APPLICATION OF DISCRETE MAXIMUM PRINCIPLE TO OPTIMIZATION PROBLEMS OF MULTIPLE STAGE ROCKETS

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The Discrete Maximum Principle has been applied to solve a few optimization problems of multiple staged rockets by including the gravity into the performance equations. The problem of finding the minimum mass in order to obtain a specified velocity at the end of powered phase has been solved under various assumptions about the structure factors both when the stages are arranged in series as well as in parallel. The problem when the objective function to be minimised is the cost per pound of the payload has also been investigated.

A number of authors e.g., Goldsmith[1], Schurmann[2], Weisbord[3], Hall & Zambelli[4], Ragoac & Patterson[5] etc., have solved the weight minimisation problem of a staged rocket by various techniques and under various simplifying assumptions. One of the common simplifying assumptions is the neglect of gravity factor which is taken into account by suitably reducing the mission velocity required. But the loss in velocity due to gravity is quite considerable for large boosting rockets and recently Tawakley[6] has emphasised the importance of taking the loss in velocity due to gravity directly into the performance equations. He has solved the problem of finding the total minimum mass required and its distribution in various stages so as to obtain a required mission velocity at the end of powered phase, when the material and propellant for the rocket system and the number of stages are fixed in advance. The Lagrangian maximum principle was used to solve this problem.

Recently a number of other useful methods have been devised e.g., Pontryagin's maximum principle[7] for solving optimal control problems and this method is particularly useful for continuous processes. The maximum principle for discrete processes has been given by Katz[8] and successfully applied by Fan, *et. al.*[9] in solving optimization problems of staged rocket. The advantage of discrete maximum principle in multiple stage processes is that its application gives directly a general recurrence relation in control variables and the optimum value of the objective function is obtained by solving the performance equation of the system together with the recurrence relation. But Fan, *et. al.*[9] in their analysis have neglected the gravity factor. In the present paper the author has applied the discrete maximum principle for solving the optimization problems of multiple stage rocket by taking gravity directly into the performance equation. In section I, the simplest case when exhaust velocities and structure factors for each stage are constant but different for each stage is considered. In section II, the problem has been solved after removing the restriction that structure factor is constant for each stage and employing a scaling factor for it i.e., structure factor is considered to be linearly dependent on stages weight. Section III treats the same problem by introducing a power law approximation for structure factors. In section IV the problem when the stages are arranged in parallel and all the engines are working simultaneously to provide thrust has been solved. The importance in economising in cost is of great significance in missile design and the case when the total cost of the vehicle system is minimum is considered in section V.

THE DISCRETE MAXIMUM PRINCIPLE

The discrete maximum principle for solving optimal problems of multiple stage process can in brief be described as follows :

Let x denote the k-dimensional state variable and θ the t-dimensional control variable, then if the state variable in the nth stage be given by

$$x^n = T^n (x^{n-1}, \theta^n) \qquad n = 1, 2, \ldots \ldots \ldots, N \tag{1}$$

the problem is to choose a sequence of θ^n at each step which will maximise or minimise x^n_m.

The procedure in short to solve this problem is to introduce a k-dimensional adjoint vector z^n and a Hamiltonian H^n satisfying the following equations :

$$z^{n-1} = \sum_{j=1}^{s} \frac{\partial T^n_j \left(x^{n-1}, \theta^n \right)}{\partial x_i^{n-1}} z^n_j \qquad \begin{array}{l} i = 1, 2, \ldots \ldots \ldots \\ n = 1, 2, \ldots \ldots \ldots, N \end{array} \tag{2}$$

$$H^n = \sum_{j=1}^{s} z^n_j \, T^n \left(x^{n-1} \theta^n \right) \tag{3}$$

It can easily be shown that

$$x^n = \frac{\partial H^n}{\partial z^n} \quad , \quad z^{n-1} = \frac{\partial H^n}{\partial x^{n-1}} \tag{4}$$

The optimization problem becomes that of finding a sequence θ^n to satisfy the following condition

$$H^n = \sum_{j=1}^{s} z^n_j \, T^n_j \left(x^{n-1}, \theta^n \right) = \text{min. or max.} \tag{5}$$

for which

$$\frac{\partial H^n}{\partial x^n} = 0$$

OPTIMIZATION PROBLEMS OF STAGED ROCKETS

The discrete maximum principle is now applied to solve the optimization problems of multistage rockets. The stages are numbered as 1 to N from top to bottom as shown in Fig. 1. In section I to III the problem of finding the minimum mass in order to obtain a specified mission velocity has been solved under various assumptions about the structure factors when the stages are arranged in series. Section IV deals with the same problem when the stages are arranged in parallel. Section V solves the problem when the objective function to be minimised is the cost per pound of the payload.

Section I

The structure factors and exhaust speeds for each stage are considered to be known constants but different in each stage. Here we take the stage payload weight ratio $\lambda^n \equiv (W_0^n / W_0^{n-1})$ as the control variable. The total payload weight ratio x_1^n of the nth stage and the total speed of the nth stage x_2^n are taken as the state variables. Therefore equations corresponding to (1) are

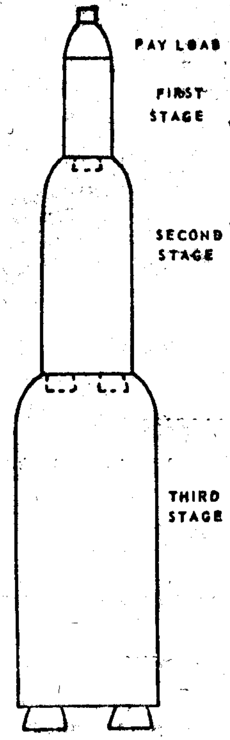
$$x_1^n = x_1^{n-1} \lambda^n \tag{6}$$

$$x_2^n = x_2^{n-1} - c^n \left[\log r^n - \frac{g^n}{f^n} \frac{r^n - 1}{r^n} \right]$$

where r^n is the mass-ratio of the nth stage rocket and f^n/g^n is the ratio of thrust to total initial weight i.e., the initial thrust acceleration of the nth stage.

But r^n in terms of λ^n can be written as

$$r^n = \frac{\lambda^n}{1 - \epsilon^n(1 - \lambda^n)}$$



Typical multiplestage rocket

Therefore (7) can be re-written as

$$x_2^n = x_2^{n-1} - c^n \left[\log \frac{\lambda^n}{1 - \epsilon^n(1 - \lambda^n)} - \frac{g^n}{f_n} \frac{(\lambda^n - 1)(1 - \epsilon^n)}{\lambda^n} \right] \tag{7a}$$

Here the following boundary conditions hold

$$x_1^\circ = 1, \quad x_2^N = 0, \; x_2^\circ = V$$

Now from (2) the adjoint vectors in this case are :

$$\left.\begin{aligned} z_1^{n-1} &= \lambda^n z_1^n \quad &(i) \\ z_2^{n-1} &= z_2^n \quad &(ii) \end{aligned}\right\} \tag{8}$$

According to (3) the Hamiltonian is

$$H^n = x_1^{n-1} \lambda^n z_1^n + z_2^n \left[x_2^{n-1} - c^n \left\{ \log \frac{\lambda^n}{1 - \epsilon^n(1 - \lambda^n)} - \frac{g^n}{f^n} \frac{(\lambda^n - 1)(1 - \epsilon^n)}{\lambda^n} \right\} \right] \tag{9}$$

Since λ^n is the control variable, therefore for optimum conditions $\partial H^n/\partial\lambda^n = 0$, $(n = 1, 2, \ldots\ldots, N)$, which gives

$$z_2^n = x_1^{n-1} z_1^n \lambda^n / c^n (1 - \epsilon^n)\left[\frac{1}{1 - \epsilon^n (1 - \lambda^n)} - \frac{g^n}{f^n} \frac{1}{\lambda^n}\right] \tag{10}$$

As a consequence of (8), we obtain the following recurrence relation to give the optimal sequence for λ^n

$$c^n(1 - \epsilon^n)\left[\frac{1}{1 - \epsilon^n(1 - \lambda^n)} - \frac{g^n}{f^n} \frac{1}{\lambda^n}\right] = c^{n-1}(1 - \epsilon^{n-1}) \times$$

$$\left[\frac{1}{1 - \epsilon^{n-1}(1 - \lambda^{n-1})} - \frac{g^{n-1}}{f^{n-1}} \frac{1}{\lambda^{n-1}}\right]$$

$$n = 2, 3, \ldots\ldots, N \tag{11}$$

The procedure to solve this recurrence relation when the values of ϵ^n's and f^n's are supposed to be known can be as follows :

Step 1—Assume an appropriate value for λ^N

Step 2—Calculate the values of λ^n, $n = 1, 2 \ldots, (N - 1)$ satisfying relation (11).

Step 3—Calculate the value of x_2^o by substituting the initial condition $x_2^N = 0$ and the values of λ^n calculated in step 2 into the equation (7a). If the value of x_2^o so calculated is identical to the required mission velocity V then the values of λ^n calculated above is the optimal sequence.

Step 4—If x_2^o is not equal to the specified mission velocity then a new value for λ^N is taken and the steps from 1 to 3 are repeated.

Section II

In the above analysis the structure ratio is taken to be constant. But Coleman[10] showed that a better optimization analysis would be to include a scaling law for structure factor which may account for its variation with stage size. In this section we take a linear scaling law for the structure factor depending upon the stage weight, i.e.,

$$\epsilon^n = A^n w^n + B^n \tag{12}$$

where A^n and B^n are constants for each stage depending upon the propellants and thrust levels under consideration. Here we take w^n as the control variable and as before x_1^n the stage total payload ratio and x_2^n the speed in the nth stage as the state variable. Therefore in this case

$$r^n = \frac{\sum_{0}^{n} w^j}{\sum_{0}^{n-1} w^j + w^n \epsilon^n} \tag{13}$$

Since

$$w^o x_1^n = \sum_{0}^{n} w^j \tag{14}$$

Therefore

$$r^n = \frac{w^o x_1^n}{w^o x_1^{n-1} + w^n \epsilon^n} = \frac{w^o x_1^n}{w^o x_1^{n-1} + w^n(A^n w^n + B^n)} \tag{15}$$

Here the relations corresponding to (1) are

$$x_1^n = x_1^{n-1} + \frac{w^n}{w^o} \tag{16}$$

$$x_2^n = x_2^{n-1} - c^n \left[\log \frac{w^o x_1^n}{w^o x_1^{n-1} + w^n(A^n w^n + B^n)} - \frac{g^n}{f^n} \left\{ 1 - \frac{w^o x_1^{n-1} + w^n(A^n w^n + B^n)}{w^o x_1^n} \right\} \right] \tag{16a}$$

with the boundary conditions that

$$x_1^o = 1,\ x_2^o = V,\ x_2^N = 0$$

The adjoint vectors are given by

$$z_1^{n-1} = z_1^n - z_2^n c^n \left[\frac{w^o}{w^o x_1^{n-1} + w^n} - \frac{w^o}{w^o x_1^{n-1} + w^n (A^n w^n + B^n)} + \frac{g^n}{f^n} \left\{ \frac{w^o}{w^o x_1^{n-1} + w^n} - \frac{w^o \{ w^o x_1^{n-1} + w^n(A^n w^n + B^n)\}}{(w^o x_1^{n-1} + w^n)^2} \right\} \right] \tag{17}$$

$$z_2^{n-1} = z_2^n \tag{17a}$$

The Hamiltonian for this case is

$$H^n = z_1^n \left(x_1^{n-1} + \frac{w^n}{w^o} \right) + z_2^n \left[x_2^{n-1} - c^n \left\{ \log \frac{w^o x_1^n}{w^o x_1^{n-1} + w^n (A^n w^n + B^n)} - \frac{g^n}{f^n} \left(1 - \frac{w^o x_1^{n-1} + w^n (A^n w^n + B^n)}{w^o x_1^{n-1} + w^n} \right) \right\} \right]$$

The conditions giving the optimum sequence for w^n for minimum x_1^N is

$$\partial H^n / \partial w^n = 0,\ (n = 1, 2, \ldots\ldots N) \text{ i.e.}$$

$$z_1^n = w^o z_2^n c^n \left[\frac{1}{w^o x_1^{n-1} + w^n} - \frac{2A^n w^n + B^n}{w^o x_1^{n-1} + w^n (A^n w^n + B^n)} + \frac{g^n}{f^n} \left\{ \frac{2A^n w^n + B^n}{w^o x_1^{n-1} + w^n} - \frac{w^o x_1^{n-1} + w^n (A^n w^n + B^n)}{(w^o x_1^{n-1} + w^n)^2} \right\} \right]$$

As a consequence of (17) and (17a) this can be re-written as

$$c^n \left[\frac{1}{w^o x_1^{n-1} + w^n (A^n w^n + B^n)} - \frac{2A^n w^n + B^n}{w^o x_1^{n-1} + w^n(A^n w^n + B^n)} + \frac{g^n}{f^n} \left\{ \frac{2A^n w^n + B^n}{w^o x_1^{n-1} + w^n} - \frac{1}{w^o x_1^{n-1} + w^n} \right\} \right]$$

$$= c^{n-1}\left[\frac{1}{w^o x_1^{n-2} + w^n} - \frac{2A^{n-1} w^{n-1} + B^{n-1}}{w^o x_1^{n-2} + w^{n-1}(A^{n-1} w^{n-1} + B^{n-1})} + \frac{g^{n-1}}{f^{n-1}}\right.$$

$$\left.\left\{\frac{2A^{n-1} w^{n-1} + B^{n-1}}{w^o x_1^{n-2} + w^{n-1}} - \frac{w^o x_1^{n-2} + w^{n-1}(A^{n-1} w^{n-1} + B^{n-1})}{(w^o x_1^{n-2} + w^{n-1})^2}\right\}\right]$$

Now with the help of (12) and (15) this can be simplified as

$$\frac{c^n(1 - r^n \epsilon^n)}{1 - \epsilon^n}\left(1 - 2\epsilon^n + B^n\right)\left(1 - \frac{1}{r^n}\frac{g^n}{f^n}\right) = c^{n-1}\left\{1 - r^{n-1}\left(2\epsilon^n - B^n\right)\right\}$$

$$\left\{1 - \frac{1}{r^{n-1}}\frac{g^{n-1}}{f^{n-1}}\right\} \tag{18}$$

$$n = 2, 3, \ldots\ldots, N$$

which is the required recurrence relation giving the optimal sequence for the optimal weight distribution.

Section III

Here we approximate the structure factor by assuming the power law of the form

$$\epsilon^n = \alpha^n (w^n)^{\beta^n - 1} \tag{19}$$

where α^n and β^n are constants depending upon the selection of propellant, feed system and auxiliary system etc. of the stages. In this case

$$r^n = \frac{w^o x_1^n}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} \tag{20}$$

Therefore as before

$$x_1^n = x_1^{n-1} + \frac{w^n}{w^o}$$

$$x_2^n = x_2^{n-1} - c^n\left[\log\frac{w^o x_1^n}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} - \frac{g^n}{f^n}\left\{1 - \frac{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}}{w^o x_1^n}\right\}\right]$$

Hence the components of the adjoint vectors are

$$z_1^{n-1} = z_1^n - z_2^n c^n\left[\frac{w^o}{w^o x_1^{n-1} + w^n} - \frac{w^o}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} + \frac{g^n}{f^n}\right.$$

$$\left.\left\{\frac{w^o}{w^o z_1^{n-1} + w^n} - \frac{w^o\{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}\}}{(w^o x_1^{n-1} + w^n)^2}\right\}\right] \tag{21}$$

$$z_2^{n-1} = z_2^n \tag{21a}$$

and the Hamiltonian is

$$H^n = \left(x_1^{n-1} + \frac{w^n}{w^o}\right) z_1^n + \left[x_2^{n-1} - c^n \left\{ \log \frac{w^o x_1^{n-1} + w^n}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} - \frac{g^n}{f^n} \left(1 - \frac{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}}{w^o x_1^{n-1} + w^n}\right)\right\}\right] z_2^n$$

and the condition for the optimum sequence of w^n from (5) is

$$z_1^n = w^o z_2^n c^n \left[\frac{1}{w^o x_1^{n-1} + w^n} - \frac{\alpha^n \beta^n (w^n)^{\beta^n - 1}}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} + \frac{g^n}{f^n} \left\{ \frac{\alpha^n \beta^n (w^n)^{\beta^n - 1}}{w^o x_1^{n-1} + w^n} - \frac{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}}{(w^o x_1^{n-1} + w^n)^2} \right\}\right]$$

Making use of the conditions (21) and (21a), the above becomes as

$$c^n \left[\frac{1}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} - \frac{\alpha^n \beta^n (w^n)^{\beta^n - 1}}{w^o x_1^{n-1} + \alpha^n (w^n)^{\beta^n}} + \frac{g^n}{f^n} \left\{ \frac{\alpha^n \beta^n (w^n)^{\beta^n - 1}}{w^o x_1^{n-1} + w^n} - \frac{1}{w^o x_1^{n-1} + w^n} \right\}\right]$$

$$= c^{n-1} \left[\frac{1}{w^o x_1^{n-2} + w^{n-1}} - \frac{\alpha^{n-1} \beta^{n-1} (w^{n-1})^{\beta^{n-1} - 1}}{w^o x_1^{n-2} + \alpha^{n-1} (w^{n-1})^{\beta^{n-1} - 1}} + \frac{g^{n-1}}{f^{n-1}} \left\{ \frac{\alpha^{n-1} \beta^{n-1} (w^{n-1})^{\beta^{n-1} - 1}}{w^o x_1^{n-2} + w^{n-1}} - \frac{w^o x_1^{n-2} + \alpha^{n-1} (w^{n-1})^{\beta^{n-1}}}{(w^o x_1^{n-2} + w^{n-1})^2} \right\}\right]$$

Simplifying it further in terms of the mass-ratio, the recurrence relation to give the optimum mass-distribution is given by

$$\frac{c^n (1 - \beta^n \epsilon^n)(1 - r^n \epsilon^n)}{(1 - \epsilon^n)} \left(1 - \frac{1}{r^n} \frac{g^n}{f^n}\right) = c^{n-1} \left(1 - \beta^{n-1} \epsilon^{n-1} r^{n-1}\right) \left(1 - \frac{1}{r^{n-1}} \frac{g^{n-1}}{f^{n-1}}\right) \tag{22}$$

$$n = 2, 3, \ldots\ldots, N$$

If we neglect gravity terms the above reduces to

$$\frac{c^n (1 - \beta^n \epsilon^n)(1 - r^n \epsilon^n)}{(1 - \epsilon^n)} = c^{n-1} \left(1 - \beta^{n-1} \epsilon^{n-1} r^{n-1}\right) \left(1 - \frac{1}{r^{n-1}} \frac{g^{n-1}}{f^{n-1}}\right)$$

$$n = 2, 3, \ldots\ldots N$$

which is identical to Coleman's[10] results obtained for the case of three stage rocket.

Section IV

Now we consider the case when the rockets are arranged in parallel instead of in series as in the above sections. Here all the rockets are supposed to be working simultaneously to provide thrust to the whole systems. We assume that the engine weight of any stage is proportional to the maximum thrust of the engine. Thus if F^n is the total thrust in any stage then the engine weight is given by

$$\overset{n}{W_e} = A^n \, W_0{}^n \left(\alpha^n - \frac{\alpha^{n-1}}{\lambda^n} \right) \tag{23}$$

where $\alpha^n = F \; / \, W_0{}^n$ with the condition that $\alpha^\circ = 0$. Therefore in this case

$$\frac{1}{r^n} = (1 - \epsilon^n) \left[A^n \, \alpha^n + \frac{1}{\lambda^n} \, (1 - A^n \, \alpha^{n-1}) \right] + \epsilon^n \tag{24}$$

Here again we consider λ^n as the control variable and the stage total payload weight ratio $x_1{}^n$ and the speed at the end of nth stage $x_2{}^n$ as the state variable, so that

$$x_1{}^n = x_1{}^{n-1} \, \lambda^n$$

$$x_2{}^n = x_2{}^{n-1} + c^n \left[\log \left\{ \frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} + A^n \, \alpha^n \, (1-\epsilon^n) + \epsilon^n \right\} \right.$$

$$\left. + \frac{g^n}{f^n} \left\{ 1 - \frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} - A^n \, \alpha^n \, (1-\epsilon^n) - \epsilon^n \right\} \right]$$

With the help of (4) the components of the adjoint vectors are

$$z_1{}^{n-1} = z_1{}^n \, \lambda^n \tag{25}$$

$$z_2{}^{n-1} = z_2{}^n \tag{25a}$$

Here the Hamiltonian is

$$H^n = x_1{}^{n-1} \lambda^n \, z_1{}^n + x_2{}^{n-1} \, z_2{}^n + c^n \, z_2{}^n \left[\log \left\{ \frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} + A^n \, \alpha^n \, (1-\epsilon^n) \right. \right.$$

$$\left. \left. + \epsilon^n \right\} + \frac{g^n}{f^n} \left\{ 1 - \frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} - A^n \, \alpha^n \, (1-\epsilon^n) - \epsilon^n \right\} \right]$$

Therefore to find the optimum sequence for λ^n we must have $\partial H^n \, / \, \partial \lambda^n = 0$, i.e.

$$z_2{}^n = c^n \, x_1{}^{n-1} \, \lambda^n \, z_1{}^n \left[\frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{(1-\epsilon^n)(1-A^n \, \alpha^{n-1}) + \lambda^n \left\{ A^n \, \alpha^n \, (1-\epsilon^n) + \epsilon^n \right\}} \right.$$

$$\left. - \frac{g^n}{f^n} \, \frac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} \right]^{-1}$$

As a consequence of the relations (25) and (25*a*), the above can be re-written as

$$c^n \, (1-\epsilon^n)(1-A^n \, \alpha^{n-1}) \, \lambda^{n-1} \left[\frac{1}{\dfrac{(1-\epsilon^n)(1-A^n \, \alpha^{n-1})}{\lambda^n} + A^n \, \alpha^n \, (1-\epsilon^n) + \epsilon^n} - \frac{g^n}{f^n} \right]$$

$$= c^{n-1} (1-\epsilon^{n-1})(1-A^{n-1} \alpha^{n-2}) \lambda^n$$

$$\left[\frac{1}{\dfrac{(1-\epsilon^{n-1})(1-A^{n-1} \alpha^{n-2})}{\lambda^{n-1}} + A^{n-1} \alpha^{n-1} (1-\epsilon^{n-1}) + \epsilon^{n-1}} - \frac{g^{n-1}}{f^{n-1}} \right]$$

$$n = 2, 3, \ldots \ldots, N$$

If we express the above in terms of the mass-ratio r^n it can be simplified as

$$c^n \left[1 - \left\{A^n \alpha^n (1-\epsilon^n) + \epsilon^n\right\} r^n\right]\left[1 - \frac{1}{r^n}\frac{g^n}{f^n}\right]$$

$$= c^{n-1}\left[1 - r^{n-1}\left\{A^{n-1}\alpha^{n-1}(1-\epsilon^{n-1}) + \epsilon^{n-1}\right\}\right]\left[1 - \frac{1}{r^{n-1}}\frac{g^{n-1}}{f^{n-1}}\right]$$

$$n = 2, 3, \ldots\ldots, N \quad (26)$$

which gives the general recurrence relation for the optimum distribution of the total weight in the various stages.

Section V

Chase[11] has presented an analysis for optimum staging when the objective function is the minimum cost-per-pound of payload vehicle under the assumption of a linearized cost equation for each stage. This equation consists of a fixed cost plus a variable cost which is dependent on the propellant loading in each stage, i.e.,

$$Rs^n = A^n + B^n W^n{}_p \quad (27)$$

But Chase's analysis is very cumbersome and does not give any explicit relationships giving the optimum conditions. Here we apply the discrete maximum principle in order to obtain such relationships. We take the control variable as the weight of the stage w^n while the state variables are $x^n{}_1$ ($\equiv$ total cost of the n stages), $x^n{}_2$ the stage total payload weight ratio and $x^n{}_3$ the velocity added by the n stages. Therefore,

$$x_1^n = x_1^{n-1} + A^n + B^n w^n (1-\epsilon^n)$$

$$x_2^n = x_2^{n-1} + \frac{w^n}{w^\circ}$$

$$x_3^n = x_3^{n-1} - c^n\left[\log\frac{w^\circ x_2^n}{w^\circ x_2^{n-1} + w^n\epsilon^n} - \frac{g^n}{f^n}\left\{1 - \frac{w^\circ x_2^{n-1} + w^n\epsilon^n}{w^\circ x_2^n}\right\}\right]$$

in this case the boundary conditions are

$$x_1^\circ = 1, \quad x_2^\circ = 1, \quad x_3^\circ = V, \quad x_3^N = 0$$

Hence the components of the adjoint vectors are given by

$$z_1^{n-1} = z_1^n \quad (28)$$

$$z_2^{n-1} = z_2^n - z_3^n c^n w^n\left[\frac{1}{w^\circ x_2^{n-1} + w^n} - \frac{1}{w^\circ x_2^{n-1} + w^n\epsilon^n} + \frac{g^n}{f^n}\left\{\frac{1}{w^\circ x_2^{n-1} + w^n} - \frac{w^\circ x_2^{n-1} + w^n\epsilon^n}{(w^\circ x_2^{n-1} + w^n)^2}\right\}\right] \quad (28a)$$

$$z_3^{n-1} = z_3^n \quad (28b)$$

while the Hamiltonian is

$$H^n = \left[x_1^{n-1} + A^n + B^n(1-\epsilon^n)w^n\right]z_1^n + \left[x_2^{n-1} + \frac{w^n}{w^\circ}\right]z_2^n + \left[x_3^{n-1} - c^n\left\{\log\frac{w^\circ x_2^{n-1} + w^n}{w^\circ x_2^{n-1} + w^n\epsilon^n} - \frac{g^n}{f^n}\left(1 - \frac{w^\circ x_2^{n-1} + w^n\epsilon^n}{w^\circ x_2^{n-1} + w^n}\right)\right\}\right]z_3^n \quad (29)$$

Here again for optimum conditions : $\partial H^n / \partial w^n = 0$, i.e.,

$$z_2^n = w^\circ z_3^n c^n \left[\frac{1}{w^\circ x_2^{n-1} + w^n} - \frac{\epsilon^n}{w^\circ x_2^{n-1} + w^n \epsilon^n} + \frac{g^n}{f^n} \left\{ \frac{\epsilon^n}{w^\circ x_2^{n-1} + w^n} - \frac{w^\circ x_2^{n-1} + w^n \epsilon^n}{(w^\circ x_2^{n-1} + w^n)^2} \right\} \right] - B^n (1 - \epsilon^n) w^\circ z_1^n$$

Therefore making use of relations (28) and (28a), we have

$$w^\circ z_3^n c^n \left[\frac{\epsilon^n - 1}{w^\circ x_2^{n-1} + w^n \epsilon^n} - \frac{g^n}{f^n} \left\{ \frac{\epsilon^n - 1}{w^\circ x_2^{n-1} + w^n} \right\} \right] + w^\circ z_3^{n-1} c^{n-1} \left[\frac{1}{w^\circ x_2^{n-2} + w^{n-1}} - \frac{\epsilon^{n-1}}{w^\circ x_2^{n-2} + w^{n-1} \epsilon^{n-1}} + \frac{g^{n-1}}{f^{n-1}} \left\{ \frac{\epsilon^{n-1}}{w^\circ x_2^{n-2} + w^{n-1}} - \frac{w^\circ x_2^{n-2} + w^{n-1} \epsilon^{n-1}}{(w^\circ x_2^{n-2} + w^{n-1})^2} \right\} \right] + B^n (1 - \epsilon^n) w^\circ - B^{n-1} (1 - \epsilon^{n-1}) w^\circ = 0 \quad (30)$$

Now with the help of relation (28b), the above can be written as

$$\left[B^{n-1} (1 - \epsilon^{n-1}) - B^n (1 - \epsilon^n) \right] \left[\frac{c^{n-1} (\epsilon^{n-1} - 1)}{w^\circ x_2^{n-2} + w^{n-1}} \left(1 - \frac{g^n}{f^n} \right) + c^{n-2} \left\{ \frac{1}{w^\circ x_2^{n-3} + w^{n-2}} - \frac{\epsilon^{n-1} - 1}{w^\circ x_2^{n-3} + w^{n-2} \epsilon^{n-2}} + \frac{g^{n-2}}{f^{n-2}} \left(\frac{\epsilon^{n-2}}{w^\circ x_2^{n-3} + w^{n-2}} - \frac{w^\circ x_2^{n-3} + w^{n-2} \epsilon^{n-2}}{(w^\circ x_2^{n-3} + w^{n-2})^2} \right) \right\} \right]$$

$$= \left[B^{n-2} (1 - \epsilon^{n-2}) - B^{n-1} (1 - \epsilon^{n-1}) \right] \left[\frac{\epsilon^n (\epsilon^n - 1)}{w^\circ x_2^{n-1} + w^n} \left(1 - \frac{g^n}{f^n} \right) - c^{n-1} \left\{ \frac{1}{w^\circ x_2^{n-2} + w^{n-1}} - \frac{\epsilon^n - 1}{w^\circ x_2^{n-2} + w^{n-1} \epsilon^{n-1}} + \frac{g^{n-1}}{f^{n-1}} \left(\frac{\epsilon^n - 1}{w^\circ x_2^{n-2} + w^{n-1}} - \frac{w^\circ x_2^{n-2} + w^{n-1} \epsilon^{n-1}}{(w^\circ x_2^{n-2} + w^{n-1})^2} \right) \right\} \right] \quad (31)$$

Making use of the following relations

$$w^\circ x_2^n = w^\circ x_2^{n-1} + w^n$$

$$w^\circ x_2^n = \frac{w^n r^n (1 - \epsilon^n)}{r^n - 1}$$

the above can be simplified as

$$\left[c^n (1 - r^n \epsilon^n) \left(1 - \frac{g^n}{f^n} \right) - c^{n-1} \left(1 - r^{n-1} \epsilon^{n-1} \right) \left(1 - \frac{g^{n-1}}{f^{n-1}} \frac{1}{r^{n-1}} \right) \right]$$

$$\left[B^{n-2} (1 - \epsilon^{n-2}) - B^{n-1} (1 - \epsilon^{n-1}) \right] = \frac{r^{n-1} (1 - \epsilon^{n-1})}{1 - \epsilon^{n-1} r^{n-1}}$$

$$\left[c^{n-1}\left(1-\epsilon^{n-1} r^{n-1}\right)\left(1-\frac{g^{n-1}}{f^{n-1}}\right)-c^{n-2}\left(1-\epsilon^{n-2} r^{n-2}\right)\left(1-\frac{g^{n-2}}{f^{n-2}}\frac{1}{r^{n-2}}\right)\right]$$

$$\left[B^{n-1}\left(1-\epsilon^{n-1}\right)-B^{n}\left(1-\epsilon^{n}\right)\right] \tag{32}$$

In the above analysis we have considered the structure factors ϵ^n to be constants for each stage, but the analysis could have been carried out by taking scaling laws for it as given by (12) and (19).

Particular cases—(*a*) In case c^n, ϵ^n are independent of n, i.e.

$$\left.\begin{aligned} c^n &= c \\ \epsilon^n &= \epsilon \end{aligned}\right\} n = 1, 2, \ldots, N \tag{33}$$

and gravity is completely ignored the general recurrence relation (32) reduces to

$$(r^{n-1} - r^n)\,(B^{n-2} - B^{n-1}) = \frac{r^{n-1}(1-\epsilon)}{1-\epsilon\, r^{n-1}}\,(r^{n-2} - r^{n-1})\,(B^{n-1} - B^n)$$

For the case of a three stage rocket the above gives a relation between r^1, r^2, r^3, B^1, B^2, B^3 and ϵ of the form

$$(r^3 - r^2)\,(B^2 - B^1) = \frac{r^2(1-\epsilon)}{1-\epsilon\, r^2}\,(r^1 - r^2)\,(B^2 - B^3)$$

(*b*) If in addition to (33), the following also hold good

$$\left.\begin{aligned} A^n &= A \\ B^n &= B \\ g^n/f^n &= g/f \end{aligned}\right\} n = 1, 2, \ldots, N \tag{34}$$

then from (28*b*) and (30), we obtain

$$(1 - r^n \epsilon)\left(1 - \frac{1}{r^n}\,\frac{g}{f}\right) = (1 - r^{n-1}\epsilon)\left(1 - \frac{1}{r^{n-1}}\,\frac{g}{f}\right) \quad n = 2, 3, \ldots\ldots, N$$

which in the simplified case when gravity is completely ignored reduces to the well-known result that r^n should also be independent of n i.e. the mass-ratios for each stage must be identical, as also obtained by Malina & Summerfield[12].

NUMERICAL ILLUSTRATION

In order to illustrate the Discrete Maximum Principle we take a simple problem of a three stage rocket and consider only the case discussed in section I. The data for the rocket given below does not specify any particular mission.

$c_1 = 16000$ ft/sec.	$\epsilon^1 = .25$	$g^1/f^1 = \cdot 5$	$V_b = 26500$ ft/sec.
$c_2 = 12000$ ft/sec.	$\epsilon^2 = .20$	$g^2/f^2 = \cdot 4$	$W_0^\circ = 1000$ lbs.
$c_3 = 8000$ ft/sec.	$\epsilon^3 = .15$	$g^3/f^3 = \cdot 3$	

With the help of equation (7*a*) and the recurrence relation (11) we obtain the required minimum gross initial weight and the approximate weight and velocity break-up due to the three stages. For the sake of comparison both the sets of values when the initial thrust acceleration of each stage is included and when it is neglected from the performance equations, are given in Table 1.

TABLE 1

WEIGHT AND VELOCITY DISTRIBUTION FOR A THREE STAGE ROCKET

Quantity	Definition	Initial thrust acceleration included	Initial thrust acceleration neglected
W_0^1	Initial weight of the first stage	6017 lb.	4500 lb.
W_0^2	Initial weight of the second stage	33526 lb.	15750 lb.
W_0^3	Initial weight of the third stage (Initial gross weight required)	120694 lb.	22318 lb.
x_2^0	Velocity imparted to the payload	26500 ft/sec.	26500 ft/sec.
x_2^1	Velocity imparted to the first stage	15732 ft/sec.	12464 ft/sec.
x_2^2	Velocity imparted to the second stage	6137 ft/sec.	2300 ft/sec.

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