LAUNCH ANGLES WITH OPTIMUM TRAJECTORIES FOR ROCKET INTERCEPTION AT PREASSIGNED DESTINATIONS

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The problem of interception of a rocket vehicle by optimal path at a preassigned point on the destination orbit by a commuter rocket vehicle launched from the peri-centre of the launch orbit is solved by evaluating launch angle and orbital parameters of the transfer trajectory. The analysis is general in so far as it can be applied to any system of non-coaxial coplanar elliptic orbits under all gravitational fields. The equations have been reduced for axially aligned launch and destination orbits; various particular cases of coaxial orbits are analysed. Interception through Hohmann transfer trajectories have also been considered. The theory is illustrated by taking a practical case of Earth Mars transfer.

NOMENCLATURE

h = twice the aerial velocity.

E = eccentricity.

θ = vectorial angle.

r = radius vector.

K = gravitational constant $\times$ gravitating mass.

α = angle made at the force centre between the lines joining the pericentres of the launch and destination orbits to the force centre (focus).

β = angle of inclination of the major axis of the transfer orbit.

t = time.

t_T° = time of launch of the interceptor vehicle.

a = semi major axis.

t^f_T = time of interception of the interceptor vehicle.

V = velocity.

θ^f_d = interception angle, i.e., the angle between the radii vector passing through the pericentre (launch point) of the launch orbit and the preassigned interception point on the destination orbit.

t^f_d = time when vectorial angle of the destination vehicle is θ^f_d

θ°_d = launch angle i.e., vectorial angle of the destination vehicle at launching time of the interceptor rocket.

t°_d = time when vectorial angle of the destination vehicle is θ°_d

Subscripts

l = corresponds to launch orbit.

T = corresponds to transfer orbit.

d = corresponds to destination orbit.

H = relates to Hohmann transfer trajectory.

Superscripts

f = denotes values at the time of interception.

0 = denotes values at the launch time of interceptor vehicle.

p = denotes values at the pericentre of the launch orbit.

$(-)$ = relates to optimum transfer trajectory of the first kind.

$(=)$ = relates to optimum transfer trajectory of the second kind.

c = relates to circular orbital velocity.

The optimum transfer paths between coplanar circular orbits have been found by Hohmann to consist of the family of cotangential ellipses. In ref. (1) solutions or approximate solutions are found to the equations determining the mode of transfer of a rocket between two coplanar orbits, with minimum fuel expenditure, for different cases. In practice, however, conditions may necessitate inter-vehicle transfer without applying corrective thrust during free-coasting phase of the interceptor rocket vehicle rather than inter-orbital transfer. To achieve this purpose of interception of an orbiting destination rocket at a required space-point on the destination orbit, the knowledge of the precise timing of the launch rocket is essential. Paiewonsky[2] treats the problem of inter-vehicle transfer for a very restricted case of coplanar concentric circular orbits taking only Hohmann transfer trajectories.

In the present paper a general theory is developed for inter-vehicle transfer between non-coaxial coplanar elliptic orbits. The problem is to intercept a rocket vehicle through optimal paths at a preassigned location on the destination orbit. This is achieved by evaluating the launch angle, that is, the angle attained by the destination rocket vehicle when the launch rocket should be fired at the peri-centre of the launch orbit so as to intercept the former at the preassigned destination point. The optimum transfer trajectories are classified into two groups : (*i*) "Optimum transfer trajectories of the first kind"—these require minimum fuel expenditure at the launch point and (*ii*) "Optimum transfer trajectories of the second kind"—these consume minimum fuel at the interception point to attain requisite velocity for rendezvous with the destination rocket. Elements of the transfer trajectories of both the kinds have been determined. Investigation is done for cases when launch and destination orbits are coaxial elliptic or concentric circles. Interception manoeuvres through Hohmann transfer trajectories between axially aligned coplanar orbits is also analysed from which result given by Paiewonsky[2] is derived as a particular case. Lastly a numerical example for optimum transfer trajectory of the second kind taking Earth's and Mars' orbit as launch and destination orbit respectively is solved to illustrate the theory. All applications of rocket thrust are taken to be impulsive in nature. It is shown that gradient variation of the launch angle and the orbital parameters (except $\bar{\bar{\beta}}$) of the transfer trajectory is smaller for low and high values of the interception angle as compared to that of the intermediate values of the interception angle. Also the rate of increment of the launch angle and orbital parameters (except $\bar{\bar{\beta}}$) of the transfer trajectory is comparatively higher in the beginning which gradually decreases as interception angle increases. It is further shown that the rotational rate of the major axis of the transfer trajectory with respect to the interception angle is almost uniform.

NON-COAXIAL COPLANAR ELLIPTIC ORBITS

Let α be the angle made at the force centre between peri-lines (perigee and perihelion lines in the case of Earth and Sun respectively) of the launch and destination orbits.

Equations of the launch, transfer and destination orbits can be written as

$$h^2_l \, u = K [\, 1 + E_l \cos \theta \,] \tag{1}$$

$$h^2_T \, u = K [\, 1 + E_T \cos (\theta - \beta)] \tag{2}$$

$$h^2_d \, u = K [\, 1 + E_d \cos (\theta - \alpha)] \tag{3}$$

where $u = \frac{1}{r}$ and β is the angle of inclination of the major axis of the transfer orbit measured with respect to the reference line, i.e., the line joining the pericentre of launch orbit and the force centre.

If θ be the vectorial angle of the space vehicle at the time t, by principle of conservation of momentum

$$u^2 h \, \delta t = \delta \theta \tag{4}$$

Now, if the space-point at which the destination vehicle is to be intercepted by the interceptor vehicle at the time $t^f{}_d$ be given by $u^f{}_d, \theta^f$; using (3) and (4) and integrating we have :

$$\frac{h^3{}_d}{K^2} \int_{\theta^\circ{}_d}^{\theta^f{}_d} \frac{d\theta}{[1 + E_d \cos (\theta - \alpha)]^2} = t^f{}_d - t^\circ{}_d \equiv \triangle_{t^f{}_d \, , \, t^\circ{}_d} \tag{5}$$

or

$$\begin{aligned} \triangle_{t^f{}_d \, , \, t^\circ{}_d} = \frac{(a_d)^{\frac{3}{2}}}{\sqrt{K}} \Bigg[& 2 \tan^{-1} \left\{ \left(\frac{1-E_d}{1+E_d} \right)^{\frac{1}{2}} \tan \left(\frac{\theta^f{}_d - \alpha}{2} \right) \right\} \\ & - 2 \tan^{-1} \left\{ \left(\frac{1-E_d}{1+E_d} \right)^{\frac{1}{2}} \tan \left(\frac{\theta^\circ{}_d - \alpha}{2} \right) \right\} \\ & - E_d \left(1 - E^2{}_d \right)^{\frac{1}{2}} \left\{ \frac{\sin (\theta^f{}_d - \alpha)}{1 + E_d \cos (\theta^f{}_d - \alpha)} - \frac{\sin (\theta^\circ{}_d - \alpha)}{1 + E_d \cos (\theta^\circ{}_d - \alpha)} \right\} \Bigg] \end{aligned} \tag{6}$$

$$(1 > E_d)$$

For the interceptor vehicle a procedure similar as above yields

$$\begin{aligned} \triangle_{t^f{}_T \, , \, t^\circ{}_T} = \frac{(a_T)^{\frac{3}{2}}}{\sqrt{K}} \Bigg[& 2 \tan^{-1} \left\{ \left(\frac{1-E_T}{1+E_T} \right)^{\frac{1}{2}} \tan \left(\frac{\theta^f{}_T - \beta}{2} \right) \right\} + 2 \tan^{-1} \left\{ \left(\frac{1-E_T}{1+E_T} \right)^{\frac{1}{2}} \tan \left(\frac{\beta}{2} \right) \right\} \\ & - E_T \left(1 - E^2{}_T \right)^{\frac{1}{2}} \left\{ \frac{\sin (\theta^f{}_T - \beta)}{1 + E_T \cos (\theta^f{}_T - \beta)} + \frac{\sin \beta}{1 + E_T \cos \beta} \right\} \Bigg] \end{aligned} \tag{7}$$

For interception the following boundary conditions are to be satisfied,

$$\left. \begin{aligned} t^\circ{}_d &= t^\circ{}_T \quad && (i) \\ t^f{}_d &= t^f{}_T \quad && (ii) \\ \theta^f{}_d &= \theta^f{}_T \quad && (iii) \\ \text{and} \qquad u^f{}_d &= u^f{}_T \quad && (iv) \end{aligned} \right\} \tag{8}$$

And hence from 8(*i*) and 8(*ii*)

$$\triangle_{t^f_T,\, t^\circ_T} = \triangle_{t^f_d,\, t^\circ_d} \tag{9}$$

Equation (9) gives the value of θ°_d for any given θ^f_d if the elements (h_T, E_T and β) of the transfer trajectory are known. The problem now is to evaluate the elements of the optimum transfer trajectory.

ELEMENTS FOR OPTIMUM TRANSFER TRAJECTORY

Let the following parameters be defined as

$\gamma = \dfrac{u^f_d}{u^p_l}$ = ratio of the peri-distance of the launch orbit to the distance of the pre-assigned interception point on the destination orbit measured from the force centre (focus).

ω = launch departure angle, i.e., the angle between the tangents drawn to the launch and transfer trajectories at pericentre of the launch orbit.

Now if V_l is the launch velocity required to reach the destination point (u^f_d, θ^f_d), we have[3]

$$\gamma = \frac{K\, u^p_l}{V^2_l}\left(\frac{1 - \cos \theta^f_d}{\cos^2 \omega}\right) + \frac{\cos\left(\omega + \theta^f_d\right)}{\cos \omega} \tag{10}$$

$$\left|\omega\right| \leqslant \frac{\pi}{2}$$

Equation (10) can be written as

$$V^2_l = \frac{K\, u^p_l\left(1 - \cos \theta^f_d\right) \sec^2 \omega}{\left(\gamma + \sin \theta^f_d \tan \omega - \cos \theta^f_d\right)} \tag{11}$$

Let V^p be the orbital velocity at pericentre of the launch orbit. Then the square of the velocity change required for the interceptor rocket to proceed along the transfer trajectory will be

$$\left(\triangle V\right)^2 = V^2_l + \left(V^p\right)^2 - 2\, V_l\, V^p \cos \omega \tag{12}$$

For optimum transfer trajectory of the first kind, velocity change should be minimum. Hence (12) yields

$$\frac{d\left(\triangle V\right)^2}{d\,\omega} = 2\,\frac{d\,V_l}{d\,\omega}\left[\,V_l - V^p \cos \omega\,\right] + 2\,V_l\,V^p \sin \omega = 0 \tag{13}$$

Putting

$$K\, u^p_l\left(1 - \cos \theta^f_d\right) = A \text{ and } \left(\gamma - \cos \theta^f_d\right) = B\,,$$

using (11) and (13), simplifying and rearranging the terms, we have

$$C_0 (\tan \omega)^4 + C_1 (\tan \omega)^3 + C_2 (\tan \omega)^2 + C_3 \tan \omega + C_4 = 0 \qquad (14)$$

where

$$\left.\begin{aligned} C_0 &= A \sin^2 \theta^f{}_d \\ C_1 &= 4\, A\, B \sin \theta^f{}_d \\ C_2 &= 2\, A\, [\, 2\, B^2 - \sin^2 \theta^f{}_d \,] \\ C_3 &= -\sin \theta^f{}_d\, [\, 4\, A\, B + \sin^2 \theta^f{}_d\, (V^p)^2 \,] \\ C_4 &= \sin^2 \theta^f{}_d [\, A - B\, (V^p)^2 \,] \end{aligned}\right\} \qquad (14A)$$

The quartic equation (14) after numerical solution will give the optimum departure angle $\overline{\omega}$, which when substituted in (11) yields the optimum launch speed $\overline{V}_l$. Utilizing the fact that the transfer trajectory has to pass through launch and destination points, (2) gives

$$\overline{h}^2{}_T\, u^p{}_l = K \left[\, 1 + \overline{E}_T \cos \overline{\beta} \,\right] \qquad (15)$$

$$\overline{h}^2{}_T\, u^f_d = K \left[\, 1 + \overline{E}_T \cos \left(\theta^f_d - \overline{\beta} \right) \right] \qquad (16)$$

where (—) denotes values corresponding to optimum transfer trajectory of the first kind. Also from dynamical considerations, for elliptic transfer trajectories, V_l can be expressed as

$$\overline{V}^2{}_l = K \left[\, 2\, u^p_l - \frac{K \left(1 - \overline{E}^2{}_T \right)}{\overline{h}^2{}_T} \right] \qquad (17)$$

or

$$\overline{h}^2{}_T = \frac{K \left(1 - \overline{E}^2{}_T \right)}{2\, u^p{}_l - \dfrac{\overline{V}^2_l}{K}} \qquad (18)$$

Substituting $\overline{h}^2{}_T$ from (18) into (15), we have

$$\overline{\beta} = \cos^{-1} \left[\frac{\overline{V}^2_l - K\, u^p_l \left(1 + \overline{E}^2{}_T \right)}{\left(2\, K\, u^p_l - \overline{V}^2{}_l \right) \overline{E}_T} \right] \qquad (19)$$

Equations (16), (18) and (19) yield

$$\left[\frac{\left(1-\bar{E}^2_T\right)}{2u^p{}_l-\dfrac{\bar{V}^2{}_l}{K}}\right] u^f_d = \left[1+\bar{E}_T\cos\left\{\theta^f_d - \cos^{-1}\left(\frac{\bar{V}^2{}_l - K\,u^p{}_l\left(1+\bar{E}^2_T\right)}{\left(2\,K\,u^p{}_l-\bar{V}^2{}_l\right)\bar{E}_T}\right)\right\}\right] \quad (20)$$

Using (3), equation (20) is transformed to

$$\frac{K^2\left(1-\bar{E}^2_T\right)\left[1+E_d\cos\left(\theta^f_d-\alpha\right)\right]}{h^2{}_d\left(2\,K\,u^p{}_l-\bar{V}^2{}_l\right)} = \left[1+\bar{E}_T\cos\left\{\theta^f_d-\cos^{-1}\left(\frac{\bar{V}^2{}_l-K\,u^p_l\left(1+\bar{E}^2_T\right)}{\left(2\,K\,u^p_l-\bar{V}^2{}_l\right)\bar{E}_T}\right)\right\}\right] \quad (21)$$

Equation (21) gives $\bar{E}_T$ for a preassigned destination point. Substitution of $\bar{E}_T$ in (18 and (19) will yield the remaining elements of the transfer orbit. Having known $\bar{E}_T$, $\bar{h}_T$ and $\bar{\beta}$, and substituting these for E_T, h_T and β respectively in (9) will give the required θ^o_d for preassigned θ^f_d because $\bar{a}_T$ can also be expressed in terms of $\bar{E}_T$ and $\bar{h}_T$ by the relationship

$$\bar{a}_T\;K\left(1-\bar{E}^2_T\right) = h^2{}_T \quad (22)$$

It is evident from (9), (18), (19), (21) and (22) that the launch angle is dependent upon six parameters, namely, E_d, h_d, α, u^p_l, θ^f_d and $\bar{V}_l$. Because of ellipticity of the transfer orbit, $\bar{V}_l$ is conditioned by the relationship obtained from (18) as

$$\bar{V}^2{}_l < 2\,K\,u^{p2}$$

We shall now analyse the case to find out the launch angle for rendezvous of the interceptor rocket with a space vehicle at a given injection point on the destination orbit by optimum transfer trajectory of the second kind. For this case equation (9) giving the launch angle will remain unchanged but the equations determining the elements of the transfer trajectory will be modified. If $\bar{\bar{V}}_d$ be the optimum destination approach velocity of the interceptor rocket

$$\bar{\bar{V}}^2{}_d = K \left[2\, \overset{f}{\underset{d}{u}} - \frac{K\left(1 - \bar{\bar{E}}^2{}_T\right)}{\bar{\bar{h}}^2{}_T} \right]$$

or

$$\bar{\bar{h}}^2{}_T = \frac{K\left(1 - \bar{\bar{E}}^2{}_T\right)}{2\, u^f{}_d - \dfrac{\bar{\bar{V}}^2{}_d}{K}} \tag{23}$$

For optimum transfer trajectory of the second kind (15) and (16) will be transformed into

$$\bar{\bar{h}}^2{}_T\, \overset{p}{\underset{l}{u}} = K \left[1 + \bar{\bar{E}}_T \cos \bar{\bar{\beta}} \right] \tag{24}$$

$$\bar{\bar{h}}^2{}_T\, \overset{f}{\underset{d}{u}} = K \left[1 + \bar{\bar{E}}_T \cos \left(\overset{f}{\theta}_d - \bar{\bar{\beta}} \right) \right] \tag{25}$$

From (23) and (24)

$$\bar{\bar{\beta}} = \cos^{-1} \left[\frac{\bar{\bar{V}}^2{}_d - 2\,K\, \overset{f}{u}_d + K\, \overset{p}{u}_l \left(1 - \bar{\bar{E}}^2{}_T\right)}{\left(2\,K\, \overset{f}{u}_d - \bar{\bar{V}}^2{}_d\right) \bar{\bar{E}}_T} \right] \tag{26}$$

Substituting from (23) and (26) in (25) and rearranging the terms, we get

$$\frac{K\left(1 - \bar{\bar{E}}^2{}_T\right) \overset{f}{u}_d}{2\,K\, \overset{f}{u}_d - \bar{\bar{V}}^2{}_d} = \left[1 + \bar{\bar{E}}_T \cos \left\{ \overset{f}{\theta}_d - \cos^{-1} \left(\frac{\bar{\bar{V}}^2{}_d - 2\,K\, u^f{}_d + K\, \overset{p}{u}_l \left(1 - \bar{\bar{E}}^2{}_T\right)}{\left(2\,K\, \overset{f}{u}_d - \bar{\bar{V}}^2{}_d\right) \bar{\bar{E}}_T} \right) \right\} \right] \tag{27}$$

where $u^f{}_d$, by (3), in (27) stands for

$$\overset{f}{u}_d = K \left[1 + E_d \cos \left(\overset{f}{\theta}_d - \alpha \right) \right] \Big/ h_d^2 \tag{28}$$

Having determined $\bar{\bar{E}}_T$ from (27), $\bar{\bar{h}}_T$ and $\bar{\bar{\beta}}$ can be known from (23) and (26) respectively. Substitution of the values of $\bar{\bar{E}}_T$, $\bar{\bar{h}}_T$ and $\bar{\bar{\beta}}$ in place of E_T, h_T and β

respectively in (9), launch angle for a given interception angle can be determined by utilizing the equation

$$\bar{\bar{a}}_T \; K \left(1 - \bar{\bar{E}}^2_T \right) = \bar{\bar{h}}^2_T \tag{29}$$

Knowing the transfer orbit any required quantity like departure velocity, departure angle at the launch point etc. can be evaluated. The above analysis shows that in this case also the number of parameters determining launch angle are six viz., E_d, h_d, α, u^f_d, θ^f_d and $\bar{\bar{V}}_d$. The restriction imposed upon $\bar{\bar{V}}_d$ due to elliptic nature of the transfer orbit can be obtained from (23) as

$$\bar{\bar{V}}^2_d < 2\,K\,\overset{f}{u}_d$$

$\bar{\bar{V}}_d$ can be evaluated by the same procedure and corresponding set of equations as explained earlier for finding out $\overline{V_l}$, if the following alterations and definitions are adopted in equations (11) and (12):

$$\left.\begin{array}{lll} \dfrac{1}{\gamma} & \text{for} & \gamma \\ 2\,\pi - \overset{f}{\theta}_d & \text{for} & \overset{f}{\theta}_d \\ \overset{f}{u}_d & \text{for} & \overset{p}{u}_l \\ V_d & \text{for} & V_l \\ \cos(\omega - \omega^s) & \text{for} & \cos\omega \text{ in (12)} \end{array}\right] \tag{30}$$

and V^s in place of V^p

where V_d and V^s respectively are the destination approach velocity of the interceptor rocket and the orbital velocity of the space vehicle moving along the destination orbit at the injection point, ω^s the angle between V^s and local horizontal and $(\omega - \omega^s)$ is the destination approach angle defined as the angle between the tangents drawn to the transfer and destination orbits at the injection point.

COAXIAL ELLIPTIC ORBITS

If coaxial elliptic orbits are taken, (1) and (2) remain unaltered while (3) takes the form

$$h^2_d \; u = K \left[1 + E_d \;\cos\theta \right] \tag{31}$$

Equation (9) still holds good for evaluation of launch angle with the difference that while $\triangle \overset{f}{t}_T$, $\overset{o}{t}_T$ suffers no alteration, $\triangle \overset{f}{t}_d$, $\overset{o}{t}_d$ is now expressed as

$$\overset{f}{\triangle}_{t_d}, \overset{o}{t}_d = \frac{(a_d)^{\frac{3}{2}}}{\sqrt{K}} \left[2\tan^{-1}\left\{ \left(\frac{1-E_d}{1+E_d}\right)^{\frac{1}{2}} \tan\frac{\overset{f}{\theta}_d}{2} \right\} - 2\tan^{-1}\left\{ \left(\frac{1-E_d}{1+E_d}\right)^{\frac{1}{2}} \tan\frac{\overset{o}{\theta}_d}{2} \right\} - E_d \left(1 - E^2_d\right)^{\frac{1}{2}} \frac{\sin\overset{f}{\theta}_d}{1 + E_d\cos\overset{f}{\theta}_d} + E_d\left(1 - E^2_d\right)^{\frac{1}{2}} \frac{\sin\theta^{\circ}_d}{1 + E_d\cos\theta^{\circ}_d} \right] \tag{32}$$

The equations to find out elements of the optimum transfer trajectories of both the kinds can be deduced from those of non-coaxial case by putting in the latter $\alpha = 0$. Three particular cases of coaxial orbits are:

(*i*) Circular launch orbit and elliptic destination orbit.

(*ii*) Elliptic launch orbit and circular destination orbit.

(*iii*) Circular launch and destination orbits.

For (*i*) the equation determining launch angle will be the same as that for coaxial elliptic orbits. Elements of the optimum transfer trajectory of the first kind are given by (18), (19) and (21) by substituting $\alpha = 0$ and replacing u_l^p by u_l — the reciprocal of launch circular orbit radius. It should be observed that to evaluate $\bar{V}_l$ in equations (11) to (14A), γ now stands for u^f_d / u_l ; and V^p will have to be substituted by V_l^c , the circular orbital velocity corresponding to the launch orbit. Proceeding in a similar fashion, launch angle and elements for optimum transfer trajectory of the second kind can also be evaluated.

However, in case (*ii*) the equation for evaluation of θ°_d shall undergo modification. Equations (7), (9) and (32) can be combined to yield θ°_d thus :

$$\theta^\circ_d = \theta^f_d - \left[\sqrt{K} \quad (u_d)^{\frac{3}{2}} \ \triangle^{f}_{t_T,} \quad t^o_T \right] \tag{33}$$

where

u_d = reciprocal of destination circular orbit radius. Equations (18), (19) and (21) will determine the elements of the optimum transfer trajectory of the first kind after putting

$$\alpha = E_d = 0$$

in (21) and in the equations determining $\bar{V}_l$ viz., (11) to (14A); γ will be inter preted as

$$\gamma = \frac{u_d}{u^p{}_l}$$

For optimum transfer trajectory of the second kind, in equations (30), V^s signifies in this case $V^c{}_d$ the circular orbital velocity corresponding to the destination orbit and obviously $\omega^s = 0$ in case (ii).

If the launch and destination orbits are taken circular, on substitution of (7) in (33) and simplification, θ°_d can be expressed as

$$\theta^\circ_d = \theta^f_d - (a_T\, u_d)^{\frac{3}{2}} \Bigg[2 \tan^{-1} \left\{ \left(\frac{1 - E_T}{1 + E_T} \right)^{\frac{1}{2}} \tan \left(\frac{\theta^f_d - \beta}{2} \right) \right\}$$
$$+ 2 \tan^{-1} \left\{ \left(\frac{1 - E_T}{1 + E_T} \right)^{\frac{1}{2}} \tan \frac{\beta}{2} \right\}$$
$$- E_T \left(1 - E^2_T \right)^{\frac{1}{2}} \left\{ \frac{\sin (\theta^f_d - \beta)}{1 + E_T \cos \theta^f_d} + \frac{\sin \beta}{1 + E_T \cos \beta} \right\} \Bigg] \tag{34}$$

since by virtue of equation (8)

$$\theta^f{}_T = \theta^f{}_d$$

The optimum transfer trajectory of the first kind is characterized by the equations:

$$K\, u_d \left(1 - \bar{E}^2{}_T\right) = \left(2\, K\, u_l - \bar{V}^2{}_l\right) \left[1 + \bar{E}_T \cos \left\{ \theta^f_d - \cos^{-1} \left(\frac{\bar{V}^2{}_l - K\, u_l\, (1 + \bar{E}^2{}_T)}{(2\, K\, u_l - \bar{V}^2{}_l)\, \bar{E}_T} \right) \right\} \right] . \tag{35}$$

$\left(\text{Since } h^2{}_d = \dfrac{K}{u_d}\right)$

$$\bar{\beta} = \cos^{-1}\left[\frac{\bar{V}_l{}^2 - K\, u_l\, (1 + \bar{E}^2{}_T)}{(2\, K\, u_l - \bar{V}_l{}^2)\, \bar{E}_T} \right] \tag{36}$$

$$\bar{h}^2{}_T = \frac{K\, (1 - \bar{E}^2{}_T)}{2\, u_l - \dfrac{\bar{V}_l{}^2}{K}} \tag{37}$$

where in (11) to (14A); $u_l{}^p$, γ and V^p will be substituted by u_l, $\left(\dfrac{u_d}{u_l}\right)$ and $V_l{}^c$ respectively for evaluation of $\bar{V}_l$.

From equations (23), (26) and (27), elements for optimum transfer trajectory of the second kind are obtained as

$$\bar{\bar{h}}{}^2{}_T = \frac{K(1 - \bar{\bar{E}}{}^2{}_T)}{2u_d - \dfrac{V^2{}_d}{K}} \tag{38}$$

$$\bar{\bar{\beta}} = \cos^{-1}\left[\frac{\bar{\bar{V}}{}^2{}_d - 2Ku_d + Ku_l\, (1 - \bar{\bar{E}}{}^2{}_T)}{(2Ku_d - \bar{\bar{V}}{}^2{}_d)\, \bar{\bar{E}}_T} \right] \tag{39}$$

and

$$Ku_d\, (1 - \bar{\bar{E}}{}^2{}_T) = (2Ku_d - \bar{\bar{V}}{}^2{}_d) \left[1 + \bar{\bar{E}}_T \cos \left\{ \theta^f{}_d - \cos^{-} \left(\frac{\bar{\bar{V}}{}^2{}_d - 2Ku_d + Ku_l\, (1 - E^2{}_T)}{(2Ku_d - \bar{\bar{V}}{}^2{}_d)\, \bar{\bar{E}}_T} \right) \right\} \right] . \tag{40}$$

While substituting (30) for determination of $\bar{\bar{V}}_d$, it should be observed that $1/\gamma$, $u^f{}_d$ and V^s now signify $(u_l\ /\ u_d)$, u_d and $V^c{}_d$ respectively and $\omega^s = 0$ in this case also. The above analysis will also hold good when the interception manoeuvres are to be conducted from outer launch orbits to inner destination orbits since no energy level consideration of the orbits has been used in the analysis.

HOHMANN TRANSFER TRAJECTORIES

For coaxial elliptic launch and destination orbits, putting

$$\alpha = \beta = 0$$

and

$$\theta^f_d = \pi$$

in (9) and using [8(*iii*)] we have

$$\pi \left(a_{T,H}\right)^{\frac{3}{2}} = \left(a_d\right)^{\frac{3}{2}} \left[\pi - 2 \tan^{-1} \left\{ \left(\frac{1-E_d}{1+E_d}\right)^{\frac{1}{2}} \tan \frac{\theta^\circ_d}{2} \right\} + E_d \left(1-E^2_d\right)^{\frac{1}{2}} \left(\frac{\sin \theta^\circ_d}{1+E_d \cos \theta^\circ_d}\right)\right] \tag{41}$$

Substituting $\alpha = \beta = 0$ in (2), we have for Hohmann transfer trajectory

$$h^2_{T,H}\ u^p_l = K\,(1 + E_{T,H}) \tag{42}$$

and

$$\frac{h^2_{T,H}}{[a_d\,(1+E_d)]} = K\,(1 - E_{T,H}) \tag{43}$$

equations (42) and (43) yield

$$h^2_{T,H} = \frac{[2K a_d\,(1+E_d)]}{[u_l\ a_d\,(1+E_d)+1]} \tag{44}$$

$$E_{T,H} = \frac{[u^p_l\ a_d\,(1+E_d)-1]}{[u^p_l\ a_d\,(1+E_d)+1]} \tag{45}$$

Utilizing equations (44) and (45)

$$a_{T,H} = \frac{[u^p_l\ a_d\,(1+E_d)+1]}{2u^p_l} \tag{46}$$

Substitution of equation (46) in (41) transforms the latter thus

$$\pi\left[u_l^p\ a_d\,(1+E_d)+1\right]^{\frac{3}{2}} = \left(2u_l^p\ a_d\right)^{\frac{3}{2}}\left[\pi - 2\tan^{-1}\left\{\left(\frac{1-E_d}{1+E_d}\right)^{\frac{1}{2}} \tan\frac{\theta^\circ_d}{2}\right\} + E_d\left(1-E^2_d\right)^{\frac{1}{2}}\left(\frac{\sin\theta^\circ_d}{1+E_d\cos\theta^\circ_d}\right)\right] \tag{47}$$

(47) gives the launch angle for Hohmann transfer trajectory. Elements of the transfer trajectory are determined from (44) and (45). In case of Hohmann transfers for coaxial elliptic launch and destination orbits, (47) proves the dependence of launch angle of two factors : (*i*) ratio of the semi-major axis of the destination orbit to peri-distance of the launch orbit (*ii*) eccentricity of the destination orbit. Substituting

$$E_d = 0$$

and

$$u^p_l\ a_d = \frac{u_l}{u_d}$$

in (47) we obtain the result derived by Paiewonsky[2] for coplanar concentric circular launch and destination orbits :

$$\theta^\circ_d = \pi\left[1 - \left(\frac{u_l + u_d}{2u_l}\right)^{\frac{3}{2}}\right]$$

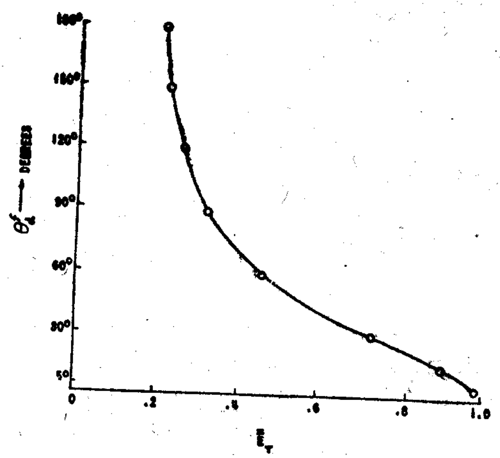

Fig. 1—Relation between interception angle and eccentricity

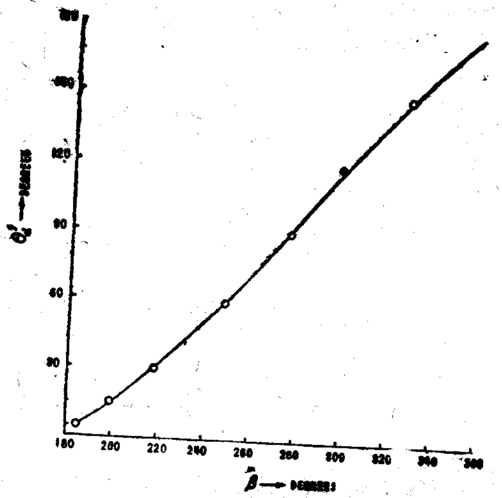

Fig. 2—Relation between interception angle and inclination angle of the major axis

EARTH MARS TRANSFER

The theory is illustrated by evaluating the launch angles and the orbital parameters of the corresponding transfer trajectories for Earth Mars transfer for the case of optimum transfer trajectory of the second kind. Earth and Mars orbits are taken as circular and coplanar and their orbital radii as ($1\cdot49\times10^8$) km. and ($2\cdot28\times10^8$) km. respectively. The Martian circular orbital velocity is taken to be equal to 24·10 km/sec. Characteristics for optimum trajectory of the first kind can be similarly calculated.

TABLE 1

CHARACTERISTICS FOR EARTH MARS TRANSFER THROUGH OPTIMUM TRANSFER TRAJECTORY OF THE SECOND KIND

Interception angle θ_d^f	Eccentricity ($\bar{\bar{E}}_T$)	Inclination angle of the major axis ($\bar{\bar{\beta}}$)	Semi-major axis times 10^{-8} ($\bar{\bar{a}}_T\, 10^{-8}$)	Aerial velocity times 10^{-8} ($\frac{1}{2}\bar{\bar{h}}_T\, 10^{-8}$)	Launch angle (θ_d^o)
			km.	sq. km/sec.	
180°	0·207	360° 0′	1·879	24·487	45° 20′
150°	0·220	330° 22′	1·865	24·329	34° 57′
120°	0·260	300° 48′	1·812	23·736	27° 9′
90°	0·319	278° 27′	1·737	22·808	20° 16′
60°	0·462	248° 37′	1·576	20·334	12° 19′
30°	0·732	218° 16′	1·365	13·403	0° 58′
15°	0·912	199° 1′	1·220	8·275	—(7° 49′)
5°	0·990	185° 56′	1·149	2·761	—(16° 31′)

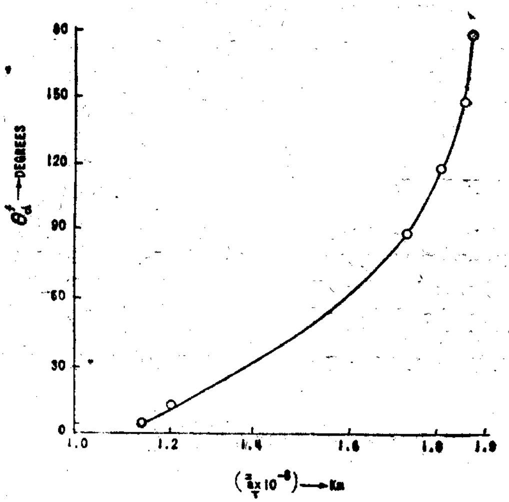

Fig. 3—Relation between length of semi-major axis and interception angle

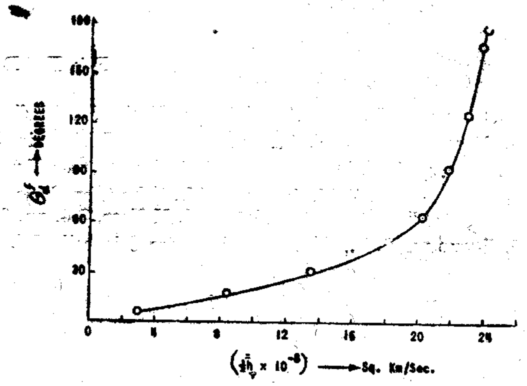

Fig. 4—Variation of aerial velocity with interception angle

Fig. 1–5 represent variation of orbital parameters of the transfer trajectory and launch angle with respect to the interception angle which varies from 0 to π. Their study reveals following interesting results :

(*i*) Launch angle and orbital parameters (except $\overline{\overline{\beta}}$) of the transfer trajectory vary in such a manner that the rate of gradient change is considerably small for lower and higher values of the interception angles. For intermediate values of the interception angle gradient variation rate is comparatively higher (Fig. 1, 3, 4, 5).

(*ii*) Launch angle and the orbital parameters (except $\overline{\overline{\beta}}$) vary fastly in the beginning but the variation gradually slows down with increasing values of interception angle.

(*iii*) Major axis of the transfer trajectory rotates with almost constant rate as interception angle varies from 0 to π (Fig. 2).

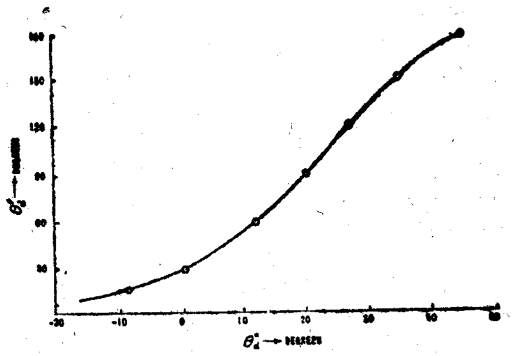

Fig. 5—Variation of launch angle with interception angle

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