MACHINE INTERFERENCE PROBLEM WITH POSTPONABLE INTERRUPTIONS

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The usual machine interference problem is considered along with the interruption process to the service activities of the operators. If an interruption occurs when the server is attending a machine, the former has to wait till the service of the latter is completed. Some of the characteristics of the process are examined by considering the tour of the server's state coordinates in the state phase space.

The 'machine interference' problem discussed by Palm[1], Naor[2], Takacs[3] *et al.* assumes that the operator is always available for restarting the machines provided he is not already engaged in the repair activities. This assumption is referred to as the complete availability of the server.

In many practical situations, however, the above assumption is not true. For example, fetching of raw materials, tools etc. in general, forms an interruption process for the repair activities. Consequently a theoretical model has been proposed in the paper taking into account the interruptions to the repair activities.

In an earlier paper[4], the author has studied the machine interference problem with interruptions of preemptive type which was termed as 'ancillary duty' of the process. In this paper, the machine interference problem will be studied with interruptions that are of postponable type. The above discipline is widely known as 'head-of-the-line' priority discipline in priority queuing literature. Cobham[5] introduced this discipline and the other notable workers in this area are Kesten & Runnenberg[6], Miller[7], Jaiswal[8] and Gaver[9].

It may be mentioned that there have been many attempts to relax the complete availability assumptions in various ways. Particular reference can be made to Benson[10], who introduced ancillary duty along with repair duties, and Ben Isrel and Naor[11] who proposed the 'Ends down', model in the machine interference problem.

ASSUMPTIONS

An operator is incharge of N machines that run continuously but fail from time to time and need servicing. The queuing model, considered here, has the following assumptions.

Arrival process of Interruptions—If x denotes the time measured from the cleaning of an interruption to its subsequent arrival, then the sequence of x's are identically and independently distributed random variables having the common probability distribution

$$Pr\{x \leqslant \tau\} = \begin{cases} 1 - e^{-\lambda_1^* \tau} & \text{if } \tau \geqslant 0 \\ 0 & \text{if } \tau < 0 \end{cases} \qquad (1)$$

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when the server is busy and has the distribution; and

$$Pr\{x \leqslant \tau\} = \begin{cases} 1-e^{-\lambda_1^* \tau} & if\ \tau \geqslant 0 \\ 0 & if\ \tau < 0 \end{cases} \quad (2)$$

when the server is idle. The server is idle if and only if all the machines are running and no interruption is waiting or being served. The introduction of two different arrival rates of nterruption process helps us to discuss two particular cases namely $\lambda_1^* = \lambda_1$ and $\lambda_1^* = 0$. The former case is defined as 'Independent Interruption Model' since the interruption occurs independent of server's position while the latter is termed as 'Active Interruption Model' where the interruption occurs only when the server is busy. It is further assumed that not more than one interruption can be present at a particular time.

Clearing Process of Interruptions—The clearing times of the successive occurrences of interruptions are identically distributed random variables $\{X^1_n\}$ having the common probability density $S_1(x)$ where $Pr\{X^1_n \leqslant x\} = \int_0^x S_1(\tau)\, d\tau$.

The arrival process of Machines— The running times of each machine are identically distributed independent random variables having the exponential distribution with mean $\frac{1}{\lambda_2}$. Thus if a machine is free to arrive at time, t, then it will fail in the interval $(t, t+dt)$ with probability $\lambda_2\, dt + 0\,(dt)$.

The servicing process of Machines—The service times of the successive failures of the machines are identically distributed, independent random variables having the common density $S_2(x)$ where the probability that the service time is less than or equal to x is given by $\int_0^x S_2(\tau)\, d\tau$.

We shall assume that the above four processes are statistically independent.

Queue discipline—The interruptions are always attended prior to the service of the machines and if an interruption occurs during the service of a machine, the former has to wait till the latter's service completion. Since the service of the interruption is postponed till the service completion of a machine, this is termed as postponable interruption model.

Let us define

$$\eta_i(x) = \left[S_i(x)/\{1-F_i(x)\}\right] \qquad (i = 1,2) \qquad (3)$$

where $$F_i(x) = \int_0^x S_i(\tau)\, d\tau \qquad (i=1,2)$$

Clearly $\eta_i(x)$ is the conditional probability density of a service completion during the interval $(x, x+\triangle)$ given that it is not completed up to time x. Further the Laplace transform of any real valued positive function $f(t)$ is denoted as $\bar{f}(s)$ where

$$\bar{f}(s) = \int_0^\omega e^{-st} f(t)\, dt \qquad (4)$$

provided the integral on the right hand side is convergent. Thus $\bar{S}_i[s]$ $(i=1,2)$ is the Laplace transform of $S_i(x)$ $[i=1,2]$ for $Re(s) \geqslant 0$

AUXILIARY LEMMAS

We shall prove that

Lemma 1: If $a_1, a_2, \ldots\ldots\ldots a_N$ and $b_0, b_1 \ldots\ldots\ldots b_{N-1}$ are the sequences such that

$$b_r = \sum_{j=r}^{N-1} \binom{j}{r} a_{N-j} \qquad (r=0, 1, 2, \ldots\ldots N-1) \tag{5}$$

then

$$a_l = \sum_{k=0}^{l-1} (-)^k \binom{N-l+k}{k} b_{N-l+k} \quad (l=1,2, \ldots\ldots N) \tag{6}$$

Further

$$\sum_{l=1}^{N} a_l \, \alpha^l = \sum_{k=0}^{N-1} \alpha^{N-k} (1-\alpha)^k \, b_k \tag{7}$$

The proof of this lemma has been given by Thiruvengadam and Jaiswal[12].

Similarly we have

Lemma 2: If $a'_0 \; a'_1, \ldots\ldots\ldots\ldots\ldots a'_N$ and $b'_0 \ldots\ldots\ldots\ldots b'_N$ are two sequences such that

$$b'_r = \sum_{j=r}^{N} \binom{j}{r} a'_{N-j} \qquad (r=0, 1,2, \ldots\ldots N) \tag{8}$$

then

$$a'_l = \sum_{k=0}^{l} (-)^k \binom{N-l+k}{k} b'_{N-l+k} \quad (l=0, 1,2, \ldots\ldots N) \tag{9}$$

Further

$$\sum_{l=0}^{N} a'_l \, \alpha^l = \sum_{k=0}^{N} \alpha^{N-k} (1-\alpha)^k \, b'_k \tag{10}$$

If we define a random varibale C (which is the time interval between two successive entries of the machines in the facility when interruptions are allowed to occur) to be a 'completion time', then the distribution of C is given by

Lemma 3: If $\qquad Pr \{ t \leqslant C \leqslant t + dt \} = \mathcal{V}(t)\, dt \qquad$ (11)

and $\qquad \overline{\mathcal{V}}[s] = \int_0^\infty e^{-st} \mathcal{V}(t)\, dt \qquad$ which is convergent

for $\qquad Re(s) \geqslant 0$ then

$$\overline{\mathcal{V}}[s] = \bar{S}_2[\lambda_1 + s] + \bar{S}_1[s]\,(\bar{S}_2(s) - \bar{S}_2[\lambda_1 + s]) \tag{12}$$

Proof: It is easy to see that

$$\mho(t) = e^{-\lambda_1 t} S_2(t) + \int_o^t S_2(\tau)(1 - e^{-\lambda_1 \tau}) \; S_1(t-\tau)\, d\tau \tag{13}$$

For, in order that the completion time lies between t and $t + dt$, we should have either the service time of the machine lies between $(t, t+dt)$ with no arrival of an interruption upto time t or the service time of the machine lies between $(\tau, \tau + d\tau)$ with an arrival of interrupton before time τ and the service time of the interruption should lie between $(t-\tau, t-\tau+dt)$. Thus we get (13). Taking the Laplace transform of (13) we get (12).

Remark—From (12), it is easy to see

$$\int_o^\infty t \; \mho(t)\, dt = \begin{cases} \eta_2 + \eta_1 (1 - \bar{S}_2[\lambda_1]) & \text{if } \eta_i < \infty \quad (i=1,2) \\ \infty & \text{if any } \eta_i = \infty \end{cases} \tag{14}$$

where $\eta_i = \int_o^\infty x\, S_i(x)\, dx \qquad [i=1,2]$

FORMULATION OF EQUATIONS

Following Keilson and Kooharion [13], and considering the tour of the server's state coordinates in the state phase space, we define the following densities for the process:

$P_m(x,t)\,dx$ $(m=0,1,2, \ldots\ldots N)$—The probability that at time t, there are 'm' machines waiting for service while the server is busy in clearing the inerrupption with the elapsed service time lying between x and $x + dx$.

$Q_m(x,t)\,dx$ $(m=1,2, \ldots\ldots N)$—The probability that at time t, there are 'm' machines in the system with the elapsed service time of a machine under service lying between x and $x + dx$ and the interruption which arrived during the service period of the machine is waiting.

$R_m(x,t)\,dx$ $(m=1,2, \ldots\ldots\ldots N)$—The probability that at time t the system is in the same state as in (2) except that no interruption is waiting.

$E_o(t)$—The probability that at time t, the server is idle.

The above state probabilities are mutually exclusive and exhaustive. They provide the Markovian characterisation of the process.

Let us assume that the steady state probability densities exist and denote them by droping the argument t. Thus for example we have $\lim t\to\infty\, P_m(x,t) = P_m(x)$. By continuity arguments we have the following differential difference equations connecting the various state densities in steady state

$$\left\{\frac{\partial}{\partial x} + [N-m)\,\lambda_2 + \eta_1(x)]\right\} P_m(x) = (N-m+1)\,\lambda_2\, P_{m-1}(x) \tag{15}$$

$$\left\{\frac{\partial}{\partial x} + [(N-m)\,\lambda_2 + \eta_2(x)]\right\} Q_m(x) = (N-m+1)\,\lambda_2\, Q_{m-1}(x) + \lambda_1 R_m(x) \tag{16}$$

$$\left\{\frac{\partial}{\partial x} + [\lambda_1 + (N-m)\,\lambda_2 + \eta_2(x)]\right\} R_m(x) = (N-m+1)\,\lambda_2\, R_{m-1}(x) \tag{17}$$

and

$$[\lambda_1^* + N\lambda_2] \ E_o = \int_0^\infty R_1(x)\eta_2 x\,dx + \int_0^\infty P_o(x)\,\eta_1(x)\,dx \tag{18}$$

The above equations are to be solved subject to the following boundary conditions

$$P_m(0) = \int_0^\infty Q_{m+1}(x)\,\eta_2(x)\,dx + \delta_{o,m}\,\lambda_1^*\,E_o. \tag{19}$$

$$P_N(0) = 0 \tag{20}$$

$$Q_m(0) = 0 \ \ \textit{for all } m \tag{21}$$

$$R_m(0) = \int_0^\infty R_{m+1}(x)\,\eta_2(x)\,dx + \int_0^\infty P_m(x)\,\eta_1(x)\,dx + \delta_{1,m}\,N\lambda_2\,E_o \tag{22}$$

and

$$R_N(0) = \int_0^\infty P_N(x)\,\eta_1(x)\,dx \tag{23}$$

Further let us introduce the following generating function of the above steady state probabilities

$$\pi(\alpha,\beta) = E_o + \sum_{m=1}^{N} \alpha^m \int_0^\infty R_m(x)\,dx + \beta\left[\sum_{m=1}^{N}\alpha^m\int_0^\infty Q_m(x)\,dx + \sum_{m=0}^{N}\alpha^m\int_0^\infty P_m(x)\,dx\right] \tag{24}$$

which is convergent in the entire α and β planes.

STEADY STATE QUEUE LENGTH DISTRIBUTION

The solution of the set of equations obtained above will give us the steady state queue length distribution. We prove that

Theorem 1—If $\eta_i < \infty$ $(i = 1,2)$, then the generating function of the steady state queue length distribution is given by

$$\pi(\alpha,\beta) = E_o\left[1 + \beta\sum_{m=0}^{N}\alpha^{N-m}(1-\alpha)^m T_m + \sum_{m=0}^{N-1}\alpha^{N-m}(1-\alpha)^m R_m\left\{1+\beta\left(\frac{\lambda_1}{m\lambda_2}+k_m-1\right)\right\}\right] \tag{25}$$

where

$$E_o = [(1-\bar{S}_2[\lambda_1]) - \lambda_1^*\,\eta_2 + \chi_o\,\{\eta_2 + \eta_1(1-\bar{S}_2[\lambda_1])\}]^{-1} \tag{26}$$

$$T_n = \chi_n \frac{1 - \bar{S}_1 (n \lambda_2)}{n \lambda_2} \tag{27}$$

$$R_n = \frac{C_{n-1}}{\lambda_1 + n\lambda_2} \left(\sum_{l=o}^{n} \frac{\chi_l \bar{S}_1 [l \lambda_2]}{C_{l-1}} - \gamma_n \right) \tag{28}$$

$$k_n = \frac{1 - \bar{S}_2 [n \lambda_2]}{1 - \bar{S}_2 [\lambda_1 + n \lambda_2]} \tag{29}$$

$$\chi_n = \Theta_n \left\{ \frac{\gamma_N + \beta_{N-1}}{\delta_{N+1} \; \Theta_N} - \sum_{l=n+1}^{N-1} \frac{\beta_l - \beta_{l-1}}{\delta_l \; \Theta_{l-1}} \right\} \tag{30}$$

$$\gamma_n = \begin{cases} \lambda_1^* \quad if \quad n = o \\ \lambda_1^* \sum_{l=o}^{n} \binom{N}{l} \frac{1}{C_{l-1}} + N \lambda_2 \sum_{l=o}^{n-1} \binom{N-1}{l} \frac{1}{C_l} \end{cases} \tag{31}$$

$$\alpha_n = \begin{cases} \dfrac{(1 - \bar{S}_2 [n \lambda_2]) + k_n \bar{S}_1 [n \lambda_2]}{C_{n-1} (1 - k_n) + C_{n-2} (1 - k_{n-1})} & if\ n=1,2,\ldots(N-1) \\ [C_{N-2} \; (1 - k_{N-1})]^{-1} & if\ n = N \end{cases} \tag{32}$$

$$\beta_n = \begin{cases} 0 \ if\ n=0 \\ \dfrac{\binom{N}{n} \lambda_1^* - \gamma_n \; C_{n-1} (1 - k_n) - \gamma_{n-1} C_{n-2} \; (1 - k_{n-1})}{C_{n-1} (1 - k_n) + C_{n-2} (1 - k_{n-1})} \\ \dfrac{\lambda_1^* - \gamma_{N-1} \; C_{N-2}}{C_{N-2} \; (1 - k_{N-1})} & if\ n=N \end{cases} \tag{33}$$

$$\delta_n = \begin{cases} 1 \ if\ n=1. \\ \alpha_{n-1} + \dfrac{S_1 [(n-1) \lambda_2]}{C_{n-2}} & if\ n=2, \ldots(N+1) \end{cases} \tag{34}$$

$$C_r = \begin{cases} 1 \ if\ r = -1 \\ \prod_{l=o}^{r} \dfrac{\bar{S}_2 [l \lambda_2 + \lambda_1]}{1 - \bar{S}_2 [l \lambda_2 + \lambda_1]} \end{cases} \tag{35}$$

and

$$\Theta_n = \begin{cases} 1 \ if\ n=0 \\ \dfrac{\delta_1, \delta_2 \ldots\ldots \delta_n}{\alpha_1 \alpha_2 \ldots\ldots \alpha_n} & \text{otherwise} \end{cases} \tag{36}$$

Proof:—Employing the transforms

$$A_n (x) = \sum_{j=n}^{N} \binom{j}{n} P_{N-j} (x)$$

$$B_n (x) = \sum_{j=n}^{N-1} \binom{j}{n} Q_{N=1} (x) \tag{37}$$

and

$$b_n(x) = \sum_{j=n}^{N-1} \binom{j}{n} R_{N-j}(x)$$

the equations (15) to (18) can be written as

$$\left\{\frac{\partial}{\partial x} + \left[n\,\lambda_2 + \eta_1(x)\right]\right\} A_n(x) = 0 \tag{38}$$

$$\left\{\frac{\partial}{\partial x} + \left[n\,\lambda_2 + \eta_2(x)\right]\right\} B_n(x) = \lambda_1\, b_n(x) \tag{39}$$

$$\left\{\frac{\partial}{\partial x} + \left[\lambda_1 + n\,\lambda_2 + \eta_2(x)\right]\right\} b_n(x) = 0 \tag{40}$$

and

$$[\lambda_1^* + N\lambda_2]\, E_o = \int_0^\infty A_N(x)\,\eta_1(x)\,dx + \int_0^\infty b_{N-1}(x)\,\eta_2(x)\,dx \tag{41}$$

The solution of the equations (38)to (40) are given by

$$A_n(x) = A_n(0)\ exp\ \{-n\,\lambda_2\,x - \int_0^x \eta_1(x)\,dx\} \tag{42}$$

$$B_n(x) = b_n(0)\,(1 - e^{-\lambda_1 x})\ exp\ \{-n\,\lambda_2\,x - \int_0^x \eta_2(x)\,dx\} \tag{43}$$

and

$$b_n(x) = b_n(0)\ exp\ \{-[\lambda_1 + n\,\lambda_2]\,x - \int_0^x \eta_2(x)\,dx\} \tag{44}$$

In deriving (43) we have used (21). The boundary conditions (19) to (23), on employing the above mentioned transforms, can be written as

$$A_N(0) = \int_0^\infty B_{N-1}(x)\,\eta_2(x)\,dx + \lambda_1^* E_o \tag{45}$$

$$A_n(0) = \int_0^\infty [B_n(x) + B_{n-1}(x)]\,\eta_2(x)\,dx + \binom{N}{n} \lambda_1^* E_o \tag{46}$$

$$A_o(0) = \int_0^\infty B_o(x)\,\eta_2(x)\,dx + \lambda_1^* E_o \tag{47}$$

$$b_n(0) = \int_0^\infty \left[b_n(x) + b_{n-1}(x)\right] \eta_2(x)\,dx + \binom{N-1}{n} N\lambda_2 E_o \tag{48}$$

$$+ \int_0^\infty A_n(x)\,\eta_1(x)\,dx - \binom{N}{n} \left[\lambda_1^* + N\lambda_2\right] E_o$$

and

$$b_o(0) = \int_o^\infty b_o(x)\,\eta_2(x)\,d_x + \int_o^\infty A_o(x)\,\eta_1(x)\,d^x - \overset{*}{\lambda}_1 E_o \tag{49}$$

In deriving (48) and (49) we have used (41).

Now substituting (44) in (48), we get

$$b_n(0)\{1-\bar{S}_2[\lambda_1+n\lambda_2]\} = b_{n-1}(0)\,\bar{S}_2[\lambda_1+(n-1)\lambda_2]$$

$$+A_n(0)\,\bar{S}_1[n\lambda_2] - \binom{N}{n}\left[\overset{*}{\lambda}_1+N\lambda_2\right]E_o + \binom{N-1}{n}N\lambda_2 E_o \tag{50}$$

Defining

$$C'_n = \begin{cases} \dfrac{1}{1-\bar{S}_2[\lambda_1]} & \text{if } n=o \\ \dfrac{1}{1-\bar{S}_2[\lambda_1]} \displaystyle\prod_{l=1}^{n} \dfrac{\bar{S}_2[\lambda_1+(l^{-1})\lambda_2]}{1-\bar{S}_2[\lambda_1+l\lambda_2]} & \end{cases} \tag{51}$$

and dividing (50) by $C'n$ both sides, we obtain

$$\frac{b_n(0)}{C'_n} = \frac{b_{n-1}(0)}{C'_{n-1}} + \frac{A_n(0)\,\bar{S}_1[n\lambda_2]}{C_{n-1}} - \binom{N}{n}\frac{[\overset{*}{\lambda}_1+N\lambda_2]E_o}{C_{n-1}}$$

$$+ N\lambda_2 E_o\binom{N-1}{n}\frac{1}{C_{n-1}} \tag{52}$$

where Cn is given by (35).

Changing n to $(n-1)$..................1, adding all the equations, and substituting the value of $b_{o(0)}$ from (49), we get

$$\frac{b_n(0)}{C'_n} = \sum_{l=o}^{n}\frac{A_l(0)\,\bar{S}_1[l\lambda_2]}{C_{l-1}} - \gamma_n E_o \tag{53}$$

where γ_n is given by (31).

Rearranging the terms in (52), we get

$$\frac{b_n(0)}{C'_n} = \frac{b_{n+1}(0)}{C'_{n+1}} - \frac{A_{n+1}(0)\,\bar{S}_1[(n+1)\lambda_2]}{C_n}$$

$$-(\gamma_{n+1}-\gamma_n)E_o \tag{54}$$

Changing n to $(n+1)$, ...$(N\text{-}2)$, substituting the value of $b_{n-1}(0)$ from (41) and finally equating the resulting expression with (53), we obtain

$$\sum_{l=0}^{N}\frac{A_l(0)\,\bar{S}_1[l\lambda_2]}{C_{l-1}} = \gamma_N E_o \tag{55}$$

From (45 to 47) it is easy to get the following equations

$$A_n\ (0) = b_n\ (0)\ \{\bar{S}_2\ [n\ \lambda_2] - \bar{S}_2[\lambda_1 + n\lambda_2]\} + \binom{N}{n} \overset{*}{\lambda}_1\ E_o$$

$$+ b_{n-1}\ (0)\ \{\bar{S}_2\,[(n-1)\ \lambda_2] - \bar{S}_2\ [\lambda_1 + (n-1)\ \lambda_2]\} \tag{56}$$

$$A_N\,(0) = b_{N-1}\ (0)\ \{\bar{S}_2\ [(N-1)\ \lambda_2] - \bar{S}_2\ [\lambda_1 + (N-1)\ \lambda_2]\} + \overset{*}{\lambda}_1\,E_o \tag{57}$$

and

$$A_o\ (0) = b_o\ (0)\ \{1 - \bar{S}_2\ [\lambda_1]\ \} + \overset{*}{\lambda}_1\ E_o \tag{58}$$

The equation (58) can be easily seen to be an identity.

Substituting the necessary values from (53), one can easily obtain from (56) and (57)

$$\alpha_n\ A_n\ (0) = \sum_{l=o}^{n-1} \frac{A_l\ (0)\ \bar{S}_1\ [l\ \lambda_2]}{C_{l-1}} + \beta_n\ E_o \tag{59}$$

and

$$\alpha_N\,A_N\,(0) = \sum_{l=0}^{N-1} \frac{A_l\ (0)\bar{S}_1\,[l\lambda_2]}{C_{l-1}} + \beta_N\ E_o \tag{60}$$

where α_n and β_n are given by (32) and (33) respectively.

Now, changing n to $(n-1)$ in (59) and subtracting the resulting equation from (59), we obtain

$$\alpha_n\ A_n\ (0) - \delta_n\ A_{n-1}\ (0) = (\beta_n - \beta_{n-1}\,)\ E_o \tag{61}$$

where δ_n is given by (34).

Using the equation (55), the equation (60) can be written as

$$A_N\ (0) = \frac{\beta_{N-1} + \gamma_N}{\delta_{N+1}} \tag{62}$$

Dividing the equation (61) by Θ_n and rearranging the terms, it is easy to obtain

$$\frac{A_n\ (0)}{\Theta_n} = \frac{A_{n+1}\ (0)}{\Theta_{n+1}} - \frac{\beta_{n+1} - \beta_n}{\delta_n\ \Theta_n}\ E_o \tag{63}$$

where Θ_n is given by (36). Changing n to $(n+1)$ ...... $(N-1)$ adding all the equations, and substituting the value of $A_N\,(0)$ from (62), we get

$$\frac{A_n\ (0)}{\Theta_n} = E_o \left\{ \frac{\beta_{N-1} + \gamma_N}{\delta_{N+1}\ \Theta_N} - \sum_{l=(n+1)}^{N-1} \frac{\beta_{l+1} - \beta_l}{\delta_l\ \Theta_{l-1}} \right\} \tag{64}$$

Thus all the constants are expressed in terms of E_o .

Further the value of E_o is determined such that

$$\left\{ E_o + \sum_{m=o}^{N} \int_o^\infty P_m\,(x)\ dx + \sum_{m=1}^{N} \int_o^\infty [Q_m\,(x) + R_m\,(x)]\ dx \right\} = 1 \tag{65}$$

The above equation can be written as

$$\{E_o + \int_o^\infty [A_o (x) + B_o (x) + b_o (x)] \, dx\} = 1 \tag{66}$$

Substituting the necessary values from the solution obtained above, we get E_o and is given by (26). In the above, it may be noted that the existence of finite first moments of service time distributions is assumed. Using the Lemma given in this paper, the generating function of the steady state probabilities can be written as

$$\pi (\alpha, \beta) = E_o + \beta \sum_{m=o}^{N} \alpha^{N-m} (1-\alpha)^m \int_o^\infty A_m (x) \, dx$$

$$+ \sum_{m=0}^{N-1} \alpha^{N-m} (1-\alpha)^m \int_o^\infty \Big[b_m (x) + \beta B_m (x) \Big] dx. \tag{67}$$

This equation can be simplified to (25) after substituting the solution obtained above. This completes the proof of the theorem.

Remark 1—The operational efficiency is defined as the proportion of time the operator is busy either in clearing off the interruption or engaged in repair activities and is given by

$$0. E. = 1 - E_o \tag{68}$$

The equation (68) can be computed with the help of the value given in (26).

Remark 2—If we define the random variable $s(t)$ to be the number of machines not in working order at time t and

$$\pi_n (t) = P_r \{s (t) = n\} \qquad (n = o, 1, 2 \ldots . N) \tag{69}$$

then assuming that the steady state probabilities π_n where $\lim_{t \to \infty} \pi_n (t) = \pi_n$) exists, it is easy to see that

$$\pi_n = \begin{cases} E_o + \int_o^\infty P_o (x) \, dx \quad if \; n=o \\ \int_o^\infty \Big[P_n (x) + Q_n (x) + R_n (x) \Big] dx \quad \text{otherwise} \end{cases} \tag{70}$$

Defining

$$E_o \, \theta_n = \sum_{j=n}^{N-1} \binom{j}{n} \pi_{N-j} \tag{71}$$

we get

$$E_o \, \theta_n = \int_o^\infty \Big[A_n (x) - \binom{N}{n} A_n (x) + B_n (x) + b_n (x) \Big] dx \tag{72}$$

Substituting necessary values from Theorem 1, and using the lemma 1, we have the following

Theorem 2—If $\eta_i < \infty$, the queue length generating function of steady state probabilities is given by

$$\sum_{n=o}^{N} \pi_n \, \alpha^n = E_o \left\{ (1+L) + \sum_{n=o}^{N-1} \alpha^{N-n} (1-\alpha)^n \, \theta_n \right\} \tag{73}$$

where

$$L = \frac{\gamma_N + \beta_{N-1}}{\delta_{N+1}} \quad \frac{1-\bar{S}_1 [N\lambda_2]}{N\lambda_2}$$

$$\theta_n = \begin{cases} \chi_o \, \eta_1 - L + R_o \dfrac{\lambda_1 \eta_2}{1-\bar{S}_2 [\lambda_1]} & \text{if } n = o \\ \chi_n \dfrac{1-\bar{S}_1 [n \lambda_2]}{n \lambda_2} - L + \dfrac{k_n}{n\lambda_2} R_n (\lambda_1 + n\lambda_2) \end{cases} \tag{75}$$

and E_o is given by (26).

Further, the expected number of machines not in working order is given by

$$\sum n \, \pi_n = N - \theta_1 E_o - N (1 + L) E_o \tag{76}$$

If we define the machine availability $(M.A)$ to be the proportion of machines in working order, then we have

$$M.A = \frac{N - \Sigma_n \, \pi_n}{N} = \frac{(1 + L) N + \theta_1}{N} \tag{77}$$

Remark 3—When $\overset{*}{\lambda}_1 = 0$, the above results reduce to active interruption case and when $\overset{*}{\lambda}_1 = \lambda_1$, the results of independent interruption case can be obtained.

Remark 4—The proportion of time the operator is engaged in clearing off the interruption, is given by

$$E (m) = \sum_{m=o}^{N} \int_o^{\infty} P_m (x) \, dx + \sum_{m=1}^{N} \int_o^{\infty} Q_m (x) \, dx \tag{78}$$

which reduces to

$$E (m) = E_o \left\{ \chi_o \, \eta_1 + \frac{\chi_o - \overset{*}{\lambda}_1}{1-\bar{S}_2 [\lambda_1]} \left(\eta_2 + (1-\bar{S}_2 [\lambda_1]) \right) \right\} \tag{79}$$

STEADY STATE OCCUPATION TIME DISTRIBUTIONS

We can obtain the distribution of the following random variables by suitably augmenting the times required for service with the corresponding state coordinates of the server defined earlier.

(*i*) $\Omega_1 (t)$—The occupation time of an interruption at time t, *i. e.* the time that an interruption has to wait if it joins at time t.

(*ii*) $\Omega_2 (t)$—The occupation time of a machine at time t, *i.e.* the time that a machine will have to wait if it joins at time t.

The above random variables evidently give the distribution of the virtual waiting time of an interruption and a machine respectively.

Further let us define

$$W_i\ (Z, t)\, dZ = Pr\,\{Z \leqslant \Omega_i\ (t) \leqslant Z + dZ\} \qquad (i = 1, 2) \tag{80}$$

and assume that

$$\lim_{t \to \infty}\ W_i\ (Z, t) = W_i\ (Z) \qquad (i = 1, 2) \tag{81}$$

exist irrespective of the initial conditions.

We shall prove that

Theorem 3 — If $\overline{W}_1\ (s) = \int_o^\infty e^{-sZ}\ W_1\ (Z)\, dZ$ (82)

which is convergent for $Re\ (s) \geqslant 0$, then

$$\overline{W}_1\ (s) = E_o\ \Big\{ (1 + \chi_o\ \frac{1 - \overline{S}_1\ [s]}{s} + (\chi_o - \overset{*}{\lambda}_1) \tag{83}$$

$$\left[\overline{S}_1\,[s]\ \left(\frac{1 - \overline{S}_2\ [s]}{s} - \frac{\overline{S}_2\,[s] - \overline{S}_2\,[\lambda_1]}{\lambda_1 - s}\right) + \frac{\overline{S}_2\,[s] - \overline{S}_2\,[\lambda_1]}{\lambda_1 - s}\right]\Big\}$$

where E_o and χ_o are given by theorem 1.

Proof:—Let us suppose that at time t an interruption joins and the arrival of interrruption process stops. The interruption which has joined at time t has to wait for a time (i) equal to the unfinished portion of the service of an interruption if it joins when the interruption is under service or (ii) equal to the sum of the unfinished service time of a machine and the service time of an interruption if it joins when the service of a machine is in process and an interrruption is already waiting or (iii) equal to the unfinished service time of the machine if it joins when the service of a machine is in progress.

It is evident from the definition of the state coordinates given in an earlier section that the probability densities of the above events are $P_n\ (x, t)\ Q_n\ (x,t)$ and $R_n\ (x,t)$ respectively. Consequently we have

$$W_1\ (Z) = E_o\ \delta\ (Z) + \sum_{n=o}^{N} \int_o^\infty P_n\ (x)\, dx\ \frac{S_1\ (x + Z)}{1 - F_1\ (x)} \tag{84}$$

$$+ \sum_{n=1}^{N} \int_o^\infty Q_n\ (x)\, dx \left[\int_o^Z \frac{S_2\ (x + \tau)}{1 - F_2\ (x)}\ S_1\ (Z - \tau)\, d\tau \right.$$

$$+ \sum_{n=1}^{N} \int_o^\infty R_n\ (x)\, dx\ \frac{S_2\ (x + Z)}{1 - F_2\ (x)}$$

By using (37), (84) can be written as

$$W_1(Z) = E_o\,\delta(Z) + \int_o^\infty A_o(x)\,\frac{S_1(x+Z)}{1-F_1(x)}\,dx$$

$$+ \int_o^\infty B_o(x)\left(\int_o^Z \frac{S_2(x+\tau)}{1-F_2(x)}\,S_1(Z-\tau)\,d\tau\right)dx \tag{85}$$

$$+ \int_o^\infty b_o(x)\,\frac{S_2(x+Z)}{1-F_2(x)}\,dx$$

Taking the Laplace transform, and substituting the necessary values from Theorem 1 we obtain (83) from (85).

This completes the proof of the theorem.

Theorem 4—If

$$\overline{W}_2(s) = \int_o^\infty e^{-sZ}\,W_2(Z)dZ \tag{86}$$

which is convergent for $Re(s) > 0$, then

$$\overline{W}_2(s) = E_o\,\Big\{1 + \sum_{n=o}^{N-1} (\overline{\mho}[s])^{N-1-n}\,(1-\overline{\mho}[s])^n\,s_n(s) \tag{87}$$

$$+ \sum_{n=o}^{N} (\overline{\mho}[s])^{N-n}\,(1-\overline{\mho}[s])^n\,t_n(s)\Big\}$$

where

$$s_n(s) = \left\{\sum_{l=o}^{n} \frac{\chi_l\,\overline{S}_1[l\lambda_2]}{C_{l-1}} - \gamma_n\right\} \tag{88}$$

$$\times\,\frac{\left(\overline{S}_2[\lambda_1+s]-\overline{S}_2[\lambda_1+n\lambda_2]\right)+\overline{S}_1[s]\left(\left\{\overline{S}_2[s]-\overline{S}_2[n\lambda_2]\right\}+\overline{S}_2[\lambda_1+s]-\overline{S}_2[\lambda_1+n\lambda_2]\right)}{n\lambda_2 - s}$$

$$t_n(s) = \chi_n \frac{\bar{S_1}[s] - \bar{S_1}[n\lambda_2]}{n\lambda_2 - s} \tag{89}$$

and χ_n γ_n and C_r are given in Theorem 1 while $\bar{\mho}[s]$ is given by Lemma 3.

Proof—The density of the occupation time distribution $W_2(Z)$ and the server's state coordinates are connected by means of the following relation

$$\begin{aligned} W_2(Z) = E_o\,\delta(Z) &+ \sum_{n=o}^{N} \int_o^\infty P_n(x)\,dx \int_o^Z \frac{S_1(x+\tau)}{1-F_1(x)}\,\mho^{n*}(Z-\tau)\,d\tau \\ &+ \sum_{n=1}^{N} \int_0^\infty Q_n(x)\,dx \int_0^Z V_1(\tau)\,\mho^{(n-1)*}(Z-\tau)\,d\tau \\ &+ \sum_{n=1}^{N} \int_0^\infty R_n(x)\,dx \int_0^Z V_2(\tau)\,\mho^{(n-1)*}(Z-\tau)\,d\tau \end{aligned} \tag{90}$$

where $\Omega r^*(y)$ is the r-fold convolution of $\mho(t)$ with itself. In (90) we have

$$Pr\left\{\tau_1 < u\right\} = \int_0^u V_1(\tau)\,d\tau \tag{91}$$

and

$$Pr\left\{\tau_2 < u\right\} = \int_0^u V_2(\tau)\,d\tau \tag{92}$$

where τ_1 is the time required to finish the remaining portion of the service of the machine and the service of the interruption and τ_2 is the time required to finish the remaining portion of the service if the interruption is allowed to occur. Therefore, we have

$$V_1(u) = \int_0^u \frac{S_2(x+\tau)}{1-F_2(x)}\,S_1(u-\tau)\,d\tau \tag{93}$$

and

$$\bar{V}_1(s) = \int_0^\infty e^{-su}\,V_1(u)\,du = \bar{S}_1[s] \int_0^\infty e^{-su}\,\frac{S_2(x+u)}{1-F_2(x)}\,du \tag{94}$$

Further we note that the distribution of τ_2 is that of the completion time distribution of the machine, similar to one given in Lemma 3. Here we have the density of the service time distribution of the machine given by $\frac{S_2(x+u)}{1-F_2(x)}$ $(0 < \mu < \infty)$ for a given x. Therefore, the Laplace transform of the density, $V_2(u)$ is given by

$$\bar{V}_2(s) = \bar{\Omega}(\lambda_1 + s) + \bar{S}_1[s]\,(\bar{\Omega}(s) - \bar{\Omega}(\lambda_1 + s)) \tag{95}$$

where

$$\bar{\Omega}(s) = \int_0 e^{-su} \frac{S_2(x+u)}{1-F_2(x)}\, du \tag{96}$$

It may be noted that in deriving (90) we have used an argument similar to that employed in (84).

Taking the Laplace transform and employing the results given in Lemmas 1 and 2, we obtain (87) after a little manipulation and using the results given in Theorem 1.

This completes the proof of the theorem.

DISCUSSION

In the above sections, we have investigated some aspects of machine interference problem with postponable interruption. The time-dependent generating function of the queue length distribution and occupation time distribution can also be obtained in the transformed space.

This problem can also be generalised to the case of finite number of priority units. In order to compare the effect of priority discipline, different operational parameters such as mean queue length, idle probability etc. can be numerically computed for the preemptive resume[4] as well as head-of-the-line case.

Throughout this paper, we have assumed the existence of densities of the service time distributions *etc.*, but this is not a serious limitation of the study. For those distributions which do not have densities, can be viewed as the limiting form of the distributions for which the densities exist or the operations of integration and differentiation performed in this paper can be viewed in a generalised sense.

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