[7]. Most of them derived the distributions through Laplace transformation. In this paper, we derive them by integrating the joint distribution of sum and difference of two weighted *chi*-squares. Section I is devoted to the distribution of sum and difference of two non-central gamma variates. In Section II, the distribution of sum of r non-central gamma variates and the difference of two such sums is considered. Finally in Section III, we have obtained the density functions of definite and indefinite quadratic forms.

I—DISTRIBUTION OF SUM AND DIFFERENCE OF NON-CENTRAL GAMMA VARIATES

Let $\chi^2_{m,d}$ denote a non-central *chi*-square variate with m degrees of freedom and non-central parameter d^2 whose probability density function is given by

$$f(y) = y^{\frac{m}{2}-1} e^{-\frac{y+d^2}{2}} 2^{-\frac{m}{2}} \sum_{i=0}^{\infty} \left[(d^2 y)^i \Big/ 2^{2i}\, i!\, \Gamma\left(\frac{m}{2}+i\right) \right] \qquad (1)$$

for $y > 0$, and zero otherwise.

Then the probability density function of $x = \alpha y$, for $\alpha > 0$ is given by

$$a^P e^{-\left(ax+\frac{d^2}{2}\right)} \sum_{i=0}^{\infty} \left[(d^2 a)^i x^{P+i-1} \Big/ 2^i\, i!\, \Gamma(P+i) \right] \qquad (2)$$

where $P = \dfrac{m}{2}$ and $a^{-1} = 2\,\alpha$

Then x can be named as non-central gamma variate with parameters a, P and non-centrality parameter d^2.

Let us define

$$U_1 = x_1 - x_2$$
$$V_1 = x_1 + x_2$$

where x_i ($i=1, 2$) be two independent non-central gamma variates with parameters a_i, P_i and d^2_i ($i = 1, 2$).

The joint distribution of U_1 and V_1 is given by

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_{jk}\, 2^{1-P_1-P_2-j-k} (U_1+V_1)^{P_1+j-1} (V_1-U_1)^{P_2+k-1} e^{-\left[\frac{a_1-a_2}{2} U_1 + \frac{a_1+a_2}{2} V_1\right]} \tag{3}$$

where

$$q_{jk} = \left(q_j^{(1)} \; q_k^{(2)} \; a_1^{P_1+j} a_2^{P_2+k}\right) \left(\overline{|P_1+j} \;\; \overline{|P_2+k}\right)^{-1}$$

and

$$q^{(i)} = e^{-\frac{d^2_i}{2}} \left(\frac{d^2_i}{2}\right)^j \Big/ \underline{|j} \qquad (i = 1,2).$$

If $h_{2P_1,\,2P_2}(U_1)$ denotes the density function of U_1, then for $U_1 > 0$

$$h_{2P_1,\,2P_2}(U_1) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_{jk}\, 2^{1-P_1-P_2-j-k} e^{-\frac{(a_1-a_2)\,U_1}{2}} \int_{U_1}^{\infty} (U_1+V_1)^{P_1+j-1}$$
$$(V_1-U_1)^{P_2+k-1} e^{-\frac{(a_1+a_2)V_1}{2}} \, d\, V_1 \tag{4}$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_{jk}\, e^{-a_1 U_1} U_1^{P_1+P_2+j+k-1} \int_0^{\infty} e^{-(a_1+a_2)U_1 t}\, t^{P_2+k-1} (1+t)^{P_1+j-1} dt$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_{jk} \overline{|P_2+k}\; e^{-a_1U_1} U_1^{P_1+P_2+j+k-1}\, \Psi\left[P_2+k, P_1+P_2+j+k; (a_1+a_2)\, U_1\right]$$

where, we have from Erdelyi[8]

$$\Psi(a, b; x) = (\overline{|a})^{-1} \int_0^{\infty} e^{-xt}\, t^{a-1} (1+t)^{b-a-1} dt$$

for $a > 0$, $x > 0$ is a confluent hypergeometric function. Similarly for $U_1 < 0$, integrating (3) *w.r.t.* V_1 over the limits ($-U_1$, ∞), we have

$$h_{2P_1\,2P_2}(U_1) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_{jk} \overline{|P_1+j}\; e^{a_2U_1} (-U_1)^{P_1+P_2+j+k-1}$$
$$\Psi\left[P_1+j, \;\; P_1+P_2+j+k; -(a_1+a_2)\, U_1\right]. \tag{5}$$

(4) and (5) can also be written as

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} Q_{ij} \exp(-a_1 U_1) U_1^{P_1+P_2+i+j-1} \quad \text{for } U_1 \geqslant 0$$

$$\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} Q'_{ij} \exp(a_2 U_1)(-U_1)^{P_1+P_2+i+j-1} \quad \text{for } U_1 \leqslant 0$$

where Q_{ij} and Q'_{ij} are some functions of P_1, P_2, d_1, d_2, a_1 and a_2.

The probability density function of V_1 when $a_1 \geqslant a_2$ is given by

$$g^{(V_1)}_{2P_1, 2P_2} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{jk} 2^{1-P_1-P_2-j-k} e^{-\frac{(a_1+a_2)V_1}{2}} \int_{-V_1}^{V_1} e^{-\frac{(a_1-a_2)U_1}{2}} (U_1+V_1)^{P_1+j-1}(V_1-U_1)^{P_2+k-1} dU_1 \tag{6}$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{jk} 2^{1-P_1-P_2-j-k} V_1^{P_1+P_2+j+k-1} e^{-\frac{(a_1+a_2)V_1}{2}} \int_{-1}^{1} e^{-\frac{(a_1-a_2)V_1 t}{2}} (1+t)^{P_1+j-1}(1-t)^{P_2+k-1} dt$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{jk} B(P_1+j, P_2+k) e^{-a_1 V_1} V_1^{P_1+P_2+j+k-1} \; {}_1F_1\left[P_2+k; P_1+P_2+j+k; (a_1-a_2)V_1\right].$$

Since we have from Slater[9]

$${}_1F_1(a, b; x) = B^{-1}(b-a, a) e^{\frac{x}{2}} 2^{1-b} \int_{-1}^{1} e^{-\frac{xS}{2}} (1+S)^{b-a-1} (1-S)^{a-1} dS$$

when $a_1 \leqslant a_2$, we can write

$$g^{(V_1)}_{2P_1, 2P_2} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} q_{jk} B(P_1+j, P_2+k) e^{-a_2 V_1} V_1^{P_1+P_2+j+k-1} \; {}_1F_1\left[P_1+j; P_1+P_2+j+k; (a_2-a_1)V_1\right].$$

Corollary (1) : The probability density function of the difference (U) and the sum (V) of two independent gamma variates with parameters (a_1, P_1) and (a_2, P_2) are given by

$$h(U) = \begin{cases} \dfrac{a_1^{P_1} \; a_2^{P_2}}{\overline{|P_1}} \; e^{-a_1 U} \; U^{P_1+P_2-1} \; \Psi\left[P_2, P_1 + P_2; (a_1 + a_2) U\right] & \text{for } U \geqslant 0 \\ \dfrac{a_1^{P_1} \; a_2^{P_2}}{\overline{|P_2}} \; e^{a_2 U} \; (-U)^{P_1+P_2-1} \; \Psi\left[P_1, P_1 + P_2; -(a_1 + a_2) U\right] & \text{for } U_1 \leqslant 0 \end{cases} \tag{7}$$

$$g(V) = \begin{cases} \dfrac{a_1^{P_1} \; a_2^{P_2}}{\overline{|P_1+P_2}} \; e^{-a_1 V} \; V^{P_1+P_2-1} \; {}_1F_1\left[P_2; P_1 + P_2; (a_1 - a_2) V\right] & \text{for } a_1 > a_2 \\ \dfrac{a_1^{P_1} \; a_2^{P_2}}{\overline{|P_1+P_2}} \; e^{-a_2 V} \; V^{P_1+P_2-1} \; {}_1F_1\left[P_1; P_1 + P_2; (a_2 - a_1) V\right] & \text{for } a_1 \leqslant a_2 \end{cases} \tag{8}$$

(8) was also obtained by Kabe[6].

r^{th} moment of V is given by

$$E(V^r) = \frac{a_2^{P_2} \; \overline{|P_1 + P_2 + r}}{a_1^{P_2+r} \; \overline{|P_1 + P_2}} \; {}_2F_1\left[P_2, P_1 + P_2 + r; P_1 + P_2; 1 - \frac{a_2}{a_1}\right].$$

For $a_1 > a_2$, we can write the probability density function of $Z = (a_1 - a_2) V$ as

$$G_1(P, q; \alpha; z) = \frac{\alpha^P \; (1 + \alpha)^{q-P}}{\overline{|q}} \; e^{-(1+\alpha)z} \; z^{q-1} \; {}_1F_1\left[P; q; z\right] \tag{9}$$

for $z > 0$ and the parameters are $P = P_2$, $q = P_1 + P_2$ and $\alpha = \dfrac{a_2}{a_1 - a_2} > 0.$

This is nothing but the generalised exponential family considered by Bhattacharya[10] while dealing with confluent hypergeometric functions and accident proneness.

Corollary (2) : Let χ^2_{ni} $(i = 1, 2)$ be two independent *chi*-square variates with n_i $(i = 1,2)$ degrees of freedom. Then the density function of $U = \alpha \, \chi^2_{n_1} - \beta \, \chi^2_{n_2}$ for $\alpha > 0, \beta > 0$ is

$$h(U) = \begin{cases} \left[\dfrac{C(n_1, n_2)}{\overline{\left|\dfrac{n_1}{2}\right.}}\right] e^{-\frac{U}{2\alpha}} \; U^{\frac{n_1+n_2}{2}-1} \; \Psi\left[\dfrac{n_2}{2}, \dfrac{n_1 + n_2}{2}; \left(\dfrac{1}{2\alpha} + \dfrac{1}{2\beta}\right) U\right] & \text{for } U \geqslant 0 \\ \left[\dfrac{C(n_1, n_2)}{\overline{\left|\dfrac{n_2}{2}\right.}}\right] e^{\frac{U}{2\beta}} \; (-U)^{\frac{n_1+n_2}{2}-1} \; \Psi\left[\dfrac{n_1}{2}, \dfrac{n_1 + n_2}{2}; -\left(\dfrac{1}{2\alpha} + \dfrac{1}{2\beta}\right) U\right] & \text{for } U < 0 \end{cases} \tag{10}$$

where

$$C^{-1}(n_1, n_2) = 2^{\frac{n_1+n_2}{2}} \; \alpha^{\frac{n_1}{2}} \; \beta^{\frac{n_2}{2}}.$$

This result was obtained by Robinson[11] through Laplace transformation and by Press[7] through direct integration. For $\beta > \alpha$ the probability density function of $V = \alpha\chi^2_{n_1} + \beta\chi^2_{n_2}$ is given by

$$g^{(V)}_{n_1, n_2} = \left[\frac{C(n_1, n_2)}{\Gamma\left(\frac{n_1 + n_2}{2}\right)} \right] e^{-\frac{V}{2\alpha}} V^{\frac{n_1+n_2}{2}-1} {}_1F_1 \left[\frac{n_2}{2}, \frac{n_1 + n_2}{2}, \left(\frac{1}{2\alpha} - \frac{1}{2\beta}\right) V \right] \tag{11}$$

the expected value of $V^{\frac{1}{2}}$ is

$$E(V^{\frac{1}{2}}) = \left(\frac{2^{\frac{1}{2}} \alpha^{\frac{n_2+1}{2}} \Gamma\left(\frac{n_1 + n_2 + 1}{2}\right)}{\beta^{\frac{n_2}{2}} \Gamma\left(\frac{n_1 + n_2}{2}\right)} \right) {}_2F_1 \left[\frac{n_2}{2} ; \frac{n_1 + n_2 + 1}{2} ; \frac{n_1 + n_2}{2} ; 1 - \frac{\alpha}{\beta} \right]. \tag{12}$$

This value, but in a different form, was used by Press[12] in connection with the comparison of lengths of confidence intervals for Behrens-Fisher problem.

II—DISTRIBUTION OF THE SUM OF r NON-CENTRAL GAMMA VARIATES

Let x_i $(i = 1, 2, \ldots\ldots r)$ be r independent non-central gamma variates with parameters a_i, P_i and d^2_i $(i = 1, 2 \ldots\ldots r)$.

From (6) we can write the probability density function of $V_1 = x_1 + x_2$ as

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} C_{jkl}\, e_k^{-a_1 V_1}\, V_1^{P_1+P_2+j+k+l-1}$$

where

$$C_{jkl} = q_j^{(1)} q_k^{(2)} a_1^{P_1+j} a_2^{P_2+k} (P_2 + k)_l (a_1 - a_2)^l / \Gamma(P_1 + P_2 + j + k + l).$$

Let $a_1 > a_i$ $(i = 2, 3, \ldots\ldots r)$.

The probability density function of $V_2 = V_1 + x_3$ can be obtained as in Section I, by integrating the joint distribution of V_2 and U_2, $(U_2 = V_1 - x_3)$, *w.r.t.* U_2 over the limits $(-V_2, V_2)$ and is given by

$$g(V_2) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{C_{jkl}\, e^{-\frac{d^2_3}{2}}\, a_3^{P_3+m}\, d_3^{2m}}{2^m\, \Gamma(P_3 + m)\, \underline{|m}}$$

$$\left\{ B(P_1 + P_2 + j + k + l, P_3 + m)\; e^{-a_1 V_2}\; V_2^{\sum_{i=1}^{3} P_i + j + k + l - 1} \right.$$

$$ {}_1F_1 \left[P_3 + m, \sum_{i=1}^{3} P_i + j + k + l : (a_1 - a_3) V_2 \right] \Bigg\} $$

$$ = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{ q_j^{(1)} \; q_k^{(2)} \; q_m^{(3)} \; a_1^{P_1+j} \, a_2^{P_2+k} \, a_3^{P_3+m} }{ \left| \overline{\sum_{i=1}^{3} P_i + j + k + m} \right. } $$

$$ \Bigg\{ e^{-a_1 V_2} \; V_2^{\sum_{i=1}^{3} P_i + j + k + m - 1} \qquad \Phi_2 \Bigg[P_2 + k, P_3 + m; $$

$$ \sum_{i=1}^{3} P_i + j + k + m ; (a_1 - a_2) V_2, (a_1 - a_3) V_2 \Bigg] \Bigg\} \qquad (13) $$

where

$$ q_m^{(3)} = \left(e^{-\frac{d_3^2}{2}} \; d_3^{2m} \right) \left(2^m \, \underline{|m} \right)^{-1} $$

and

$$ \Phi_2 (\alpha, \beta; \gamma; z_1, z_2) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{ (\alpha)_{i_1} (\beta)_{i_2} z_1^{i_1} z_2^{i_2} }{ (r)_{i_1 + i_2} \, \underline{|i_1} \; \underline{|i_2} } $$

is a confluent hypergeometric function in two variables. Similarly, by extending, this to r variates, the density function of $V = \sum_{i=1}^{r} x_i$ is given by

$$ g(V) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{ir=0}^{\infty} \Bigg[\left\{ \prod_{j=1}^{r} q_{ij}^{(j)} \, a_j^{(P_j + i_j)} \right\} \left(\left| \overline{\sum_{j=1}^{r} (P_j + i_j)} \right. \right)^{-1} e^{-a_1 V} $$

$$V^{\sum_{j=1}^{r}(P_j+i_j-1)} \quad \Phi_2\left[P_2+i_2, P_3+i_3, \ldots, P_r+i_r\,; \sum_{j=1}^{r}(P_j+i_j)\,;\,(a_1-a_2)V, (a_1-a_3)V, \ldots, (a_1-a_r)V\right]\Bigg] \tag{14}$$

where $\Phi_2\left[\alpha_1, \alpha_2, \ldots \alpha_n\,; V; z_1, z_2 \ldots z_n\right]$ is a confluent hypergeometric function in n variables, and

$$q^{(j)}_{i_j} = \left(e^{-\frac{d^2_j}{2}}\, d_j^{2 i_j}\right)\left(2^{i_j}\, |\underline{i_j}\right)^{-1} \qquad ; \; j = 1, 2 \ldots r.$$

(14) can also be written as

$$g(V) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty}\cdots\sum_{t_r=0}^{\infty} Q_r P_r\, e^{-a_1 V}\, V^{\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=2}^{r} t_j - 1} \tag{15}$$

where

$$Q_r = \prod_{j=1}^{r}\left\{ q^{(j)}_{ij}\, a_j^{p_j+i_j} \right\}$$

$$P_r = \left[\prod_{j=2}^{r} (P_j+t_j)\, t_j\, (d_1-a_j)^{t_j} \Big/ |\underline{t_j}\right]\left\{ \left|\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=2}^{r} t_j\right.\right\}^{-1}. \tag{16}$$

It is clear that the density function of the sum of r non-central gamma variates can be written as infinite sum of weighted densities of gamma variates.

Let V^* be the sum of s non-central gamma variates x^*_i $(i = 1, 2, \ldots\ldots\ldots s)$ with parameters a^*_i, P^*_i and d^{*2}_i $(i = 1, 2 \ldots\ldots\ldots s)$.

Let us assume that $a_1^* \geqslant a^*_i$ $(i = 2, 3, \ldots\ldots\ldots s)$.

Following the procedure adopted in Section I, the probability density function of $U = V - V^*$ is given by

$$h(U)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty}\sum_{i_1^*=0}^{\infty}\sum_{i_2^*=0}^{\infty}\cdots\sum_{i_s^*=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty}\cdots\sum_{t_r=0}^{\infty}\sum_{t^*_2=0}^{\infty}\sum_{t^*_3=0}^{\infty}\cdots\sum_{t^*_s=0}^{\infty}$$

$$\Bigg\{ Q_r\ P_r\ Q^*_s\ P^*_s \left|\overline{\sum_{j=1}^{s}(P^*_j+i_j^*)+\sum_{j=2}^{s}t^*_j}\right. e^{-a_1 U}$$

$$U^{\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=1}^{s}(P^*_j+i_j^*)+\sum_{j=2}^{r}t_j+\sum_{j=2}^{s}t_j^*-1}$$

$$\Psi\Bigg[\sum_{j=1}^{s}(P^*_j+i^*_j)+\sum_{j=2}^{s}t^*_j\ ;\ \sum_{j=1}^{r}(P_j+i_j)+$$

$$\sum_{j=1}^{s}(P^*_j+i^*_j)+\sum_{j=2}^{r}t_j+\sum_{j=2}^{s}t^*_j\ ;\ (a_1+a^*_1)\,U\Bigg]\Bigg\} \tag{17}$$

for $U>0$, and

$$h(U)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_r=0}^{\infty}\sum_{i_1^*=0}^{\infty}\sum_{i_2^*=0}^{\infty}\cdots\sum_{i^*_s=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{t_3=0}^{\infty}\cdots\sum_{t_r=0}^{\infty}\sum_{t_2^*=0}^{\infty}\sum_{t_3^*=0}^{\infty}\cdots\sum_{t^*_s=0}^{\infty}$$

$$\Bigg\{ Q_r\ P_r\ Q^*_S\ P^*_S \left|\overline{\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=2}^{r}t_j}\right. e^{a_2 U}$$

$$(-U)^{\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=1}^{s}(P^*_j+i^*_j)+\sum_{j=2}^{r}t_j+\sum_{j=2}^{s}t^*_j-1}$$

$$\Psi\Bigg[\sum_{j=1}^{r}(P_j+i_j)+\sum_{j=2}^{r}t_j\ ;\ \sum_{j=1}^{r}(P_j+i_j)$$

$$+\sum_{j=1}^{s}(P^*_j+i^*_j)+\sum_{j=2}^{r}t_j+\sum_{j=2}^{s}t^*_j\ ;\ -(a_1+a_1^*)\,U\Bigg]\Bigg\} \tag{18}$$

for $U<0$

where Q^*_s and P^*_s are similar to Q_r and P_r.

It seems that the density function of the difference of the two sums of non-central gamma variates can be written as the infinite sum of weighted densities of the difference of two gamma variates.

Corollary (1) : The probability density function of the sum of r gamma variates is given by

$$g(V) = \left[\prod_{i=1}^{r} a_i^{P_i} \right] \left(\overline{\left| \sum_{j=1}^{r} P_j \right.} \right)^{-1} e^{-a_1 V} V^{\sum_{j=1}^{r} P_j - 1} \Phi_2 \left[P_2, P_3, \ldots P_r ; \sum_{j=1}^{r} P_j ; \right.$$

$$\left. (a_1 - a_2) V, (a_1 - a_3) V, \ldots (a_1 - a_r) V \right] \tag{19}$$

$$\text{for } a_1 > a_i \quad (i = 2, 3, r).$$

This result coincides with Kabe's[6].

Corollary (2) : Let Z_1 and Z_2 be two independent generalised exponential families with parameters P'_i, q'_i and α_i ($i = 1, 2$) whose density functions follow from (9).

Then the probability density function of $Z = Z_1 + Z_2$ is given from (14) as

$$P(Z) = \left\{ \prod_{i=1}^{2} \alpha_i^{P_i'} (1 + \alpha_i)^{q'_i - P_i'} \right\} Z^{q'_1 + q'_2 - 1} e^{-(1 + \alpha_1) Z} \Phi_2 \left[P'_1, q_2' - \right.$$

$$\left. P'_2, P'_2; q'_1 + q'_2; Z, (\alpha_1 - \alpha_2) Z, (1 + \alpha_1 - \alpha_2) Z \right]$$

When $\alpha_1 = \alpha_2 = \alpha$, it is clear that Z is distributed as a generalised exponential family with parameters $P'_1 + P'_2$, $q'_1 + q'_2$ and α.

III—DISTRIBUTION OF QUADRATIC FORMS

Let us define

$$\left.\begin{aligned} T_1 &= \alpha \left(\chi^2_{n_1, d_1} + \sum_{i=2}^{r} b_i \, \chi^2_{n_i, d_i} \right) \\ T_2 &= \beta \left(\chi^2_{n_1^*, d_1^*} + \sum_{j=2}^{s} b_j^* \, \chi^2_{n_j^*, d_j^*} \right) \\ T &= T_1 - T_2 \end{aligned}\right\} \tag{20}$$

where $\alpha, \beta > 0$; $b_i, b_j^* \geqslant 1, d_i, d_j^* > 0$ for all i and j and the *chi*-square variates $\chi^2_{n_i d_i}$, $\chi^2_{n_j^*, d_j^*}$ are all independent.

If $\sum_{i=1}^{p} \lambda_i \omega_i^2$ is an arbitrary quadratic form in which the constants λ_i are real numbers and ω_i are independent random variables each of which has a normal distribution with non-zero mean and unit variance, the form can be given the representation (20), T_1 and T_2 are each expressible as some positive definite quadratic form while T can be expressed as indefinite quadratic form.

The probability density function of T_1 can be obtained by substituting

$$a_1^{-1} = 2\alpha, \; a_j^{-1} = 2\alpha b_j \text{ and } P_j = \frac{n_j}{2} \qquad (j = 2, 3, \ldots r)$$

in (14) and is given by

$$g(T_1) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} \left\{ \left[\prod_{j=1}^{r} q_{i_j}^{(j)} \right] (2\alpha)^{-\sum_{j=1}^{r}\left(\frac{n_j}{2} + i_j\right)} \prod_{j=2}^{r} b_j^{-\left(\frac{n_j}{2} + i_j\right)} \left(\Gamma \sum_{j=1}^{r} \left(\frac{n_j}{2} + i_j\right)\right)^{-1} e^{-\frac{T_1}{2\alpha}} \right.$$

$$T_1^{\sum_{j=1}^{r}\left(\frac{n_j}{2} + i_j\right) - 1} \Phi_2 \left[\frac{n_2}{2} + i_2, \; \frac{n_3}{2} + i_3, \ldots \frac{n_r}{2} + i_r \; ; \sum_{j=1}^{r} \left(\frac{n_j}{2} + i_j\right) \right.$$

$$\left(\frac{1-\overset{-1}{b_2}}{2\alpha}\right)T, \left(\frac{1-\overset{-1}{b_3}}{2\alpha}\right)T, \cdot\cdot, \left(\frac{1-\overset{-1}{b_r}}{2\alpha}\right)T \Bigg] \Bigg\} \quad (21)$$

By expanding Φ_2 and rearranging after a little algebra, we get

$$g(T_1) = \sum_{i=0}^{\infty} f_i \, \gamma\left(\frac{1}{2\alpha}, \frac{N}{2} + i ; T_1\right) \quad (22)$$

where $N = \sum_{i=1}^{r} n_i$ and f_i are the constants given by Press[7] and $\gamma(a,P,x)$ is the density of a gamma variate x with parameters a, P. Similarly, from (17) after a little algebra, the probability density function of T can be written as

$$h(T) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f_i f_j^* \; h^{(T)}_{N+2i,\, N^*+2j} \quad (23)$$

where f_j^* are similar to f_i, $N^* = \sum_{j=1}^{s} n_j^*$ and $h^{(T)}_{N,M}$ is the density of the difference of two gamma variates ($T >$ and $T < 0$). The procedure we followed here is more simple than the characteristic function approach followed by others.

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