APPLICATION OF VARIATIONAL TECHNIQUE TO THE OPTIMIZATION PROBLEMS OF MULTIPLE STAGE ROCKETS

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The paper deals with the application of variational principle to the optimization problems of multistage rockets. Conditions have been deduced for extremising a general 'pay off'. Two particular cases have been discussed in details.

Miele[1, 2] and Breakwell[3] have made an extensive survey of the general formulation of trajectory optimization problems and of the efforts made to solve them by employing the discipline of the calculus of variations. In a recent paper Leitmann[4] has solved the general problem of extremising a certain function called the 'pay off' or 'performance index' and has derived the necessary conditions for the existence of such extrema and the nature of the extremal arc for the case of a single stage rocket. In the present paper the variational technique has been employed to a multistage rocket vehicle and conditions obtained for finding the optimum flight path under the assumption of constant thrust in each stage and no coasting period is admissible between the stages so as to extremise a given pay off. In general, the state variable mass and thrust experience step discontinuities or finite jumps at the staging points. Also, since the thrust can be controlled in magnitude and direction, the problem of finding the optimum thrust direction is considered when the vehicle is supposed to be flying in vacuum and constant gravitational field. Two particular cases of vertically ascending staged rocket are then considered and conditions deduced for finding the separation times in order to maximise the final velocity and payload mass respectively.

VARIATIONAL PROBLEM

Considering the motion in a vertical plane in a constant gravitational field and negligible atmospheric resistance, the equations of motion for a multistage rocket are

$$\left.\begin{aligned} \dot{p}_i &= \frac{c_i\,\beta_i}{m_i}\cos\theta_i \\ \dot{q}_i &= \frac{c_i\,\beta_i}{m_i}\sin\theta_i - g_i \\ \dot{x}_i &= p_i \\ \dot{y}_i &= q_i \\ \dot{m}_i &= \beta_i \end{aligned}\right\} \quad i = 1, 2, \ldots, N \qquad (1)$$

These equations contain seven physical variables $p_i, q_i, x_i, y_i, m_i, \beta_i$ and θ_i for any particular stage and there are five equations combining them. Thus there are two degrees of freedom i.e. there are two variables (say β_i and θ_i for ith stage) which can be controlled and are subject to independent admissible variations. Thus the problem is to minimise G (Z_{i0} Z_{if}, t_0, t_f) where $Z_i = p_i, q_i, x_i, y_i, m_i$ are the state variables, subject to given terminal conditions.

Equations (1) can be re-written as

$$\left.\begin{aligned} {}_1\Psi_i &\equiv \dot{p}_i - \frac{c_i\,\beta_i}{m_i}\cos\theta_i = 0 \\ {}_2\Psi_i &\equiv \dot{q}_i - \frac{c_i\,\beta_i}{m_i}\sin\theta_i + g = 0 \\ {}_3\Psi_i &\equiv \dot{x}_i - p_i = 0 \\ {}_4\Psi_i &\equiv \dot{y}_i - q_i = 0 \\ {}_5\Psi_i &\equiv \dot{m}_i + \beta_i = 0 \end{aligned}\right\} i = 1, 2, \ldots, N \quad (2)$$

Introducing Lagrangian multipliers $_j\lambda_i$ ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, 5$) and forming the fundamental function

$$F_i = \sum_{j=1}^{5} {}_j\lambda_i\ {}_j\Psi_i \quad (3)$$

the problem reduces to minimising G subject to constraint (2) and given terminal conditions or minimising

$$J = G + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} F_i\, dt \quad (4)$$

The undetermined variable Lagrangian multipliers $_j\lambda_i$ are continuous functions of time since the constraint equations must be satisfied at all points of the trajectory. They may experience discontinuity at the points of staging.

Thus our problem reduces to that of the Bolza type and according to Bliss[5] the extremal path must not only satisfy equations (2) but also the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial F_i}{\partial_j \dot{Z}_i}\right) - \frac{\partial F_i}{\partial_j Z_i} = 0 \quad \begin{aligned} i &= 1, 2, \ldots, N \\ j &= 1, 2, \ldots, 6 \end{aligned} \quad (5)$$

As a consequence of the above equations, the following first integral must also hold good

$$-F_i + \sum_{j=1}^{5} \frac{\partial F_i}{\partial_j \dot{Z}_i}\ {}_j\dot{Z}_i = k_i \quad (6)$$

where k_i is an integration constant. Since there are discontinuities in the variation of mass at the staging points, the conditions which are to be satisfied at such points are given by the 'corner conditions' i.e.

$$\left(\frac{\partial F_i}{\partial_j \dot{Z}_i}\right)_- = \left(\frac{\partial F_i}{\partial_j \dot{Z}_i}\right)_+ \quad 7)$$

where negative and positive signs imply respectively the conditions just prior to and immediately after the staging. Moreover, the transversality condition—which gives changes in boundary conditions as also change in J must be satisfied, and is given by

$$\sum_{i=1}^{N} \left[dG + \left(F - \sum_{j=1}^{5} \frac{\partial F_i}{\partial_j \dot{Z}_i} \, {}_j\dot{Z}_i \right) dt + \sum_{j=1}^{5} \frac{\partial F_i}{\partial_j \dot{Z}_i} \, d_jZ_i \right]_{t_{i-1}}^{t_i} = 0 \quad (8)$$

For the problem under consideration we have, according to expression (3), augmented function as

$$F_i \equiv {}_1\lambda_i \left(\dot{p}_i - \frac{c_i \, \beta_i}{m_i} \cos \theta_i \right) + {}_2\lambda_i \left(\dot{q}_i - \frac{c_i \, \beta_i}{m_i} \sin \theta_i + g \right)$$

$$+ {}_3\lambda_i \, (\dot{x}_i - p_i) + {}_4\lambda_i \, (\dot{y}_i - q_i) + {}_5\lambda_i \, (\dot{m}_i + \beta_i) \quad (9)$$

Therefore from (5), the extremum path must satisfy the following equations together with equations (2):

$$\left.\begin{aligned}
&\frac{d}{dt}\left({}_1\lambda_i\right) + {}_3\lambda_i = 0 && (a)\\
&\frac{d}{dt}\left({}_2\lambda_i\right) + {}_4\lambda_i = 0 && (b)\\
&\frac{d}{dt}\left({}_3\lambda_i\right) = 0 && (c)\\
&\frac{d}{dt}\left({}_4\lambda_i\right) = 0 && (d)\\
&\frac{d}{dt}\left({}_5\lambda_i\right) - \frac{c_i \, \beta_i}{m_i}\left({}_1\lambda_i \cos \theta_i + {}_2\lambda_i \sin \theta_i\right) = 0 && (e)\\
&\frac{c_i \, \beta_i}{m_i}\left({}_1\lambda_i \sin \theta_i - {}_2\lambda_i \cos \theta_i\right) = 0 && (f)\\
&i = 1, 2, \ldots, N
\end{aligned}\right\} \quad (10)$$

SOLUTION OF THE EQUATIONS

Thus the two sets of equations (2) and (10) must be integrated for given initial and boundary conditions to give the optimum trajectory. Some of the equations (10) can readily be integrated i.e. from (10c) and (10d), we have

$$\left.\begin{aligned} {}_3\lambda_i &= a_i \\ {}_4\lambda_i &= b_i \end{aligned}\right\} \quad (11)$$

Also (10a) and (10b) give

$$\begin{aligned}
{}_1\lambda_i &= {}_1\lambda_{i-1} - a_i \, (t_i - t_{i-1}) \\
&= {}_1\lambda_{i+1} + a_i \, (t_{i+1} - t_i) \\
{}_2\lambda_i &= {}_2\lambda_{i-1} - b_i \, (t_i - t_{i-1}) \\
&= {}_2\lambda_{i+1} + b_i \, (t_{i+1} - t_i)
\end{aligned} \quad (12)$$

Again since during flight $c_i \beta_i / m_i \neq 0$, equation (10f) shows that for optimum path

$$\tan \theta_i = \frac{{}_2\lambda_i}{{}_1\lambda_i} \qquad i = 1, 2, \ldots, N \tag{13}$$

Also as a consequence of condition (7), we have

$$\left({}_j\lambda_i \right)_- = \left({}_j\lambda_i \right)_+ \qquad \begin{array}{l} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, 5 \end{array}$$

which implies that at the points of separation where thrust becomes discontinuous, the Lagrange's multipliers are continuous. Thus they are continuous throughout the powered phase. Actually from (11) we see that the constants a_i and b_i are absolute constants having the same value throughout the flight period. From (13) it is clear that θ_i also does not experience any discontinuity at the points of separation.

Now relation (6) gives the first integral in this case as

$$\frac{c_i \beta_i}{m_i} \left({}_1\lambda_i \cos \theta_i + {}_2\lambda_i \sin \theta_i \right) - {}_2\lambda_i\, g + {}_3\lambda_i\, p_i + {}_4\lambda_i\, q_i - {}_5\lambda_i\, \beta_i = k_i \tag{14}$$

$$i = 1, 2, \ldots, N$$

where k_i is an integration constant. Nothing can be said about the continuity of the constant k_i across the staging points; it may have different values in different stages.

Also for this problem the transversality condition (8) gives

$$\sum_{i=1}^{N} \left[dG + {}_1\lambda_i\, dp_i + {}_2\lambda_i\, dq_i + {}_3\lambda_i\, dx_i + {}_4\lambda_i\, dy_i + {}_5\lambda_i\, dm_i - k_i\, dt \right]_{t_{i-1}}^{t_i} = 0 \tag{15}$$

Now if G is a function depending wholly on the conditions at the end of powered phase' and all the state variables at the initial point are known, the above can be written as

$$\left[dG + {}_1\lambda_N\, dp_N + {}_2\lambda_N\, dq_N + {}_3\lambda_N\, dx_N + {}_4\lambda_N\, dy_N + {}_5\lambda_N\, dm_N - k_N\, dt \right]_{t_{N-}}$$

$$+ \sum_{i=1}^{N-1} \Big[\Big\{ \left({}_1\lambda_i\, dp_i \right)_- - \left({}_1\lambda_{i+1}\, dp_{i+1} \right)_+ \Big\} + \Big\{ \left({}_2\lambda_i\, dq_i \right)_- - \left({}_2\lambda_{i+1}\, dq_{i+1} \right)_+ \Big\}$$

$$+ \Big\{ \left({}_3\lambda_i\, dx_i \right)_- - \left({}_3\lambda_{i+1}\, dx_{i+1} \right)_+ \Big\} + \Big\{ \left({}_4\lambda_i\, dy_i \right)_- - \left({}_4\lambda_{i+1}\, dy_{i+1} \right)_+ \Big\}$$

$$+ \Big\{ \left({}_5\lambda_i\, dm_i \right)_- - \left({}_5\lambda_{i+1}\, dm_{i+1} \right)_+ \Big\} - \Big\{ \left(k_i\, dt \right) - \left(k_{i+1}\, dt \right)_+ \Big\} \Big]_{t_1}$$

$$= 0 \tag{16}$$

where $(\quad)_-$ and $(\quad)_+$ represent respectively the conditions just before and immediately after staging. Since the state variables and the Lagrange's constants are continuous throughout the flight period, their variations just before and immediatel after the staging are equal and we can, therefore, write (16) as

$$\left[dG + {}_1\underset{N}{\lambda}\, dp_N + {}_2\underset{N}{\lambda}\, dq_N + {}_3\underset{N}{\lambda}\, dx_N + {}_4\underset{N}{\lambda}\, dy_N + {}_5\underset{N}{\lambda}\, dm_N - k_N\, dt \right]_{t_{N-}}$$

$$+ \sum_{i=1}^{N-1} \left[{}_5\underset{i}{\lambda} \left\{ \left(dm_i \right)_- - \left(dm_{i+1} \right)_+ \right\} - \left\{ \left(k_i \right)_- - \left(k_{i+1} \right)_+ \right\} dt \right]_{ti} = 0$$

Hence for optimality, the conditions to be satisfied at the points of staging are

(*i*) At the end point

$$\left[dG + {}_1\underset{N}{\lambda}\, dp_N + {}_2\underset{N}{\lambda}\, dq_N + {}_3\underset{N}{\lambda}\, dx_N + {}_4\underset{N}{\lambda}\, dy_N + {}_5\underset{N}{\lambda}\, dm_N - k_N\, dt \right]_{t_{N-}} = 0 \quad (17)$$

and

(*ii*) at the intermediate points

$$\left[{}_5\underset{i}{\lambda} \left\{ \left(dm_i \right)_- - \left(dm_{i+1} \right)_+ \right\} - \left\{ \left(k_i \right)_- - \left(k_{i+1} \right)_+ \right\} dt \right]_{t_i} = 0 \quad (18)$$

$$i = 1, 2, \ldots, (N-1)$$

If the staging times are not specified, the above can be re-written as

$$\left[{}_5\underset{i}{\lambda} \left\{ \left(\frac{dm_i}{dt} \right)_- - \left(\frac{dm_{i+1}}{dt} \right)_+ \right\} - \left\{ \left(k_i \right)_- - \left(k_{i+1} \right)_+ \right\} \right]_{t_i} = 0 \quad (19)$$

As a consequence of (14) this becomes

$$\left[\left(\frac{c_i\ \beta_i}{m_i} \right)_- - \left(\frac{c_{i+1}\ \beta_{i+1}}{m_{i+1}} \right)_+ \left({}_1\underset{i}{\lambda}\ \cos\ \theta_i + {}_2\underset{i}{\lambda}\ \sin\ \theta_i \right) \right.$$

$$\left. - {}_5\underset{i}{\lambda} \left\{ \beta_i - \beta_{i+1} + \left(\frac{dm_i}{dt} \right)_- - \left(\frac{dm_{i+1}}{dt} \right)_+ \right\} \right]_{t_i} = 0$$

$$i = 1, 2, \ldots, (N-1)$$

SPECIAL CASES

Case I—We now consider a particular problem of a vertically ascending two stage rocket where the initial mass, the payload mass and the total time are specified and it is required to find the staging time t_1 so that the final velocity at the end of powered phase is maximum i.e. $G = -q_2$. Since y_2 is not specified, we have in this case

$${}_3\underset{2}{\lambda} = 0, \quad {}_4\underset{2}{\lambda} = 0$$

Therefore

$${}_1\underset{1}{\lambda} = {}_1\underset{2}{\lambda} = 0$$

$${}_2\underset{1}{\lambda} = {}_2\underset{2}{\lambda} = 1$$

$$\tan\ \theta = \infty \text{ or } \theta = \frac{\pi}{2}$$

Now if σ_i ($i=1, 2$) represents the ratio of the structure mass to the mass of the propellant for any stage, then

$$\left.\begin{aligned} m_{1-} &= m_0 - \beta_1 (t_1 - t_0) \ldots\ldots (a) \\ m_{1+} &= m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) \ldots (b) \\ m_{2-} &= m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1) \ldots (c) \\ m_{2+} &= m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)(1+\sigma_2) \ldots (d) \end{aligned}\right\} \quad (20)$$

According to (17), (20c) and (20d), we have

$$(k_2)_{t_{2-}} = \beta_2 (1+\sigma_2) \left({}_5\lambda_2 \right)_{t_{2-}}$$

Therefore, as a consequence of (14), we obtain

$$\left({}_5\lambda_2 \right)_{t_{2-}} = \frac{1}{\beta_2 (2+\sigma_2)} \left\{ \frac{c_2 \beta_2}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)} - g \right\} \quad (21)$$

Also from (10e) and (21) we deduce that

$$\left({}_5\lambda_2 \right)_{t_{1-}} = \left({}_5\lambda_1 \right)_{t_{1-}} = c_2 \left[\frac{1}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1)} - \frac{1}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)} \right]$$
$$+ \frac{1}{\beta_2 (2+\sigma_2)} \left\{ \frac{c_2 \beta_2}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)} - g \right\} \quad (22)$$

But from (19) we get

$$\left({}_5\lambda_1 \right)_{t_{1-}} = \frac{1}{\beta_1 (1+\sigma_1) - \beta_2} \left[\frac{c_1 \beta_1}{m_0 - \beta_1 (t_1 - t_0)} - \frac{c_2 \beta_2}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1)} \right] \quad (23)$$

Therefore (22) and (23) give

$$c_2 \left[\frac{1}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1)} - \frac{1}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)} \right]$$
$$+ \frac{1}{\beta_2 (2+\sigma_2)} \left\{ \frac{c_2 \beta_2}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1) - \beta_2 (t_2 - t_1)} - g \right\}$$
$$= \frac{1}{\beta_1 (1+\sigma_2) - \beta_2} \left\{ \frac{c_1 \beta_1}{m_0 - \beta_1 (t_1 - t_0)} - \frac{c_2 \beta_2}{m_0 - \beta_1 (t_1 - t_0)(1+\sigma_1)} \right\}$$

If $t_0 = 0$, then

$$\frac{1}{\beta_2 (2+\sigma_2)} \left\{ \frac{c_2 \beta_2}{m_0 - \beta t_1 (1+\sigma_1) - \beta_2 (t_2 - t_1)} - g \right\} - \frac{c_2 \beta_2 (t_2 - t_1)}{\{m_0 - \beta_1 t_1 (1+\sigma_1)\} \{m_0 - \beta_1 t_1 (1+\sigma_1) - \beta_2 (t_2 - t_1)\}}$$
$$= \frac{1}{\beta_1 (1+\sigma_1) - \beta_2} \left\{ \frac{c_1 \beta_1}{m_0 - \beta_1 t_1} - \frac{c_2 \beta_2}{m_0 - \beta_1 t_1 (1+\sigma_1)} \right\} \quad (24)$$

Since t_2 is already given this expression gives t_1 for the optimum condition. In the particular case when $c_1 = c_2 = c$, $\beta_1 = \beta_2 = \beta$ and $\sigma_1 = \sigma_2 = \sigma$ this reduces to give

$$\frac{1}{\beta (\sigma + 2)} \left\{ \frac{c \beta}{m_0 - \beta t_1 (1+\sigma) - \beta (t_2 - t_1)} - g \right\}$$
$$= \frac{c \beta \sigma (t_2 - t_1)}{\{m_0 - \beta t_1 (1+\sigma)\} \{m_0 - \beta t_1 (1+\sigma) - \beta (t_2 - t_1)\}}$$
$$- c \beta \sigma t_1 / (m_0 - \beta t_1) \{m_0 - \beta t_1 (1+\sigma_1)\} \quad (25)$$

The above analysis can easily be extended to cover the case when the number of stages is more than two.

Case II—Another problem of the similar nature is : 'given the initial conditions to find the maximum payload which may obtain a specified velocity at the end of powered phase whose duration is not fixed. The height to be attained is also given to be free'.

In this case $G = - m_2$

Therefore from (17), (20c) and (20d), we have

$$\left[\left({}_5\lambda_2 - 1 \right) dm_2 + {}_2\lambda_2 \; dq_2 + {}_4\lambda_2 \; dy_2 + \{k_2 - \beta_2 \; (1 + \sigma_2)\} \; dt \right]_{t_2-} = 0$$

Since m_2 and t_2 are not known and y_2 is free, we must have

$$\left.\begin{aligned} \left({}_4\lambda_2 \right) t_{2-} &= 0 & (a) \\ \left({}_5\lambda_1 \right) t_{2-} &= 1 & (b) \\ \left(k_2 \right) t_{2-} &= \beta_2 \; (1+\sigma_2) & (c) \end{aligned}\right\} \qquad (26)$$

With the help of relations (14), (26b), (26c), we deduce that

$$m_{2-} = \frac{c_2 \, \beta_2}{g+\beta_2 \; (2+\sigma_2)} = m_0 - \beta_1 \; (t_1 - t_0) \; (1+\sigma_1) - \beta_2 \; (t_2 - t_1) \qquad (27)$$

Again on integrating equation (10c) and applying the condition that $\left({}_5\lambda_2 \right)_{t_2-} = 1$, we obtain

$$\left({}_5\lambda_2 \right)_{t_1+} = \left({}_5\lambda_1 \right)_{t_1-} = 1 + c_2 \left[\frac{1}{m_0 - \beta_1 \, (t_1 - t_0) \, (1+\sigma_1)} - \frac{1}{m_0 - \beta_1 \, (t_1 - t_0) \, (1+\sigma_1) - \beta_2 \, (t_2 - t_1)} \right]$$

Hence from (19), we have

$$\left[\frac{c_1 \, \beta_1}{m_0 - \beta_1 \; t_1 - t_0)} - \frac{c_2 \, \beta_2}{m_0 - \beta_1 \, (t_1 - t_0) \, (1+\sigma_1)} \right] = \left[\beta_1 \, (1+\sigma_1) - \beta_2 \right] \left[1 + c_2 \left\{ \frac{1}{m_0 - \beta_1 \, (t_1 - t_0) \, (1+\sigma_1)} - \frac{1}{m_0 - \beta_1 \, (t_1 - t_0) \, (1+\sigma_1) - \beta_2 \, (t_2 - t_1)} \right\} \right] \qquad (28)$$

From (27) and (28), we can solve for t_1 and t_2 and thereby from (20) we can know the masses at the separation points as well as the required maximum payload.

The above can again be easily extended to the case when the number of stages is more than two.

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