MINIMUM TIME BALLISTIC TRANSFER TRAJECTORIES

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(*Received* 8 *June* 1967)

With a specified characteristic velocity, the transfer trajectory for the minimum transfer time has been investigated for the general case when the initial and final orbits are elliptic. Particular cases when either the initial or final or both the orbits are circular follow immediately.

Optimum transfer trajectories between two orbits have been considered mainly from the point of view of minimum characteristic velocity as this corresponds to minimum energy expenditure[1]. Recently Lee & Florence[2] considered transfer trajectories which gave minimum transfer time for the particular case when the initial and final orbits were circular. As elliptic orbits are more important, it is proposed to generalise Lee & Florence approach to cases where the initial and final orbits are elliptic. Since circle is a particular case of an ellipse, other cases where either the initial or final or both the orbits are circular follow immediately.

MATHEMATICAL MODEL

Let the initial, transfer and final orbits be taken as

$$h_1^2 = Kr\,[1+e_1 \cos\theta] \tag{1}$$

$$H^2 = Kr\,[1+E \cos(\theta-\theta_1+\alpha)] \tag{2}$$

$$h_2^2 = Kr\,[1+e_2 \cos(\theta-\xi)] \tag{3}$$

The initial and final trajectories are non-intersecting, coplanar and non-coaxial. Let (r_1, θ_1) be the point where the space vehicle enters the transfer trajectory and (r_2, θ_2) be the point where it leaves and enters the final orbit (Fig 1). Let

$$\theta_2 = \theta_1 + \varphi \tag{4}$$

Let V represent the velocity, and $\frac{\pi}{2} + \gamma$ the angle between the velocity vector and the radius vector at any point. Let magnitudes pertaining to the initial and final orbits be denoted by suffix 1 and 2 and those on the transfer trajectory at entry and exit be represented by suffixes i and f. From any book on orbital dynamics one may show that[3]

$$\frac{r_1}{r_2} = \frac{K}{r_1\,V_i^2}\left(\frac{1-\cos\varphi}{\cos^2\gamma_i}\right) + \frac{\cos(\gamma_i+\varphi)}{\cos\gamma_i} \qquad |\gamma_i| \leqslant \frac{\pi}{2} \tag{5}$$

$$\tan \alpha = \frac{\left(\frac{r_1 V_i^2}{K}\right) \sin \gamma_i \cos \gamma_i}{\left(\frac{r_1 V_i^2}{K}\right) \cos^2 \gamma_i - 1} \tag{6}$$

$$E^2 = \left(\frac{r_1 V_i^2}{K} - 1\right)^2 \cos^2 \gamma_i + \sin^2 \gamma_i \tag{7}$$

$$a = \frac{r_1}{\left(2 - \frac{r_1 V_i^2}{K}\right)} \tag{8}$$

where a is the semi-length of the major axis of the transfer trajectory

$$V_i r_1 \cos \gamma_i = V_f r_2 \cos \gamma_f \tag{9}$$

$$V_f^2 = V_i^2 + 2K \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{10}$$

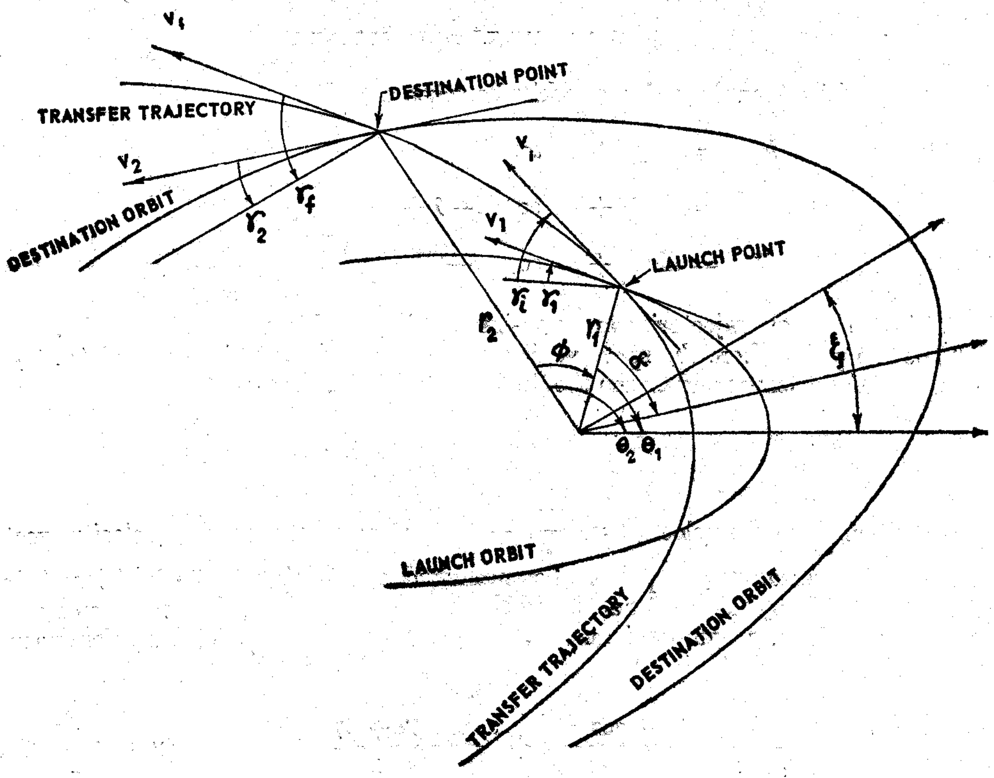

Fig 1—Geometry of transfer

Again for the launch and destination orbits

$$\tan \gamma_1 = \frac{K\, e_1\, r_1 \sin \theta_1}{h_1^2} \tag{11}$$

$$\tan \gamma_2 = \frac{K\, e_2\, r_2 \sin (\theta_1 + \varphi - \xi)}{h_2^2} \tag{12}$$

$$V_1 = \sqrt{K} \left[\frac{2}{r_1} - \frac{K(1 - e_1^2)}{h_1^2} \right]^{\frac{1}{2}} \tag{13}$$

$$V_2 = \sqrt{K} \left[\frac{2}{r_2} - \frac{K(1 - e_2^2)}{h_2^2} \right]^{\frac{1}{2}} \tag{14}$$

The characteristic impulse is given as

$$I = [V_1^2 + V_i^2 - 2V_1 V_i \cos(\gamma_1 - \gamma_i) + V_{E1}^2]^{\frac{1}{2}} + [V_2^2 + V_f^2 - 2V_2 V_f \cos(\gamma_2 - \gamma_f) + V_{E2}^2]^{\frac{1}{2}} \tag{15}$$

where $V_{E\,1}$ and $V_{E\,2}$ are the surface escape velocities of the planets in the launch and destination orbits respectively and are zero when gravity effects of the planets are neglected. Due to relations (9) and (10) it becomes

$$I = [V_1^2 + V_i^2 - 2V_1 V_i \cos(\gamma_1 - \gamma_i) + V_{E\,1}^2]^{\frac{1}{2}} + \Bigg[V_2^2 + V_i^2 + 2K\left(\frac{1}{r_2} - \frac{1}{r_1}\right) - 2V_2 \left\{ V_i\, k \cos\gamma_i \cos\gamma_2 + \sin\gamma_2 \left(V_i^2 + 2K\left(\frac{1}{r_2} - \frac{1}{r_1}\right) - (V_i\, k\cos\gamma_i)^2 \right)^{\frac{1}{2}} \right\} + V_{E\,2}^2 \Bigg]^{\frac{1}{2}} \tag{16}$$

Evidently I is a function of three independent variables θ_1, φ and γ_i

TRANSFER TIME

The time spent by the space vehicle on the transfer trajectory is given as

$$T = \sqrt{\frac{a^3}{K}} \left[(\beta_2 - \beta_1) - E(\sin\beta_2 - \sin\beta_1) \right] \tag{17}$$

where β_1 and β_2 are the eccentric anomalies of the entry and exit points on the transfer trajectory and

$$\cos\beta = \frac{E + \cos(\theta - \theta_1 + \alpha)}{1 + E\cos(\theta - \theta_1 + \alpha)} \tag{18}$$

In terms of E, a, α and φ (17) may be re-written as

$$T = \frac{a^{\frac{3}{2}}}{\sqrt{K}} \left[2\tan^{-1}\left(\frac{\sqrt{1 - E^2}\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2} + E\cos\left(\alpha + \frac{\varphi}{2}\right)} \right) - E(1 - E^2)^{\frac{1}{2}} \left\{ \frac{\sin(\varphi + \alpha) - \sin\alpha + E\sin\varphi}{[1 + E\cos(\varphi + \alpha)](1 + E\cos\alpha)} \right\} \right] \tag{19}$$

which in view of equations (1) to (8) is a function of three variables θ_1, φ and γ_i

MINIMUM TRANSFER TIME

To get the optimum value of (19) under the condition that I is a constant, one writes the Lagrangian equations as

$$\frac{\partial T}{\partial \theta_1} + \lambda \frac{\partial I}{\partial \theta_1} = 0 \tag{20}$$

$$\frac{\partial T}{\partial \varphi} + \lambda \frac{\partial I}{\partial \varphi} = 0 \tag{21}$$

$$\frac{\partial T}{\partial \gamma_i} + \lambda \frac{\partial I}{\partial \gamma_i} = 0 \tag{22}$$

and the constraint equation is

$$I - I_0 = 0 \tag{23}$$

where I_0 denotes specified value.

Eliminating λ from (20) to (22)

$$\frac{\partial T}{\partial \theta_1} \cdot \frac{\partial I}{\partial \varphi} - \frac{\partial T}{\partial \varphi} \cdot \frac{\partial I}{\partial \theta_1} = 0 \tag{24}$$

$$\frac{\partial T}{\partial \varphi} \cdot \frac{\partial I}{\partial \gamma_i} - \frac{\partial T}{\partial \gamma_i} \cdot \frac{\partial I}{\partial \varphi} = 0 \tag{25}$$

From (19) one gets

$$\frac{\partial T}{\partial \theta_1} = \frac{a^{\frac{3}{2}}}{\sqrt{K}} \left[\frac{PX(M-1)\cos^2\gamma_i}{E} + \frac{Qr_1\{(2-M)\tan\gamma_1 + X\}}{(2-M)^2} - \frac{RX(1-\cos\varphi)^2}{M^2 U \cos^2\gamma_i} \right] \tag{26}$$

$$\frac{\partial T}{\partial \varphi} = \frac{a^{\frac{3}{2}}}{\sqrt{K}} \left[\frac{PY(M-1)\cos^2\gamma_i}{E} + \frac{QY\ r_1}{(2-M)^2} - \frac{RY(1-\cos\varphi)^2}{M^2 U\cos^2\gamma_i} + \frac{(1-E^2)^{\frac{3}{2}}}{[1+E\cos(\phi+\alpha)]^2} \right] \tag{27}$$

$$\frac{\partial T}{\partial \gamma_i} = \frac{a^{\frac{3}{2}}}{\sqrt{K}} \left[\frac{PM^2\cos\gamma_i}{(1-\cos\varphi)E} \left(\sin\gamma_i - M\sin\varphi\cos\gamma_i + \sin(\varphi-\gamma_i) \right) + \frac{QZ\ r_1}{(2-M)^2} + \frac{R(1-k)(1-\cos\varphi)}{U\sin\gamma_i\cos\gamma_i} \right] \tag{28}$$

Where

$$k = \frac{r_1}{r_2}$$

$$M = \frac{r_1 V_i^2}{K}$$

$$U = (1-k)^2 \cot\gamma_i - 2(1-k)\sin\varphi + 2\tan\gamma_i(1-\cos\varphi)$$

$$P = \sqrt{1-E^2} \left[\frac{\sin\alpha(2+E\cos\alpha)}{(1+E\cos\alpha)^2} - \frac{\sin(\alpha+\varphi)[2+E\cos(\alpha+\varphi)]}{[1+E\cos(\alpha+\varphi)]^2} \right]$$

$$Q = \frac{3\sqrt{K}\,(2 - M)\,T}{2r_1\, a^{\frac{3}{2}}}$$

$$R = (1 - E^2)^{\frac{3}{2}} \left[\frac{1}{[1 + E \cos (\varphi + \alpha)]^2} - \frac{1}{(1 + E \cos \alpha)^2} \right]$$

$$X = \frac{k\, M^2 \cos^2 \gamma_i \,(\tan \gamma_2 - \tan \gamma_1)}{(1 - \cos \varphi)}$$

$$Y = \frac{M^2 \cos^2 \gamma_i}{(1 - \cos \varphi)^2} \left[(k - 1) \sin \varphi + (1 - \cos \varphi)(\tan \gamma_i + k \tan \gamma_2) \right]$$

$$Z = M \left[2 \tan \gamma_i - \frac{M \sin \varphi}{1 - \cos \varphi} \right]$$

From (16) one gets

$$\frac{\partial I}{\partial \theta_1} = \frac{1}{2} I_1^{-\frac{1}{2}} \Bigg[\left(\frac{KX}{r_1} - V_i^2 \tan \gamma_1 \right) \left(1 - \frac{V_1 \cos (\gamma_i - \gamma_1)}{V_i} \right) - 2 \Bigg\{ \frac{K \sin \gamma_1}{h_1} \left(V_1 - V_i \cos (\gamma_i - \gamma_1) \right) + V_i \sin (\gamma_i - \gamma_1) \left(V_1 - \frac{K}{r_1 V_1} \right) \Bigg\} \Bigg]$$

$$+ \frac{1}{2} I_2^{-\frac{1}{2}} \Bigg[\left(\frac{KX}{r_1} - V_i^2 \tan \gamma_1 \right) \left(1 - \frac{r_1\, h_2 \cos \gamma_i}{r_2^2\, V_i} \right) - 2 \Bigg\{ \frac{K \sin \gamma_2}{h_2} \left(2V_2 - k\, V_i \cos \gamma_i \cos \gamma_2 - G \sin \gamma_2 \right)$$

$$+ \left(1 - \frac{K}{r_2 V_2^2} \right) \left(\frac{G\, h_2}{r_2} - k\, V_2\, V_i \sin \gamma_2 \cos \gamma_i \right) - \frac{K \tan \gamma_1}{r_1}$$

$$+ \frac{k h_2\, V_i \cos \gamma_i}{r_2} \left(\tan \gamma_1 - \tan \gamma_2 \right) + \frac{V_2 \sin \gamma_2}{G} \Bigg[\frac{1}{2} \left(\frac{KX}{r_1} - V_i^2 \tan \gamma_1 \right)$$

$$\left(1 - k^2 \cos^2 \gamma_i \right) + K \left(\frac{\tan \gamma_1}{r_1} - \frac{\tan \gamma_2}{r_2} \right) - \left(k\, V_i \cos \gamma_i \right)^2 \left(\tan \gamma_1 - \tan \gamma_2 \right) \Bigg] \Bigg\} \Bigg] \qquad (29)$$

$$\frac{\partial I}{\partial \varphi} = \frac{1}{2} I_1^{-\frac{1}{2}} \left[\frac{K\,Y}{r_1} \left(1 - \frac{V_1 \cos (\gamma_i - \gamma_1)}{V_i} \right) \right] + \frac{1}{2} I_2^{-\frac{1}{2}} \Bigg[\frac{KY}{r_2^2} \left(\frac{r_2}{k} - \frac{h_2 \cos \gamma_i}{V_i} \right)$$

$$- 2 \Bigg\{ \frac{K \sin \gamma_2}{h_2} \left(2V_2 - k\, V_i \cos \gamma_i \cos \gamma_2 - \frac{k\, h_2\, V_2\, V_i \cos \gamma_i}{K} - G \sin \gamma_2 \right) + \left(1 - \frac{K}{r_2 V_2^2} \right) \left(\frac{G\, h_2}{r_2} - k\, V_2\, V_i \sin \gamma_2 \cos \gamma_i \right)$$

$$+ \frac{V_2 \sin \gamma_2}{G} \left[\frac{KY}{2r_1} (1 - k^2 \cos^2 \gamma_i) - \frac{K \tan \gamma_2}{r_2} + (kV_i \cos \gamma_i)^2 \tan \gamma_2 \right] \Bigg\} \Bigg] \qquad (30)$$

$$\frac{\partial I}{\partial \gamma_i} = \frac{1}{2} I_1^{-\frac{1}{2}} \left[\frac{KZ}{r_1} \left(1 - \frac{V_1 \cos(\gamma_i - \gamma_1)}{V_i} \right) + 2V_i V_1 \sin(\gamma_i - \gamma_1) \right]$$

$$+ \tfrac{1}{2} I_2^{-\frac{1}{2}} \left[\frac{KZ}{r_2^2} \left(\frac{r_2}{k} - \frac{h_2 \cos \gamma_i}{V_i} \right) + \frac{2r_1 h_2 V_i \sin \gamma_i}{r_2^2} \right.$$

$$\left. - \frac{V_2 \sin \gamma_2}{G} \left\{ \frac{ZK}{r_1} (1 - k^2 \cos^2 \gamma_i) + \frac{2K E k \sin \alpha}{r_2} \right\} \right] \tag{31}$$

Where

$$G = \left[V_i^2 (1 - k^2 \cos^2 \gamma_i) + 2K \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \right]^{\frac{1}{2}}$$

Substitution of (26) to (31) in (24) and (25) gives two equations in three unknowns θ_1, φ and γ_i. With the additional constraint (23), one can solve them numerically to get θ_1, φ and γ_i and then obtain all the parameters of the optimum transfer trajectory and the minimum transfer time.

PARTICULAR CASES

Three particular cases of the above problem are,

(*i*) When launch orbit is circular and destination orbit is elliptic. This can be analysed by taking $e_1 = 0$ and r_1 constant equal to radius of the launch orbit. Further simplification can be affected by taking the line joining the force centre and peri-apsis of the destination orbit as initial line for launch and destination orbits and thus $\xi = 0$.

(*ii*) When launch orbit is elliptic and destination orbit is circular. In this case $e_2 = 0$ and r_2 constant equal to radius of the destination orbit. It can also be treated proceeding as in above general case.

(*iii*) When both launch and destination orbits are circular. For this case (1) and (3) are transformed to

$$r = \frac{h_1^2}{K} = r_l \quad \text{(radius of the launch orbit)} \tag{32}$$

$$r = \frac{h_2^2}{K} = r_d \quad \text{(radius of the destination orbit)} \tag{33}$$

Due to isotropic symmetry of the launch and destination orbits about the force centre, the analysis of the transfer trajectory will be independent of the launch point. Thus the launch point can be chosen some fixed point (r_l, θ_0) and destination point becomes (r_d, θ_2). The analysis can be performed proceeding along the lines of that of the elliptic launch and elliptic destination orbit and remembering that :

$$e_1 = e_2 = 0 \tag{34}$$

and r_1 and r_2 have constant values given by (32) and (33) respectively. Since θ_1 has constant value θ_0, (20) and hence (24) vanishes and the equations determining the optimum solution reduce to two only, *i e*, equations corresponding to (25) and (23).

Taking units of length, velocity and time as in reference 2, (19) and (16) in this case can be transformed into

$$T = F_1 = \frac{1}{2\pi}\left[\frac{q}{1-E}\right]^{\frac{3}{2}}\left(\cos^{-1}\left[\frac{q-n(1-E)}{qE}\right]-\cos^{-1}\left[\frac{q-(1-E)}{qE}\right]\right.$$

$$\left.+E\left\{\sin\left[\cos^{-1}\left(\frac{q-(1-E)}{qE}\right)\right]-\sin\left[\cos^{-1}\left(\frac{q-n(1-E)}{qE}\right)\right]\right\}\right) \quad (35)$$

and

$$I = F_2 = \left[\left(3-2\sqrt{q(1+E)}\right)-\frac{(1-E)}{q}+V_{E1}^2\right]^{\frac{1}{2}}$$

$$+\left[\left\{\frac{\left(3-2\sqrt{\frac{q(1+E)}{n}}\right)}{n}-\frac{(1-E)}{q}\right\}+V_{E2}^2\right]^{\frac{1}{2}} \quad (36)$$

where q is peri-apsis distance of the transfer trajectory and n is r_d expressed in terms of r_l.

From (5) and (7) E is a function of two variables φ and γ_i. Similarly from (8) q is also a function of φ, γ_i. Therefore substituting (35) and (36) in (25)

$$\left(\frac{\partial F_1}{\partial E}\cdot\frac{\partial F_2}{\partial q}-\frac{\partial F_1}{\partial q}\cdot\frac{\partial F_2}{\partial E}\right)\left(\frac{\partial E}{\partial \varphi}\cdot\frac{\partial q}{\partial \gamma_i}-\frac{\partial E}{\partial \gamma_i}\cdot\frac{\partial q}{\partial \varphi}\right)=0$$

But

$$\left(\frac{\partial E}{\partial \varphi}\cdot\frac{\partial q}{\partial \gamma_i}-\frac{\partial E}{\partial \gamma_i}\cdot\frac{\partial q}{\partial \varphi}\right)\neq 0$$

as can be verified from (5), (7) and (8)

Hence

$$\frac{\partial F_1}{\partial E}\cdot\frac{\partial F_2}{\partial q}-\frac{\partial F_1}{\partial q}\cdot\frac{\partial F_2}{\partial E}=0 \quad (37)$$

(35), (36) and (37) are the equations obtained by Lee and Florence[2].

ACKNOWLEDGEMENTS

Authors are extremely grateful to Dr R. R. Aggarwal, Assistant Director, Defence Science Laboratory for his keen interest and useful discussions in the preparation of this paper. Our thanks are also due to Dr Kartar Singh, Director, Defence Science Laboratory for permission to publish this paper.

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