# SIACCI APPROXIMATION TO THE MEAN-TWISTED TRAJECTORY OF A PROJECTILE*

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In this paper, we have integrated the dynamical equations of the mean-twisted trajectory of a spinning shell in terms of step functions similar to those of Siacci. The air-resistance is assumed to be a known tabulated function of Mach number. Using our solutions we have calculated the drift of a 3 inch 16 lb. shell which agrees well with the observation. The corrections arising in the various elements of the trajectory are also calculated.

Our earlier solutions[1] to the differential equations of the mean-twisted trajectory‡ of a spinning shell is restricted to the case of Newtonian resistance and hence has limited application only to projectiles fired with subsonic muzzle velocities. Effectively this is a generalization of Otto-Lardilon's solution to trajectories in space. In the present paper, however, we undertake to integrate the dynamical equations of the mean-twisted trajectory where the resistance due to air is a known tabulated function of Mach number. The solutions we thus obtain are in terms of step functions similar to those of Siacci[2] but these functions naturally involve the effects of spin which are reflected through, factors of cross-wind force and the upsetting moment. Give the ballistic coefficient of a certain projectile and tabulated values of lift and moment coefficients in terms of Mach number, the present solution will enable us to determine the various trajectory elements including the drift of the projectile. Here use is made of the fact that the ratio of the lift and moment coefficients *i.e.* $f_L/f_M$ is a constant** (MAYEVSKI factor). Method of calculation of these corrections to various elements is also indicated by considering a 3 inch 16 lb. shell fired at a quadrant elevation of 30°.

## FOWLER'S DYNAMICAL EQUATIONS TO THE MEAN TWISTED TRAJECTORY

We shall assume at the outset that the projectile is stable and the deceleration of its spin due to air friction is negligible, and the projectile is impressed by the air forces, the drag and the lift given by

$$R = \rho v^2 r^2 f_R\,(a/v,\,\delta) \tag{1}$$

$$L = \rho v^2 r^2 \sin\delta\, f_L\,(v/a,\,\delta) = \mu \sin\delta \tag{2}$$

and the moment of the air forces is given by

$$M = \rho v^2 r^3 \sin\delta\, f_M\,(v/a,\,\delta) \tag{3}$$

The notations are the same as in Fowler's[3] and the same notations have been used throughout the paper unless otherwise stated.

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‡It is the trajectory in space of a spinning projectile after the initial vibrations of the projectile have damped out.

**It is true for the case when the velocity of shell does not fall below 1200 f.p.s.

After the initial vibrations of the shell damp out, the subsequent motion of the projectile is governed by the differential equations*

$$\left.\begin{aligned} m' &= -n(\omega + \Psi' \sin\theta) - l\theta' \\ n' &= m(\omega + \Psi' \sin\theta) - l\Psi' \cos\theta \end{aligned}\right\} \quad (4)$$

and

$$\left.\begin{aligned} v' &= -\frac{R}{m^*} - g\sin\theta \\ \theta' &= km - \frac{g\cos\theta}{v} \\ \Psi' \cos\theta &= kn \end{aligned}\right\} \quad (5)$$

Equations (4) give the angular motion of the shell and (5) the rectilinear motion. The direction-cosines of the axis of the shell are given by

$$\left.\begin{aligned} l &= \cos\delta \\ m &= \sin\delta\cos\phi \\ n &= \sin\delta\sin\phi \end{aligned}\right\} \quad (6)$$

where $\delta$ is the yaw and $\phi$ the angle of rotation of the plane of yaw from the vertical plane containing the velocity-vector of the centre of mass of the shell. The coordinates of the centre of mass of the shell relative to a rectangular right-handed Cartesian frame of reference O-xyz stationed at the gun position are

$$\left.\begin{aligned} x &= \int_0^t v\cos\theta\cos\Psi\, dt \\ y &= \int_0^t v\sin\theta\, dt \\ z &= \int_0^t v\cos\theta\sin\Psi\, dt \end{aligned}\right\} \quad (7)$$

Thus at any time $t$, $x$ measures the range, $y$ the height and $z$ the right-handed drift of the projectile. The parameters $k$ and $\omega$ occurring in equations (4) and (5) are given by

$$k = L/m^*v = \frac{\rho_0 r^2}{m^*} \cdot \frac{v}{f} \cdot f_L\,(v/a,\ \delta) \quad (8)$$

*Here onwards a prime stands for differentiation with respect to time. I bid (2) pp. 295-387.

$$\omega = \mu_{/AN} = \frac{\rho_0 r^3}{AN} \cdot \frac{v^2}{f} \cdot f_M \, (v/a, \delta) \tag{9}$$

and therefore

$$k_{/\omega} = \frac{AN}{m^* r} \cdot \frac{1}{v} \cdot \frac{f_L}{f_M} \tag{10}$$

Here $\rho_0$ is the density of the air at the ground, $f$ is the altitude factor for th height, so that $\rho = \rho_0 f$

We further define

$$\epsilon = \frac{k}{\omega} \cdot \frac{v}{a} = \frac{AN}{m^* r} \cdot \frac{1}{a} \cdot \frac{f_L}{f_M} \tag{11}$$

which is effectively a constant for projectiles fired from large coastal guns where the velocity seldom falls below the sonic speed. Even otherwise it is fairly steady all over the trajectory so that for all practical purposes it may be treated a constant which is small. See Table* 1.

TABLE 1

TABLE SHOWING THE VALUE OF PARAMETRE ε AT DEFFERENT POINT OF THE TRAJECTORY FOR A 3 INCH 16 lb. SHELL

| $t$ | $v$ f.s. | $\theta$ degrees. | $\omega$ rad./sec. | $k = k.\frac{a}{v}$ rad./sec | $\varepsilon \times 10^2$ |
|---|---|---|---|---|---|
| 0 | 2,000 | 30·0 | 24·8 | 0·954 | 3·840 |
| 1 | 1,726 | 29·1 | 19·0 | 0·633 | 3·343 |
| 2 | 1,515 | 28·1 | 15·0 | 0·430 | 2·868 |
| 6 | 1,075 | 22·7 | 7·50 | 0·306 | 4·080 |
| 12 | 860 | 11·7 | 3·44 | 0·145 | 4·224 |
| 18 | 751 | —2·1 | 2·56 | 0·092 | 3·623 |
| 24 | 703 | —17·2 | 2·29 | 0·083 | 3·611 |
| 30 | 705 | —31·5 | 2·43 | 0·088 | 3·617 |
| 36 | 735 | —43·7 | 2·88 | 0·104 | 3·608 |
| 37·8 | 747 | —46·8 | 3·06 | 0·111 | 3·635 |
| | | | | Average : | 3·6449 |

## APPROXIMATIONS DUE TO SMALL YAW

As Siacci approximation is valid only for low-angle fire and since we have assumed the projectile to be stable, the angle of yaw must remain small throughout the trajectory. In the present case the aerodynamic coefficients $f_R$, $f_L$ and $f_M$ are independent of yaw†.

Using equations (5) in (4) we have

$$m' = -(\omega + kn \tan \theta)\, n - km \cos \delta + (g/v) \cos \theta \cos \delta$$

$$n' = (\omega + kn \tan \theta)\, m - kn \cos \delta$$

As yaw remains small $kn \tan \theta$ may be neglected compared to $\omega$, for their ratio $\frac{k}{\omega}\,\delta \tan \theta$ is much less than unity.

* Reference[4] p. 619

† Reference[4] p. 585

Writing

$$\sigma = (g/v) \cos \theta \tag{12}$$

The above equations become

$$\left.\begin{aligned} m' &= -\omega n - km + \sigma \\ n' &= \omega m - kn \end{aligned}\right\} \tag{13}$$

A solution of these equations may be expressed as

$$m = X_2 - X_4 + X_6 - \ldots\ldots\ldots\ldots$$
$$n = X_1 - X_3 + X_5 - \ldots\ldots\ldots\ldots$$

where

$$X_1 = \sigma/\omega$$

and

$$X_{n+1} = \frac{k}{\omega} X_n + \frac{1}{\omega} X'_n \qquad (n = 1,2,3\ldots.)$$

Approximately we shall assume for our subsequent calculations

$$n = \sigma/\omega \tag{14}$$

$$m = \frac{k}{\omega^2} \sigma + \frac{1}{\omega} (\sigma/\omega)' \tag{15}$$

and we have from equations (7), writing $\cos \Psi = 1$ and $\sin \Psi = \Psi$

$$\left.\begin{aligned} x' &= v \cos \theta = q \\ y' &= v \sin \theta \\ z' &= q \Psi \end{aligned}\right\} \tag{16}$$

The approximate equations giving the motion of the shell in space may now be written out as

$$\left.\begin{aligned} q' &= -(R(v)/_{m^*}) \cos \theta - km\, q \tan \theta \\ \theta' &= -\sigma + km \\ \Psi' &= kn \sec \theta \end{aligned}\right\} \tag{17}$$

where $n$ and $m$ are given by equations (14) and (15) and the retardation function is

$$R(v)/_{m^*} = \frac{1}{10^4 C} \times v^2 P(v) \quad , \ say \tag{18}$$

C being the ballistic coefficient of the projectile.

As the plane trajectory is a first approximation of the mean-twisted trajectory for small yawing motion, the correction terms due to the cross-aerodynamic forces will be small and hence we have from the first two equations of (17)

$$\frac{d\theta}{dq} = \frac{10^4\, Cg \cos^3 \theta}{q^3\ P(q/\cos\theta)} \times \left\{ 1 - \frac{km}{\sigma} \times \frac{10^4\, Cg \tan \theta \cos^3 \theta}{q^2\, P(q/\cos\theta)} \right\} \tag{19}$$

Using equations (14), (16) and (17) we also have the following approximate equations

$$\frac{dt}{dq} = - \frac{10^4\, C \cos\theta}{q^2 P\,(q/\cos\theta)} \times \left\{1 - \frac{km}{\sigma} \times \frac{10^4\, Cg \tan\theta \cos^3\theta}{q^2\, P(q/\cos\theta)}\right\} \qquad (20)$$

$$\frac{dx}{dq} = - \frac{10^4 C \cos\theta}{q\, P(q/\cos\theta)} \times \left\{1 - \frac{km}{\sigma} \times \frac{10^4 Cg \tan\theta \cos^3\theta}{q^2\, P(q/\cos\theta)}\right\} \qquad (21)$$

$$\frac{d\Psi}{dq} = - \epsilon \left\{1 - \frac{km}{\sigma} \times \frac{10^4\, Cg \tan\theta \cos^3\theta}{q^2\, P\,(q/\cos\theta)}\right\} \times \frac{10^4\, Cg\, a \cos^3\theta}{q^4} \qquad (22)$$

$$\frac{dy}{dx} = \tan\theta \qquad (23)$$

and

$$\frac{dz}{dx} = \Psi \qquad (24)$$

The above equations of the mean-twisted trajectory are suitable for integration in terms of step functions similar to those of Siacci, provided suitable average values for $\cos\theta$ along the arc of the trajectory are introduced.

## THE HODOGRAPH EQUATION

Using equations (11), (15) and (16) we obtain

$$\frac{km}{\sigma} = \epsilon^2 \left[ \frac{a^2 \cos^2\theta}{q^2} - \frac{Bq^2\, P\left(\frac{q}{\cos\theta}\right)}{10^4\, C \cos\theta}\, \frac{d}{dq}\left(\frac{a^2 \cos^4\theta}{q^3\, f_L}\right) \right]$$

correct up to order of $\epsilon^2$ and $B$ is given by

$$B = \frac{m^* f}{\rho_0 r^2}$$

where $B$ is a constant similar to the ballistic coefficient $C$.

Multiplying equation (19) by $\sec^2\theta$ and using the above expression for $\frac{km}{\sigma}$ we have

$$\frac{d(\tan\theta)}{du} + \frac{10^8 C^2 g^2 \mu^2}{\lambda^4 u^5 P(u)} \times \epsilon^2 \chi(u) \tan\theta = \frac{C\mu}{\lambda^2} \times \frac{10^4 g}{u^3 P(u)} \left\{1 - \epsilon^2 \chi(u)\right\} \qquad (25)$$

where

$$\chi(u) = a^2/u^2 - \frac{2\mu}{\lambda C} \times \frac{Ba^2}{\alpha(u)} \times \frac{d}{du}\left(\frac{1}{u^3 f_L}\right) \qquad (26)$$

$$\alpha(u) = \frac{10^4}{u^2 P(u)} \qquad (27)$$

and $\lambda$ and $\mu$ are the Siacci average values for $\cos\theta$ having their usual meanings as in British Ballistic Tables (1940) and $u$ is the usual Pseudo-velocity given by

$$u = \frac{v \cos\theta}{\lambda} \qquad (28)$$

Equation (25) is thus the hodograph equation for the mean-twisted trajectory and this being a standard linear equation in tan $\theta$ can be integrated in a closed form. But as we have already pointed out the mean-twisted trajectory under consideration has little departure from the corresponding plane trajectory, we shall for convenience of calculations, seek the solution of equation (25) in the form

$$\tan\theta = \tan\theta_0 - \frac{C\mu}{2\lambda^2}\left[ I(u_0) - I(u) + \epsilon^2 \left\{ \mathbf{I}(u_0) - \mathbf{I}(u) \right\} \right] \quad (29)$$

where $I(u)$ is the Siacci inclination function for the corresponding plane trajectory and is given by

$$I(u) = 2g \int_{400}^{u} \frac{10^4}{u^3 P(u)}\, du$$

$$= 2g \int_{400}^{u} \frac{\alpha(u)}{u}\, du \quad (30)$$

and a straight forward calculation shows

$$\mathbf{I}(u) = -2g \int_{400}^{u} \frac{\alpha(u)\chi(u)}{u}\, du$$

$$- 2g^2 \times C\mu \left\{ \tan\theta_\circ - \frac{C\mu}{2\lambda^2} I(u_\circ) \right\} \times \int_{400}^{u} \frac{\alpha^2(u)\chi(u)}{u}\, du$$

$$- g^2 \times \frac{C^2\mu^2}{\lambda^2} \times \int_{400}^{u} \frac{\alpha^2(u)\,\chi(u)\, I(u)}{u}\, du \quad (31)$$

$\mathbf{I}(u)$ is obtained in the present form when we substitute equation (29) in equation (25). We note that the inclination function equation (30) is a solution of equation (25) when $\epsilon = 0$.

## REMAINING STEP FUNCTIONS FOR THE MEAN—TWISTED TRAJECTORY

We may now obtain the various elements of the mean-twisted trajectory in terms of step functions as follows:

In equation (21), introducing the Siacci mean values for $\cos\theta$ we have

$$\frac{ax}{du} = -\frac{10^4 C\mu}{uP(u)} \left\{ 1 - \epsilon^2 \chi(u) \frac{10^4\, C\mu^3 \tan\theta}{\lambda^2 u^2\, P(u)} \right\} \quad (32)$$

Substituting the expression for $\tan\theta$ from equation (29) in the above equation and integrating, we get the solution for $x$ up to an order of $\epsilon^2$ as

$$x = C\mu \left[ S(u_\circ) - S(u) + \epsilon^2 \left\{ \mathbf{S}(u_\circ) - \mathbf{S}(u) \right\} \right] \quad (33)$$

where $S(u)$ is the space function of Siacci and is given by

$$S(u) = \int_{400}^{u} u\,\alpha(u)\,du \tag{34}$$

and the correction function $S(u)$ is given by

$$\mathbf{S}(u) = -\frac{C^2\mu^2}{2\lambda^2} \times g \int_{400}^{u} u\,\alpha^2(u)\,\chi(u)\,I(u)\,du$$

$$-\,C\mu\,g\left(\tan\theta_\circ - \frac{C\mu}{2\lambda^2}\,I(u_\circ)\right)\int_{400}^{u} u\,\alpha^2(u)\,\chi(u)\,du \tag{35}$$

Similarly from equation (20) we get the equation

$$\frac{dt}{du} = -\frac{10^4\,C\mu}{\lambda\,u^2\,P(u)}\left\{1 - \epsilon^2\,\chi(u)\,\frac{10^4\,Cg\,\mu^3\tan\theta}{\lambda^2\,u^2\,P(u)}\right\} \tag{36}$$

Putting for $\tan\theta$ from equation (29) in the above equation and integrating we get the time function upto an order of $\epsilon^2$ as

$$t = \frac{C\mu}{\lambda}\left[T(u_\circ) - T(u) + \epsilon^2\left\{\mathbf{T}(u_\circ) - \mathbf{T}(u)\right\}\right] \tag{37}$$

where $T(u)$ is the Siacci time function for the corresponding plane trajectory and is given by

$$T(u) = \int_{400}^{u} \alpha(u)\,du \tag{38}$$

whose tabulated values are available and the correction function $\mathbf{T}(u)$ is given by

$$\mathbf{T}(u) = -\frac{1}{2}\left(\frac{C\mu}{\lambda}\right)^2 \times g\int_{400}^{u} \alpha^2(u)\,\chi(u)\,I(u)\,du$$

$$-\,C\mu g\left(\tan\theta_\circ - \frac{C\mu}{2\lambda^2}\,I(u_\circ)\right) \times \int_{400}^{u} \alpha^2(u)\,\chi(u)\,du \tag{39}$$

Coming to the altitude function, we consider equation (23) and have

$$dy = \tan\theta\,\frac{dx}{du}\,du$$

Putting for $\tan\theta$ and $\frac{dx}{du}$ from equation (29) and equation (32) respectively and retaining terms of order upto $\epsilon^2$ and integrating we have

$$y/x = \tan\theta_\circ - \frac{C\mu}{2\lambda^2}\left\{I(u_\circ) + \epsilon^2\,\mathbf{I}(u_\circ)\right\}$$

$$+\,\frac{C\mu}{2\lambda^2}\left[\frac{A(u_\circ) - A(u) + \epsilon^2\left\{\mathbf{A}(u_\circ) - \mathbf{A}(u)\right\}}{S(u_\circ) - S(u) + \epsilon^2\left\{\mathbf{S}(u_0) - \mathbf{S}(u)\right\}}\right] \tag{40}$$

where

$$A(u) = \int_{400}^{u} u\,\alpha(u)\,I(u)\,du \tag{41}$$

is the Siacci altitude function for the plane trajectory (tabulated values available) and

$$\mathbf{A}(u) = \int_{400}^{u} u\,\alpha(u)\,\mathbf{I}(u)\,du$$

$$-\frac{1}{2}\left(\frac{C\mu}{\lambda}\right)^2 \times g\int_{400}^{u} u\,\alpha^2(u)\,\chi(u)\,I^2(u)\,du$$

$$-\left\{ tan\ \theta_{\circ} - \frac{C\mu}{2\lambda^2} I(u_{\circ}) \right\} C\mu g \int_{400}^{u} u\,\alpha^2(u)\,\chi(u)\,I(u)\,du \tag{42}$$

is the correction to altitude function for the mean-twisted trajectory.

## THE DRIFT FUNCTION

Here we shall introduce two new functions which are due to the mean-twisted trajectory alone. These are (*i*) the deflection function which gives us the angular coordinate $\Psi$ and (*ii*) the drift function which gives the $z$–coordinate that measures the departure of the shell from the plane of fire at any time. As can be seen from the nature of the functions for drift and the deflection they can be evaluated by means of the elements of the plane trajectory alone.

Now considering equation (22) and retaining terms upto order $\epsilon^2$ and using Siacci mean values for cos θ we have

$$\frac{d\Psi}{du} = -\epsilon\,a\,g \times \left(\frac{\mu}{\lambda}\right)^3 \times \frac{10^4\,C}{u^4} \tag{43}$$

Integrating this equation we get

$$\Psi = C\,\epsilon\,a\,g \times \left(\frac{\mu}{\lambda}\right)^3 \left[\triangle(u_{\circ}) - \triangle(u)\right] \tag{44}$$

where

$$\triangle(u) = \int_{400}^{u} \frac{\alpha(u)}{u^2}\,du \tag{45}$$

is the deflection function.

From equations (24) and (32) we have on similar considerations

$$dz = -C^2\,\epsilon\,a\,g\,.\,\frac{\mu^4}{\lambda^3}\left[\triangle(u_{\circ}) - \triangle(u)\right] u\,\alpha(u)\,du$$

and on integrating,

$$z/x = C \epsilon a g \left(\frac{\mu}{\lambda}\right)^3 \times \frac{[D(u_o) - D(u)]}{[S(u_o) - S(u)]} \tag{46}$$

where

$$D(u) = \int_{400}^{u} [\triangle(u_o) - \triangle(u)]\, u\, \alpha(u)\, du$$

$$i.e. = \triangle(u_o)[S(u_o) - S(u)] - [d(u_o) - d(u)] \tag{47}$$

and

$$d(u) = \int_{400}^{u} \triangle(u)\, u\, \alpha(u)\, du \tag{48}$$

The tables for the deflection function $\triangle(u)$ and drift function $D(u)$, which are similar to other Siacci functions, are prepared and will hold good for any shell as the integrands of these functions are independent of shell characteristics.

EXAMPLE

Consider a 3 inch 16 lb. shell fired with a muzzle velocity 2,000 f.s. at a quadrant elevation of 30°. The ballistic coefficient $C$ estimated according to 1940 Law is 1·35. The gun has a rifling of 1 turn in 30 calibres.

$A = 0{\cdot}1329$ lb./ft.$^2$

$\frac{A}{B} = 0{\cdot}115$

$N = 1675$ rad./sec.

$\Omega = 192{\cdot}7$ rad./sec.

$\epsilon = 0{\cdot}03645$

and the distance of the C. G. from the base is 4·88 inches. We shall calculate the drift and the correction to the various elements.

(*i*) *Drift*

$$z/x = C\epsilon a g \times \left(\frac{\mu}{\lambda}\right)^3 \frac{[D(u_0) - D(u)]}{[S(u_0) - S(u)]}$$

where $u_0 = 2{,}000$ f.s. and $u$, the final pseudo-velocity is 530·1 f.s. Thus the drift in terms of the azimuthal angle is 83 minutes whereas from the experiments conducted in Portsmouth Jump Card Trials it is observed as 89 minutes.

(*ii*) *Corrected range*

$$x = C\mu\,[S(u_0) - S(u) + \epsilon^2 \{\mathbf{S}(u_o) - \mathbf{S}(u)\}]$$

$$= 1{\cdot}27845 \left\{22708{\cdot}00000 + 160{\cdot}40064\right\}$$

$$= 29{,}236 \text{ feet}$$

$$= 9{,}745 \text{ yards nearly.}$$

(*iii*) *Corrected time of flight*

$$t = \frac{C\mu}{\lambda}\left[ T\,(u_0) - T\,(u) + \epsilon^2 \left\{ \mathbf{T}\,(u_0) - \mathbf{T}\,(u) \right\} \right]$$

$$= 1\cdot44204 \left\{ 26\cdot210000 + 0\cdot262646 \right\}$$

$$= 38\cdot2 \text{ seconds.}$$

and the correction in the time of flight is +1%.

(*iv*) *Corrected angle of fall*

$$\tan\theta = \tan\theta_0 - \frac{C\mu}{2\lambda^2}\left[ I(u_0) - I(u) + \epsilon^2 \left\{ \mathbf{I}(u_0) - \mathbf{I}(u) \right\} \right]$$

$$= 0\cdot577350 - 1\cdot011236 \left\{ 1\cdot718700 - 0\cdot003468 \right\}$$

$$= -1\cdot157154$$

$$\omega = 49^\circ\ 10^m$$

is the corrected angle of fall and the correction in the angle of fall arising due to cross-aerodynamic forces is roughly 1/12 of a degree.

## CONCLUSION

Thus by this analysis, corrections to range, time of flight, angle of fall etc. can be given and the drift can also be predicted provided the shell constants are known. Further it is observed that table of deflection and drift functions provides an additional table to Siacci tables wherefrom the drift can be calculated for any shell.

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