SOLIDIFICATION OF A LIQUID SPHERE WITH A GIVEN HEAT FLUX AT THE SURFACE

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The method of the heat balance integral is used to investigate the solution of a problem involving the inward freezing of a liquid sphere. The solidification is effected under the assumption of constant heat flux from the surface. Solutions are presented by considering a two-parameter temperature profile and the position of the front is expressed in the form of a series in time parameter. Finally the results are graphically exhibited.

NOMENCLATURE

T = temperature distribution in the solid

t = time

K = thermal conductivity of the solidified phase

ρ = density of the solid

a = radius of the sphere

L = latent heat of fusion

Q = prescribed flux

k = thermal diffusivity

θ = dimensionless temperature in the solid

r = dimensionless space variable

τ = dimensionless time

$S(\tau)$ = position of the solidified front

$\epsilon(\tau)$ = dimensionless position of the moving front

β = dimensionless parameter

Unlike simple heat conduction problems, the problems involving phase change are very complicated as they involve a moving boundary whose location is not known a priori. Heat is liberated at this boundary and it is required to determine its growth with time. The first systematic study was made by Neumann[1] who presented the solution for a semi-infinite region initially at a constant temperature, greater than the melting temperature, with surface, maintained at zero temperature subsequently.

A number of papers have afterwards appeared on these problems[2—4]. But all these papers pertain to problems investigated either in planer regions or in cylindrical geometry.

In this paper, we discuss a problem concerning the solidification of a liquid sphere. Goodman's technique of the heat balance integral is used to obtain the solution of the problem. The liquid is initially maintained at the fusion temperature and the solidification is effected by maintaining a constant heat flux at the surface. A second order polynomial temperature is used for the temperature distribution in the solid and the location of the front, for small values of time, is expressed in the form of a series in ascending powers of time.

STATEMENT OF THE PROBLEM

Under the assumption of the constant thermal properties of the solid, the equations governing the radial heat flow in the solidified region together with the initial and surface conditions are given as

$$\frac{dT}{dt} = K \left[\frac{\partial^2 T}{\partial x^2} + \frac{2}{x} \frac{\partial T}{\partial x} \right] \tag{1}$$

$$T = 0 \quad ; x = a - s(t) \tag{2}$$

$$K \frac{\partial T}{\partial x} = -L\rho \frac{ds}{dt} \quad ; x = a - s(t) \tag{3}$$

$$K \frac{\partial T}{\partial x} = -Q \quad ; x = a \tag{4}$$

Also at $t = 0$, $s = 0$ and $T [a - s(t), t] = 0$ (5)

considering curves of constant temperature in the plane and combining equation (2) and (3) together, the following condition is obtained

$$\frac{\partial T}{\partial t} = \frac{K}{\rho L} \left(\frac{\partial T}{\partial x} \right)^2 \quad ; \quad x = a = s(t) \tag{6}$$

SOLUTION OF THE PROBLEM

Introducting the following dimensionless variables

$$r = \frac{a - x}{a}, \quad T = \frac{aQ\theta}{K}, \quad \tau = \frac{kt}{a^2}$$

$$\epsilon = \frac{s}{a}, \quad \beta = \frac{L\rho k}{Qa} \tag{7}$$

the basic equation (1) and boundary conditions (2) (3), (4) and (6) become :

$$\frac{\partial}{\partial r} \left[(1 - r)^2 \frac{\partial \theta}{\partial r} \right] = (1 - r)^2 \frac{\partial \theta}{\partial \tau} \tag{8}$$

$$\theta = 0 \; ; \quad r = \epsilon \tag{9}$$

$$\frac{\partial \theta}{\partial r} = \beta \frac{d\epsilon}{d\tau}; \quad r = \epsilon \tag{10}$$

$$\frac{\partial \theta}{\partial r} = 1 \; ; \quad r = 0 \tag{11}$$

$$\left(\frac{\partial \theta}{\partial r} \right)^2 = \beta \frac{\partial \theta}{\partial \tau} \; ; \quad r = \epsilon \tag{12}$$

Integrating both sides of (8) with respect to r from $r = 0$ to $r = \epsilon$ and using the boundary conditions (10) and (11), the heat balance integral can be written as

$$\beta (1-\epsilon)^2 \frac{d\epsilon}{d\tau} - 1 = \int_0^{\epsilon} (1-r)^2 \frac{d\theta}{d\tau} dr \tag{13}$$

Similarly by multiplying both sides of equation (8) by $(1-r)^2 \frac{\partial \theta}{\partial r} dr$ and integrating the resulting equation with the help of the condition (12), one gets

$$\beta (1-\epsilon)^4 \left(\frac{\partial \theta}{\partial \tau}\right)_{r=\epsilon} - 1 = 2 \int_0^{\epsilon} (1-r)^4 \frac{\partial \theta}{\partial r} \frac{\partial \theta}{\partial \tau} dr \tag{14}$$

In order to evaluate the integral expression on the right hand sides of (13) and (14), we assume the following quadratic temperature profile for

$$\theta = (r-\epsilon) + g\left(1 - \frac{r^2}{\epsilon^2}\right) \tag{15}$$

The two unknown parameters $\epsilon(\tau)$ and $g(\tau)$ can be obtained on solving the pair of first-order equations and on substitution of (15) into the integral equation (13) and (14). The initial conditions are

$$\epsilon(0) = 0, \qquad \underset{\epsilon \to 0}{Lt} (\epsilon g) = 0 \tag{16}$$

The latter condition is derived from consideration of the total thermal energy of the solidified phase. This is related to the energy thickness

$$\theta^* (\epsilon, 2) = \int_0^{\epsilon} (1-r)^2 \theta \, dr \tag{17}$$

condition (16) obviously follows from (7) because $\theta^* = 0$ when $\epsilon = 0$

Putting the value of θ from (15) in (13) and integrating, we get

$$\beta(1-\epsilon)^2 \frac{d\epsilon}{d\tau} - 1 = \left[-\frac{\epsilon^3}{3} + \epsilon^2\left(1 + \frac{2}{5} g\right) - \epsilon(1+g) + \frac{2}{3} g\right] \frac{d\epsilon}{d\tau} + \left[\frac{2}{15}\epsilon^3 - \frac{1}{2}\epsilon^2 + \frac{2}{3}\epsilon\right] \frac{dg}{d\tau} \tag{18}$$

Equation (18) on integration and simplification gives

$$\tau = \frac{\epsilon^4}{12} - \frac{\epsilon^3}{3} + \frac{\epsilon^2}{2} - \frac{\beta(1-\epsilon)^3}{3} + \frac{\beta}{3} - \left[\frac{2}{15}\epsilon^3 - \frac{1}{2}\epsilon^2 + \frac{2}{3}\epsilon\right] g \tag{19}$$

Similarly putting the value of θ from (15) in (14) and on integrating, we get

$$\beta(1-\epsilon)^4\left(\frac{2g}{\epsilon}-1\right)\frac{d\epsilon}{d\tau}-1=\left[-\frac{2}{5}\epsilon^5+\left(\frac{26}{21}g+2\right)\epsilon^4-\left(g^2+\frac{88}{15}g+4\right)\epsilon^3+\left(\frac{32}{7}g^2+\frac{54}{5}g+4\right)\epsilon^2\right.$$
$$\left.-\left(8g^2+\frac{28}{3}g+2\right)\epsilon+\left(\frac{32}{5}g^2+\frac{10}{3}g\right)-\frac{2g^2}{\epsilon}\right]\frac{d\epsilon}{d\tau}$$
$$+\left[\frac{4}{35}\epsilon^5-\left(\frac{1}{6}g+\frac{2}{3}\right)\epsilon^4+\left(\frac{32}{35}g+\frac{8}{5}\right)\epsilon^3-\left(2g+2\right)\epsilon^2\right.$$
$$\left.+\left(\frac{32}{15}g+\frac{4}{3}\right)\epsilon-g\right]\frac{dg}{d\tau} \qquad (20)$$

Eliminating τ from the pair of first order differential equations (19) and (20), we get

$$\left[\frac{4}{35}\epsilon^5-\left(\frac{1}{6}g+\frac{2}{3}\right)\epsilon^4+\left(\frac{32}{35}g+\frac{22}{15}\right)\epsilon^3-\left(2g+\frac{3}{2}\right)\epsilon^2+\left(\frac{32}{15}g+\frac{2}{3}\right)\epsilon-g\right]\epsilon\frac{dg}{d\epsilon}$$
$$=\frac{2}{5}\epsilon^6-\left(\frac{26}{21}g+\beta+2\right)\epsilon^5+\left[g^2+\left(\frac{88}{15}+2\beta\right)g+4\beta+\frac{11}{3}\right]\epsilon^4$$
$$-\left[\frac{32}{7}g^2+\left(\frac{52}{5}+8\beta\right)g+7\beta+3\right]\epsilon^3+\left[8g^2+\left(\frac{25}{3}+12\beta\right)g+6\beta+1\right]\epsilon^2$$
$$-\left[\frac{32}{5}g^2+\left(\frac{8}{3}+8\beta\right)g+2\beta\right]\epsilon-\left[2g^2+2\beta g\right] \qquad (21)$$

Equation (21) being non-linear, it is difficult to give its solution in closed analytic form. This equation can however be solved numerically by the method given by Fox & Goodwin[5] Substitution of the value of g in (19) will then give the time history of the moving front.

Since g and ϵ are connected by the first order differential equation (21), g can be expanded as a series in ϵ and the first few terms of g are thus given as :

$$g=\epsilon+\left(1-\frac{1}{3\beta}\right)\epsilon^2+\left(9-\frac{119}{10\beta}+\frac{3}{\beta^2}\right)\epsilon^3+\left(12-\frac{342}{5\beta}+\frac{907}{15\beta^2}-\frac{9}{\beta^3}\right)\frac{\epsilon^4}{6} \qquad (22)$$

Substituting the value of g from (22) in (19), we get

$$\tau=\begin{cases}\beta\epsilon-\left(\beta+\frac{1}{6}\right)\epsilon^2+\left(\frac{\beta^2}{3}-\frac{\beta}{2}+\frac{2}{9}\right)\frac{\epsilon^3}{\beta}-\left(\frac{11}{20}\beta^2-\frac{52}{42}\beta+\frac{1}{3}\right)\frac{\epsilon^4}{\beta^2}\\ -\left(\frac{43}{60}\beta^3-\frac{287}{72}\beta^2+\frac{1783}{540}\beta-\frac{19}{17}\right)\frac{\epsilon^5}{\beta^3}+\left(\frac{4}{5}\beta^3-\frac{773}{225}\beta^2+\frac{187}{72}\beta-\frac{19}{36}\right)\frac{\epsilon^6}{\beta^3}\\ -\left(\frac{4}{15}\beta^3-\frac{222}{225}\beta^2+\frac{959}{1350}\beta-\frac{19}{135}\right)\frac{\epsilon^7}{\beta^3}\end{cases} \qquad (23)$$

Equation (23) expresses τ as a function of the position of the moving front ϵ. It will be more realistic to express ϵ as a function of the time. τ. To achieve this we invert equation

(23) and the inverted series of is given by

$$\epsilon = \begin{cases} \dfrac{\tau}{\beta} + \dfrac{1}{6\beta^3}\left(6\beta + 1\right)\tau^2 + \dfrac{1}{3\beta^5}\left(4\beta^2 + 5\beta - \dfrac{7}{6}\right)\tau^3 + \dfrac{1}{\beta^7}\left(\dfrac{10}{3}\beta^3 + \dfrac{949}{180}\beta^2 - \dfrac{129}{90}\beta\right. \\ \left. + \dfrac{37}{216}\right)\tau^4 + \dfrac{1}{\beta^{10}}\left(\dfrac{34}{3}\beta^5 + \dfrac{357}{20}\beta^4 - \dfrac{1003}{360}\beta^3 + \dfrac{3439}{1080}\beta^2 - \dfrac{641}{648}\beta - \dfrac{2}{27}\right)\tau^4 \end{cases} \quad (24)$$

DISCUSSION OF THE RESULTS

Fig. 1 is the plot of the dimensionless thickness ϵ vs τ for two different values of β. It has been observed that the total time τ required for the complete solidification of the sphere for $\beta = 1$ and $\beta = 2$ has been found to be $1 \cdot 131$ and $2 \cdot 349$ respectively. This seems to be physically justified because larger values of β amounts to lesser withdrawal of heat at the surface and therefore more time is needed for solidification. Fig. 2 gives the temperature distribution in the solidified portion when the solidification depths are $\epsilon = 0 \cdot 5$ and $\epsilon = 0 \cdot 8$ for $\beta = 1$. It has been noticed that as the solidification process continues the surface temperature has to be lowered more and more to maintain a constant flux. Considering the case $\beta = 1$, one observes that the surface temperature has to be as low as $-1 \cdot 17$ when half of the solid has frozen and as low as $-1 \cdot 29$ when four-fifth of the sphere depth has frozen.

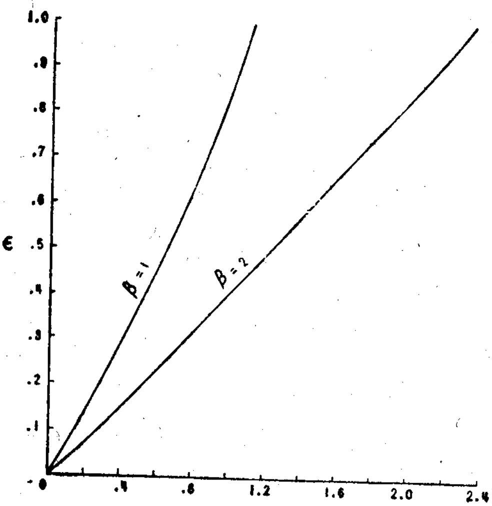

Fig. 1—Thickness of solidified region *vs* time.

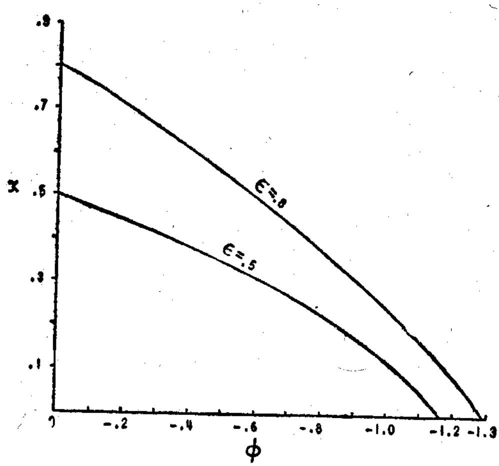

Fig. 2—Temperature distribution in the solid region for $\beta = 1$ and various values of ϵ.

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