A PROBLEM IN REPLACEMENT THEORY

A. M. KSHIRSAGAR AND S. RANGANATHAN

Defence Science Laboratory, Delhi-6

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The classical procedure of ordering new vehicles in replacement of the existing ones is unsatisfactory as it is based on the average life of a vehicle and is good only after a lapse of several years when 'stable' conditions are reached. In this paper, an alternative procedure is prescribed. It makes use of an autoregrassive equation obeyed by the number of new vehicles. This equation is obtained by using the method of Markov chains.

Suppose that there are N new vehicles, having identical 'failure-time' distributions, in a depot, and that it is the policy of the depot to replace any vehicle which fails during any year, at the end of the year by a new vehicle. Then the problem is to predict the number of new vehicles required in a year and also to predict the age distribution of the vehicles in any future year.

Let us suppose that the maximum possible age of a vehicle is $(m-1)$ years (completed). [We shall always refer to the 'completed' number of years, as the age of a vehicle.] In other words, a vehicle which has completed $(m-1)$ years is bound to fail in the mth year. Let us suppose the 'failure-time' distribution of the vehicle such that

The probability of failure in the interval

$$(i, i+1) = p_{i+1} \tag{1}$$

$$(i = 0, 1, 2, \ldots\ldots\ldots\ldots\ldots\ldots \; m-1)$$

where i is the age of the vehicle. In other words, p_{i+1} is the area under the frequency curve of the distribution, included between the ordinates at i and $(i+1)$. Consequently, $\sum_{i=0}^{m-1} p_{i+1} = 1$. If we correspond m states $(E_0, E_1 \ldots\ldots\ldots, E_{m-1})$ to the number of completed years by a vehicle, it is obvious that if a vehicle of age i does not fail during the next year, it makes a transition from E_i to E_{i+1}. However, if it fails the transition is from E_i to E_0 as at the end of the year it is replaced by a new vehicle. It is easy to see that the probability P_{ij} of the transition E_i to E_j is

$$p_{ij} = \begin{cases} \left(1 - \sum_{k=1}^{i+1} p_k\right) \Big/ \left(1 - \sum_{k=1}^{i} p_k\right) = 1 - \alpha_i, & \text{if } j = i+1 \\ \alpha_i, & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

This follows from the fact that

$$1 - \alpha_i = p_{i\ i+1} = \frac{\text{Probability of completing } (i+1) \text{ years}}{\text{Probability of completing } i \text{ years}} \tag{3}$$

Thus we have a Markov chain of m states and transition matrix $P = [p_{ij}]$ given by (2), where i refers to the previous state and j to the next state. It should be noted that, in particular $p_{00} = \alpha_0 = p_1$ and $p_{m-1,\,0} = \alpha_{m-1} = 1$. The vector of initial probabilities for this Markov chain is

$$a' = [1, 0, 0 \ldots\ldots\ldots\ldots\ldots, 0] \tag{4}$$

as we start with new vehicles.

Number of new vehicles in any future year

From the well-known results in Markovian theory, the probability of the process being in state E_0 at the end of the k years ($k=1, 2, \ldots\ldots\ldots\ldots$) i.e. after k transitions, is given by the first element of the row vector $a'\,P^k$ Expressing P in terms of its latent roots $\lambda_1=1,\ \lambda_2, \ldots\ldots\ldots\lambda_m$ and latent vectors (row and column), the expected number of new vehicles at the end of k years, will be given[1] by

$$N_0\,(k) = C_1\,\lambda_1^{\,k} + C_2\,\lambda_2^{\,k} + \ldots\ldots\ldots\ldots + C_m\,\lambda_m^{\,k}\,, \tag{5}$$

where $C_1, C_2, \ldots\ldots\, C_m$ are certain constants to be determined from the initial conditions. We thus need the latent roots of P. Now

$$\det.\,(P-\lambda I) = \begin{vmatrix} \alpha_0-\lambda & 1-\alpha_0 & 0 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ \alpha_1 & -\lambda & 1-\alpha_1 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ \alpha_2 & 0 & -\lambda & \cdot\cdot & \cdot\cdot & 0 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \alpha_{m-2} & 0 & 0 & \cdot\cdot & \cdot\cdot & -\lambda & 1-\alpha_{m-2} \\ \alpha_{m-1} & 0 & 0 & \cdot\cdot & \cdot\cdot & 0 & -\lambda \end{vmatrix} \tag{6}$$

$$= (\alpha_0 - \lambda)\,(-\lambda)^{m-1} - (1-\alpha_0)\,\Delta_{m-r} \tag{7}$$

where,

$$\Delta_{m\ \ r} = \begin{vmatrix} \alpha_{m-r} & 1-\alpha_{m-r} & 0 & \cdot\cdot & \cdot\cdot & 0 & 0 \\ \alpha_{m-r+1} & -\lambda & 1-\alpha_{m-r+1} & \cdot\cdot & \cdot\cdot & 0 & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \alpha_{m-2} & 0 & 0 & \cdot\cdot & \cdot\cdot & -\lambda & 1-\alpha_{m-2} \\ \alpha_{m-1} & 0 & 0 & \cdot\cdot & \cdot\cdot & 0 & -\lambda \end{vmatrix} \tag{8}$$

$$[r = 1, 2, \ldots\ldots, (m-1)]$$

But

$$\Delta_{m-r} = \alpha_{m-r}\,(-\lambda)^{r-1} - (1-\alpha_{m-r})\,\Delta_{m-(r-1)} \tag{9}$$

Using (9) repeatedly in (7)

$$\det.\ (P-\lambda I) = (-1)^m \left\{ \lambda^m - \alpha_0 \lambda^{m-1} - \sum_{r=1}^{m-1} (1-\alpha_0)(1-\alpha_1)\ldots(1-\alpha_{m-r+1})\,\alpha_{m-r}\,\lambda^{r-1} \right\}$$

$$= (-1)^m \left(\lambda^m - p_1 \lambda^{m-1} - p_2 \lambda^{m-2} - \ldots\ldots - p_{m-1} \lambda - p_m \right) \qquad (10)$$

as

$$(1-\alpha_0)(1-\alpha_1)\ldots\ldots(1-\alpha_{m-r+1})\,\alpha_{m-r} = p_{m-r+1} \qquad (11)$$

Taking the factor $(\lambda-1)$ out from (10), it can also be written as

$$\det.\ (P-\lambda I) = (-1)^m (\lambda-1) \sum_{r=0}^{m-1} (1-p_1-p_2-\ldots\ldots-p_r)\,\lambda^{m-(r+1)} \qquad (12)$$

Thus the number of new vehicles at the end of k years will be given by (5) where $\lambda_1=1$ and $\lambda_2, \lambda_3, \ldots\ldots\ldots\ldots\ldots \lambda_k$ are the roots of the equation

$$\sum_{r=0}^{m-1} (1-p_1-p_2-\ldots\ldots-p_r)\,\lambda^{m-(r+1)} = 0 \qquad (13)$$

Alternatively let $N_x(k)$ be the number of vehicles of age x at the end of k years ($x=0, 1, 2, \ldots\ldots, m-1$). Then it is easy to see that

$$N_x(k) = N_{x-1}(k-1)\,p_{x-1,x}$$
$$= N_{x-1}(k-1)\,(1-\alpha_{x-1});\ x \neq 0 \qquad (14)$$

But

$$N_0(k) = \sum_{x=1}^{m} \alpha_{x-1} N_{x-1}(k-1) \qquad (15)$$

From (14), we get for $x \neq 0,\ k \geqslant x$

$$N_x(k) = (1-\alpha_0)(1-\alpha_1)\ldots\ldots\ldots(1-\alpha_{x-1})\,N_0(k-x) \qquad (16)$$

and substituting this in (15)

$$N_0(k) = \sum_{x=1}^{m} (1-\alpha_0)(1-\alpha_1)\ldots\ldots\ldots(1-\alpha_{x-2})\,\alpha_{x-1} N_0(k-x)$$

$$= \sum_{x=1}^{m} p_x N_0(k-x) \qquad (17)$$

on account of (11). Thus $N_0(k)$ satisfies the autoregressive equation

$$N_0(k) = p_1 N_0(k-1) + p_2 N_0(k-2) + \ldots\ldots\ldots + p_m N_0(k-m) \qquad (18)$$

This can also be written as

$$\phi(E)\{N_0(k)\} = 0 \qquad (19)$$

where

$$\phi(\lambda) = \lambda^m - p_1\lambda^{m-1} - p_2\lambda^{m-2} - \ldots\ldots - p_m,$$

Here E is the shift-operator of the calculus of finite differences. $\phi(\lambda)$ is the same as det. $(P-\lambda I)$, apart from $(-1)^m$ and thus the solution of this autoregressive equation is the same as that of (5). Once $N_0(k)$ is known, $N_x(k), x \neq 0$ can be easily obtained from (16).

From (5) and from the fact that $/\lambda_i/ < 1$ ($i=2, \ldots\ldots, m$) it follows that $N_0(k)$ will be the sum of terms which either decay exponentially to zero or oscillate with a similar decay to zero (for complex roots).

STATIONARY STATE PROBABILITIES

The vector $\pi' = (\pi_0, \pi_1, \ldots\ldots, \pi_{m-1})$ of the stationary probabilities of the Markov chain, satisfied the equation

$$\pi' P = \pi' \tag{20}$$

i.e.

$$\text{(a)} \quad \sum_{r=0}^{m-1} \pi_r \alpha_r = \pi_0$$

$$\text{(b)} \quad \pi_r (1-\alpha_r) = \pi_{r+1}, \qquad r=0, 1, \ldots\ldots, m-2 \tag{21}$$

From (15) and from the fact that $\sum_r \pi_r = 1$ it easily follows that

$$\pi_0 = 1/\mu \text{ and } \pi_r = p_{r+1}/\mu\alpha_r \qquad (r=1, 2 \ldots, m-1) \tag{22}$$

where

$$\mu = 1 + \sum_{r=1}^{m-1} \frac{p_{r+1}}{\alpha_r} \tag{23}$$

But $P_{r+1}/\alpha_r = 1 - p_1 - p_2 - \ldots\ldots - p_r$ and hence it can be seen that

$$\mu = 1 + \sum_{r=1}^{m-1} (1 - p_1 - p_2 - \ldots\ldots - p_r)$$

$$= \sum_{r=1}^{m} r\, p_r$$

$=$ The mean life of a vehicle.

Thus the stationary probabilities are

$$\pi_0 = 1/\mu \text{ and } \pi_r = \frac{1-p_1 - \ldots - p_r}{\mu}, \tag{24}$$

$$r = 1, 2, \ldots\ldots, m-1$$

From this it follows that after a large number of transitions, i.e. after a large number of years, the number of new vehicles required will stabilize to N/μ, where μ is the average life of a vehicle.

This result, though mathematically important, is of little practical utility because, in practice, the important thing is to know the requirements in the first few years rather than after a large number of years, and plan and order accordingly in advance. The behaviour of $N_0(k)$ in the first few years is governed by the recursive relation

$$N_0(k) = p_1 N_0(k-1) + p_2 N_0(k-2) + \ldots\ldots + p_m N_0(k-m) \tag{25}$$

The values of $N_0(k)$ for the required number of years can be plotted easily from this, and the requirements determined accordingly. It can be seen that the value so obtained for smaller values of k will be much different from the 'Stable value' N/μ. The classical procedure in the Army and elsewhere is to order for only N/μ vehicles every year. This result shows that this procedure is not at all satisfactory as it will lead to undesirable shortages in some years as well as surpluses in some years. (25) will, therefore, be very useful in determining the order policy in respect of new vehicles. One has to study the amplitudes of the series $N_0(k)$ for the first few values of k for this purpose.

NUMERICAL EXAMPLE

To illustrate the behaviour of $N_0(k)$ we now consider the case where the life-distribution is normal with mean $\mu = 3$ and standard deviation $\sigma = 1$. From the table of normal probability integral, we get.

$$p_1 = 0\cdot023,\ p_2 = 0\cdot136,\ p_3 = \cdot341,\ p_4 = \cdot341,\ p_5 = 0\cdot136, \text{ and } p_6 = 0\cdot023.$$

$\lambda_1, \lambda_2, \ldots\ldots \lambda_6$ are, therefore, the roots of the equation

$$\lambda^6 - 0\cdot023\lambda^5 - 0\cdot136\lambda^4 - 0\cdot341\lambda^3 - 0\cdot341\lambda^2 - 0\cdot136\lambda - 0\cdot023 = 0$$

Taking out the root $\lambda_1 = 1$ the 'residual' equation is

$$\lambda^5 + 0\cdot977\lambda^4 + 0\cdot841\lambda^3 + 0\cdot5\lambda^2 + 0\cdot159\lambda + 0\cdot023 = 0$$

By Sturm's theorem we find that it has one real root and two pairs of complex roots. They are

$$\lambda_2 = -0\cdot408$$

$$\lambda_3, \lambda_4 = 0\cdot124 \pm 0\cdot686\, i \text{ or } 0\cdot697\,(\cos 79\cdot75^\circ \pm i \sin 79\cdot75^\circ)$$

and

$$\lambda_5, \lambda_6 = -0\cdot124 \pm 0\cdot227\, i \text{ or}$$
$$0\cdot259\,(\cos 61\cdot35^\circ \pm i \sin 61\cdot35^\circ)$$

Therefore

$$N_o(k) = C_1 + (0\cdot408)^k\ C_2 + (0\cdot697)^k\ \{C_3 \cos k\theta_1 + C_4 \sin k\theta_1\} + (0\cdot259)^k \{C_5 \cos k\theta_2 + C_6 \sin k\theta_2\} \qquad (26)$$

where $\theta_1 = 79\cdot75^\circ$ and $\theta_2 = 61\cdot35^\circ$.

The constants $C_1, \ldots\ldots\ldots\ldots C_6$ can be easily found from the initial conditions. Thus if we start with 1000 new vehicles, we obtain from (19)

$$N_0(0) = 1000 = N$$
$$N_0(1) = Np_1 = 23$$
$$N_0(2) = Np_1^2 + Np_2 = 136\cdot529$$
$$N_0(3) = Np_1^3 + 2\,Np_1 p_2 + Np_3 = 347\cdot268$$
$$N_0(4) = Np_1^4 + 3\,Np_1^2 p_2 + 2\,Np_1 p_3 + Np_2^2 + Np_4 = 375\cdot398 \qquad (27)$$
$$N_o(5) = Np_1^5 + 4\,N p_1^3 p_2 + 3\,Np_1^2 p_3 + 3\,Np_1 p_2^2 + 2\,Np_1 p_4 + 2\,Np_2 p_3 + N p_5 = 246\cdot228$$

The 'stable value' is N/μ i.e. 333·333. Thus by the classical procedure of ordering 334 vehicles every year we would have landed into difficulties.

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