OPTIMUM INTER-TERMINAL TRANSFER TRAJECTORIES

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Rocket trajectories in a gravitational field between two terminals with specified velocities at each terminal are investigated with a view to minimize the total velocity increment required in initiating the rocket along the transfer path at the first terminal and in the attainment of the given final velocity at the final terminal. The equations are transformed for transfer between circular orbits and numerical results for Earth—Mars transfer are calculated. Finally particular cases of the above problem are discussed and Stark's results is drived therefrom.

NOMENCLATURE

u_i = given initial velocity

u_1 = departure velocity with which rocket starts along the transfer trajectory just after the impulsive velocity change at the initial terminal.

u_f = given final velocity to be attained.

u_2 = approach velocity of the rocket just before the impulsive velocity change at the final terminal.

γ_1 = departure angle (the angle between u_1 and the local horizontal).

γ_2 = approach angle (the angle between u_2 and the local horizontal).

γ_i = angle between u_i and local horizontal.

γ_f = angle between u_f and local horizontal.

ρ_1 = radius vector corresponding to initial terminal (Force centre as pole).

ρ_2 = radius vector corresponding to final terminal.

ϕ = transfer angle (angle at the force centre between ρ_1 and ρ_2).

K = gravitational constant times mass of the attracting body.

Minimization of fuel consumption is an important problem for transfer trajectories. Stark[1] has discussed the problem of finding rocket trajectories between two terminals in space with a view to minimize the velocity increment added to an arbitary initial velocity at the first terminal, with the restriction that the resulting trajectory passes through the second terminal.

The present paper investigates the optimum trajectory, that is, trajectories of minimum fuel consumption between two terminals in space for a rocket with a given initial terminal velocity so as to attain a specified final terminal velocity. An optimum trajectory here is that which minimizes the total velocity increment at the initial and final terminals. An inverse square gravitational field is taken and the impulsive thrusts are applied at the initial and final terminals only. Particular cases of the above problem are also discussed in the end.

DETERMINATION OF CHARACTERISTIC VELOCITY

The square of the velocity change $\triangle u_i$ at the initial terminal will be

$$(\triangle u_i)^2 = u_i^2 + u_1^2 - 2\, u_i\, u_1 \cos(\gamma_1 - \gamma_i) \tag{1}$$

If the rocket is required to acquire a velocity u_f at the final terminal, the square of velocity change $\triangle u_f$ is given by

$$(\triangle u_f)^2 = u_f^2 + u_2^2 - 2\, u_f\, u_2 \cos(\gamma_2 - \gamma_f) \tag{2}$$

conservation of moment of momentum per unit mass gives

$$u_1\, \rho_1 \cos \gamma_1 = u_2\, \rho_2 \cos \gamma_2 \tag{3}$$

Rocket starts along the transfer path with the departure velocity u_1, moves in the gravitational field and acquires the approach velocity u_2 at the final terminal. u_1 and u_2 can be related by the equation

$$u_2^2 - u_1^2 = 2\, K \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) \tag{4}$$

Substituting from equations (3) and (4), equation (2) becomes

$$(\triangle u_f)^2 = u_f^2 - 2\, u_f \left[u_1\, d \cos \gamma_1 \cos \gamma_f + \sin \gamma_f \left\{ u_1^2 + 2\, K \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) - u_1^2\, d^2 \cos^2 \gamma_1 \right\}^{\frac{1}{2}} \right] + u_1^2 + 2\, K \left(\frac{1}{\rho_2} - \frac{1}{\rho_1} \right) = 0 \tag{5}$$

where
$$d = \frac{\rho_1}{\rho_2}$$

characteristic velocity $\triangle u$ is then given by

$$\triangle u = |\, \triangle u_i \,| + |\, \triangle u_f \,| \tag{6}$$

where $\triangle u_i$ and $\triangle u_f$ are given by equations (1) and (5) respectively.

ANALYSIS OF OPTIMUM TRAJECTORY

The relation between departure velocity and angle for a trajectory passing through the two specified terminals is expressed by[1]

$$d = \frac{K}{\rho_1\, u_1^2} \left(\frac{1 - \cos \phi}{\cos^2 \gamma_1} \right) + \frac{\cos(\gamma_1 + \phi)}{\cos \gamma_1} \qquad |\gamma_1| \leqslant \frac{\pi}{2} \tag{7}$$

which can be written as

$$u_1^2 = \frac{K\, (1 - \cos \phi)\, \sec^2 \gamma_1}{\rho_1\, (d + \sin \phi \tan \gamma_1 - \cos \phi)} \tag{8}$$

For optimum transfer trajectory

$$\frac{\partial (\triangle u)}{\partial \gamma_1} = 0 \tag{9}$$

Substitution from equations (6), (5) and (1) in equation (9) yields

$$\frac{1}{\triangle u_i}\left[P_1 - 2u_i\left\{P_2 \cos(\gamma_1 - \gamma_i) - u_1 \sin(\gamma_1 - \gamma_i)\right\}\right] +$$

$$\frac{1}{\triangle u_f}\left[P_1 - 2u_f\left[d \cos\gamma_f \left\{P_2 \cos\gamma_1 - u_1 \sin\gamma_i\right\} +\right.\right.$$

$$\left.\left.\frac{\sin\gamma_f}{2P_3}\left\{P_1(1 - d^2\cos^2\gamma_1) + 2u_1^2 d^2 \cos\gamma_1 \sin\gamma_1\right\}\right]\right] = 0 \tag{10}$$

where

$$P_1 = 2\left(\frac{K(1 - \cos\phi)\sec^2\gamma_1}{\rho_1(d + \sin\phi\tan\gamma_1 - \cos\phi)}\right)^{\frac{1}{2}} P_2$$

$$= \frac{K(1 - \cos\phi)\sec^2\gamma_1}{\rho_1(d + \sin\phi\tan\gamma_1 - \cos\phi)^2}\left[2\tan\gamma_1(d - \cos\phi) + \sin\phi(\tan^2\gamma_1 - 1)\right] \tag{11}$$

$$P_3 = \left[u_1^2 + 2k\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) - u_1^2 d^2 \cos^2\gamma_1\right]^{\frac{1}{2}}$$

(10 can be transformed into an equation of single unknown γ_1 and can be solved numerically for γ_1 which will give optimum departure angle γ_{10} (subscript '0' denotes optimum values) for specified terminals, initial velocity and final velocity to be attained. Having calculated γ_{10}, values of u_{10}, u_{20} and γ_{20} can be obtained from (8), (4) and (3) respectively.

TRANSFER BETWEEN CIRCULAR ORBITS

For transfer between circular orbits obviously $\gamma_i = \gamma_f = 0$ and (10) can be easily transformed into

$$\frac{[A\tan\gamma_1 + \sin\phi(\tan^2\gamma_1 - 1)]}{A + \sin\phi\tan\gamma_1}\left[\sqrt{\frac{B\sec^2\gamma_1}{(A + \sin\phi\tan\gamma_1)}}(\alpha + \beta) - \right.$$

$$\left.\cos\gamma_1(\alpha u_i + \beta d u_f)\right] + 2\sin\gamma_1(\alpha u_i + \beta d u_f) = 0 \tag{12}$$

where

$$B = \frac{K}{\rho_1}(1 - \cos\phi)$$

$$A = d - \cos\phi$$

$$C = 2K\left(\frac{1}{\rho_2} - \frac{1}{\rho_1}\right) + u_f^2 \tag{13}$$

$$\alpha = \left[\frac{B\sec^2\gamma_1}{(A + \sin\phi\tan\gamma_1)} + u_i^2 - 2u_i\sqrt{\frac{B}{(A + \sin\phi\tan\gamma_1)}}\right]^{-\frac{1}{2}}$$

$$\beta = \left[\frac{B\sec^2\gamma_1}{(A + \sin\phi\tan\gamma_1)} + C - 2d\,u_f\sqrt{\frac{B}{(A + \sin\phi\tan\gamma_1)}}\right]^{-\frac{1}{2}}$$

(12) when solved will give the optimum departure angle for the transfer path that requires least fuel consumption in the impulsive change over from the launch orbit to the transfer path at the initial terminal and from the transfer path to the destination orbit

at the final terminal. From the knowledge of optimum transfer angle other characteristics for the optimum transfer path can be easily calculated with the help of (3), (4) and (8). It is to be observed that in (12), d signifies the ratio between the radii of launch and destination orbits whereas u_i and u_f denote the corresponding orbital velocities.

NUMERICAL ILLUSTRATION

TABLE 1

NUMERICAL RESULTS FOR OPTIMUM EARTH—MARS TRANSFER

Transfer ingle ϕ	Departure angle γ_1	Approach angle γ_2	Departure velocity km/sec u_1	Approach velocity km/sec u_2
5°	81°45′	54°12′	25·08	4·07
15°	63°45′	41°48′	26·87	10·46
45°	30°25′	20°33′	31·10	18·83
90°	12°45′	10°8′	32·58	21·18
135°	5°12′	4°42′	32·76	21·48
180°	0°	0°	32·77	21·4

characteristics for optimum Earth—Mars transfer for various transfer angles are obtained by solving the above equations. A two dimensional solar system with circular planetary orbits is assumed. A comparison with Stark's[1] results shows that as expected $\frac{\triangle u_i}{u_i}$ is always greater than $\frac{\triangle V_{min}}{V_1}$

An observation of Table 2 shows that for any transfer angle, $\triangle u_i$ is greater than $\triangle u_f$ and it is encouraging result in the sense of structural economy for space navigation. Since the larger fraction of the fuel will be consumed up at initial terminal leaving the smaller fraction to be carried for use at the final terminal.

TABLE 2

TERMINAL VELOCITY CHANGES FOR OPTIMUM EARTH-MARS TRANSFER

Transfer angle ϕ	Velocity change at initial terminal $\triangle u_i$ km/sec	Velocity change at final terminal $\triangle u_f$ km/sec	$\frac{\triangle u_i}{u_i}$	$\frac{\triangle u_f}{u_f}$
5°	38·63	21·99	1·30	0·91
15°	30·01	17·73	1·01	0·74
45°	15·92	9·25	0·53	0·38
90°	7·50	4·94	0·25	0·21
135°	4·47	3·22	0·15	0·13
180°	3·02	2·61	0·10	0·11

The variation of $\frac{\triangle \mu_i}{\mu_i}$ and $\frac{\triangle \mu_f}{\mu_f}$ with respect to transfer angle is shown in Fig. 1.

PARTICULAR CASES

Two special cases of the above problem are (*a*) rocket trajectories between two given terminals characterized by minimum velocity increment to a given velocity at initial terminal, (*b*) rocket trajectories between two given terminals characterized by minimum velocity increment to a given velocity at the final terminal.

For trajectories of type (*a*), characteristic velocity is given by $| \triangle u_i |$ and the departure angle can be obtained by solving the equation

$$P_1 - 2u_i \{P_2 \cos (\gamma_1 - \gamma_i) - u_1 \sin (\gamma_1 - \gamma_i)\} = 0 \tag{14}$$

knowing departure angle, departure velocity can be obtained from (8). This is an alternative method to solve the problem discussed in reference[1]. Characteristic[1] velocity for rocket trajectories of type (*b*) is given by $| \triangle u_f |$ and their departure angle can be obtained by solving the equation

$$P_1 - 2u_f \left[d \cos \gamma_f \{ P_2 \cos \gamma_1 - u_1 \sin \gamma_1 \} + \frac{\sin \gamma_f}{2 P_3} \left\{ P_1 (1 - d^2 \cos^2 \gamma_1) + 2 u_1^2 d^2 \cos \gamma_1 \sin \gamma_1 \right\} \right] = 0 \tag{15}$$

Having found out departure angle, (8), (4) and (3) will give the departure velocity, approach velocity and approach angle respectively. This is a second method of attack for the solution of problem of reference[2].

Stark's result can be easily derived from (14) which can be written as

$$2u_1 \frac{\partial u_1}{\partial \gamma_1} - 2u_i \left\{ \frac{u_1}{\partial \gamma_1} \cos (\gamma_1 - \gamma_i) - u_1 \sin (\gamma_1 - \gamma_i) \right\} = 0 \tag{16}$$

Also (8) can be put down as

$$u_1^2 = \frac{B \sec^2 \gamma_1}{A + \sin \phi \tan \gamma_1} \tag{17}$$

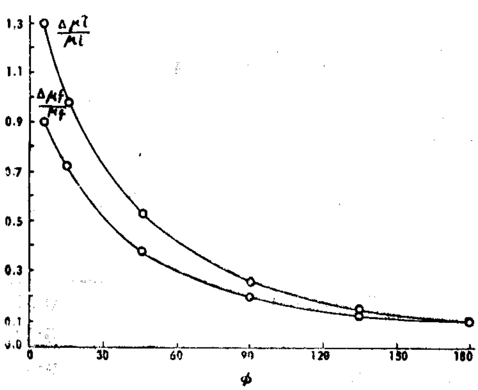

Fig. 1—Variation of $\frac{\Delta \mu_i}{\mu_i}$ and $\frac{\Delta \mu_f}{\mu_f}$ with respect to transfer angle

where

$$\frac{\partial u_1}{\partial \gamma_1} = u_1 \tan \gamma_1 - \frac{u_1^3 \sin\phi}{2B} \tag{18}$$

Substituting (18) in (16) and writing

$$u_1 = X_{opt} \sec \gamma_1$$

$$\tan \gamma_1 = \left[\frac{B}{X_{opt}^2} - A \right] \frac{1}{\sin \phi}$$

$$u_i \cos \gamma_i = X_1$$

$$u_i \sin \gamma_i = Y_1$$

we get Stark's equation[1]

$$(A^2 + \sin^2 \phi)\, X_{opt}^4 + (A\, Y_1 \sin \phi - X_1 \sin^2 \phi)\, X^3_{opt} + (B\, Y_1 \sin \phi) X_{opt} - B^2 = 0$$

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