STABILITY OF ROCKET FLIGHT DURING BURNING

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(*Received* 19 *August* 1966)

Stability of the rocket-motion during burning is discussed taking into consideration gravity, aerodynamic forces and torques. Conditions for stabilizing the rocket motion are investigated. Analysis for initial and final phases of burning is given separately. Stability regions of the projected motions on two dimensional co-ordinate planes are obtained and thereby stability region of the actual motion is derived. Stability diagrams illustrate statically and dynamically stable and unstable regions.

NOMENCLATURE

A = Cross-sectional area of the projectile

C_m = The restoring moment coefficient

C_q = Damping moment coefficient

C_D = Drag coefficient

C_l = Lift coefficient.

g = Acceleration due to gravity

G_j = Acceleration produced by the jets alone

K = Radius of gyration about the transverse axis of the rocket passing through the centre of mass

K_{jD} = Jet damping torque coefficient

l = Length of the projectile

m = Rocket mass

R_m = (R_{Mx}, R_{My}) = Linear malalignment

V = Rocket velocity

ν = (ν_x, ν_y) = Deflection of the tangent to the trajectory

ϕ = (ϕ_x, ϕ_y) = Orientation angle

θ_o = Quadrant elevation of the launcher above the horizontal

β_M = (β_{Mx}, β_{My}) = Angle between the rocket axis and the direction of the resultant force.

ρ = Density of air.

Srivastava[1] has investigated the stability of a projectile moving in a plane as well as the stability criteria of projectile motion. The analysis was applicable to a rocket after burntout. During burning phase of the rocket its motion is affected by jet forces. The object of the present paper is to discuss the stability of the rocket flight during its burning phase in three dimensional co-ordinate system. Jet thrust is assumed to be constant. The problem is analysed for both phases of rocket burning *i.e.*, initial and final. In the initial phase rocket thrust is the dominant factor whereas in the final phase the velocity also becomes equally important.

EQUATIONS OF ROCKET MOTION DURING BURNING

Following Davis *et al*[2] the co-ordinate axes are chosen as shown in the Fig. 1. The linear motion of the rocket is referred to the fixed co-ordinate system $O_o X_o Y_o Z_o$ while angular motion to the moving system $OXYZ$. All angles are represented by combination of two angles measured as arcs on a unit sphere about the centre of gravity of the rocket. For example $\phi = (\phi_x, \phi_y)$ where ϕ_x is the angle measured from the YZ—plane to the point A along the arc of great circle through A and the X —axis, and ϕ_y is the angle measured from the XZ—plane to the point A along the arc of small circle parallel to the YZ—plane. If we take ϕ and ν to be small (a valid assumption during burning phase of the rocket) we can write

$$\phi = (\varphi_x{}^2 + \phi_y{}^2)^{\frac{1}{2}} \quad (1)$$

and

$$\nu = (\nu_x{}^2 + \nu_y{}^2)^{\frac{1}{2}} \quad (2)$$

By restricting ϕ and v to be small, the three dimensional problem of the rocket motion during burning can be split up into 2 sets of equations, where quantities in one set depend

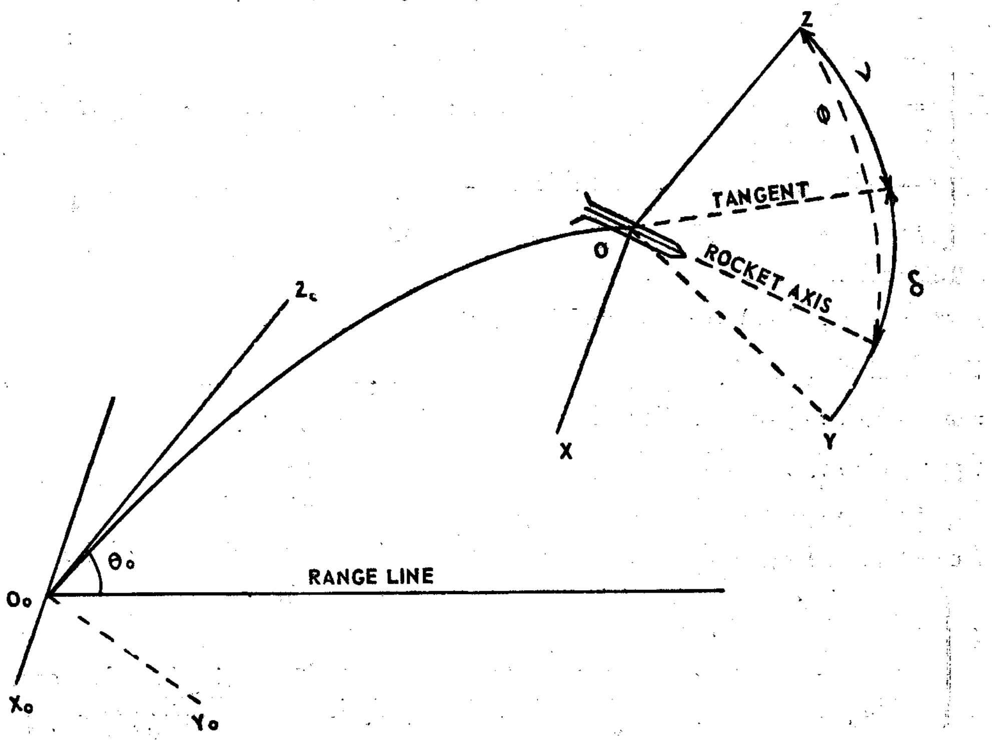

Fig. 1—Location of the co-ordinate system.

upon the X co-ordinate only and those in the other on the Y co-ordinate only. The equations of motion corresponding to the projection of the actual three-dimensional space motion on the vertical plane YZ passing through the launcher are given by

$$m\dot{V} = m\ddot{Z}_o = m\,Gj - mg\sin\theta_o + mg\cos\theta_o\,\bar{\nu}_y - \tfrac{1}{2}C_D\,\rho A V^2 \tag{3}$$

$$mV\dot{\nu}_y = m\,G_j\,(\phi_y - \nu_y) + m\,G_j\,\beta_{My} + mg\cos\theta o + mg\sin\theta o\,\nu_y + \tfrac{1}{2}C_l\,\rho A V^2\,(\phi_y - \nu_y) \tag{4}$$

and

$$m\,K^2\ddot{\phi}_y = -\,m\,G_j\,R_{My} - K_{jD}\,\dot{\phi}_y - \tfrac{1}{2}C_m\,\rho AlV^2\,(\phi_y - \nu_y) - \tfrac{1}{2}C_q\,\rho Al^2V\,\dot{\phi}_y \tag{5}$$

where

K_{jD} is the jet damping torque coefficient. The equations of motion that represent the projection of actual space motion on the XZ plane will be given by (3) and the following two equations:

$$m\,V\dot{\nu}_x = m\,G_j\,(\phi_x - \nu_x) + m\,G_j\,\beta_{Mx} + \tfrac{1}{2}C_l\,\rho A V^2\,(\phi_x - \nu_x) \tag{6}$$

$$m\,K^2\ddot{\phi}_x = -\,mG_j\,R_{Mx} - K_{jD}\,\dot{\phi}_x - \tfrac{1}{2}C_m\,\rho AlV^2\,(\phi_x - \nu_x) - \tfrac{1}{2}\,Cq\,\rho Al^2V\dot{\phi}_x \tag{7}$$

STABILITY ANALYSIS OF PROJECTED MOTION ON YZ-PLANE

From (3), (4) and (5) we can write the perturbation equations as

$$\left.\begin{cases} m\delta\dot{V} + C_D\,\rho AV\,\delta V - mg\cos\theta_o\,\delta\nu_y = o \\ [m\dot{\nu}_y - C_l\,\rho AV\,(\phi_y - \nu_y)]\,\delta V + m\,V\,\delta\dot{\nu}_y + [m\,Gj - mg\sin\theta_o + \tfrac{1}{2}C_l\,\rho AV^2]\,\delta\nu_y - [m\,Gj + \tfrac{1}{2}C_l\,\rho AV^2]\,\delta\phi_y = o \\ C_m\,\rho AlV\,(\phi_y - \nu_y) + \tfrac{1}{2}C_q\,\rho Al^2\,\dot{\phi}_y\,]\,\delta V - \tfrac{1}{2}C_m\,\rho AlV^2\,\delta\nu_y + m\,K^2\,\delta\ddot{\phi}_y + \\ (K_{jD} + \tfrac{1}{2}C_q\,\rho Al^2\,V)\,\delta\dot{\phi}_y + \tfrac{1}{2}C_m\,\rho AlV^2\,\delta\phi_y = o \end{cases}\right\} \tag{8}$$

Taking Laplace transformation

$$(m\,S + C_D\,\rho AV)\,\delta V - mg\cos\theta_o\,\delta\nu_y = o$$

$$[m\,\dot{\nu}_y - C_l\,\rho AV\,(\phi_y - \nu_y)]\,\delta V + mVS\,\delta\nu_y + (mG_j - mg\sin\theta_o + \tfrac{1}{2}C_l\,\rho AV^2)\,\delta\nu_y - (m\,Gj + \tfrac{1}{2}C_l\,\rho AV^2)\,\delta\phi_y = o$$

$$[C_m\,\rho AlV\,(\phi_y - \nu_y) + \tfrac{1}{2}C_q\,\rho Al^2\,\dot{\phi}_y\,]\,\delta V - \tfrac{1}{2}C_m\,\rho AlV^2\,\delta\nu_y + mK^2S^2\,\delta\phi_y + (K_{jD} + \tfrac{1}{2}C_q\,\rho Al^2V)\,S\,\delta\phi_y + \tfrac{1}{2}C_m\,\rho AlV^2\,\delta\phi_y = o \tag{9}$$

equations (9) can be put as

$$\begin{bmatrix} A_{11}S + A_{12} & B_{11} & O \\ A_{21} & B_{21}S + B_{22} & C_{21} \\ A_{31} & B_{31} & C_{31}S^2 + C_{32}S + C_{33} \end{bmatrix} \begin{bmatrix} \delta V \\ \delta\nu_y \\ \delta\phi_y \end{bmatrix} = 0 \tag{10}$$

where

$$A_{11} = m \qquad B_{11} = - mg \cos \theta_o$$

$$A_{12} = C_D \rho A V \qquad B_{21} = mV$$

$$A_{21} = m \dot{v}_y - C_l \ \rho AV(\phi_y - v_y) \qquad B_{22} = m G_j - m_g \sin \theta_o + \tfrac{1}{2} C_l \ \rho A V^2$$

$$A_{31} = C_m \rho A l V \ (\phi_y - v_y) + \tfrac{1}{2} C_q \ \rho A l^2 \dot{\phi}_y \qquad B_{31} = - \tfrac{1}{2} C_m \rho A l V^2$$

$$C_{21} = - m G_j - \tfrac{1}{2} C_l \ \rho A V^2 \qquad C_{31} = m K^2$$

$$C_{32} = K_{jD} + \tfrac{1}{2} C_q \ \rho \ A l^2 V \qquad C_{33} = \tfrac{1}{2} C_m \rho A l V^2 \tag{11}$$

From (10) we obtain

$$P_o S^4 + P_1 S^3 + P_2 S^2 + P_3 S + P_4 = 0 \tag{12}$$

and the coefficients are given by

$$P_o = A_{11} B_{21} C_{31}$$

$$P_1 = A_{11} B_{21} C_{32} + A_{11} B_{22} C_{31} + A_{12} B_{21} C_{31}$$

$$P_2 = A_{11} (B_{21} C_{33} + B_{22} C_{32}) + A_{12} (B_{21} C_{32} + B_{22} C_{31}) - A_{21} B_{11} C_{31}$$

$$P_3 = A_{11} (B_{22} C_{33} - B_{31} C_{21}) + A_{12} (B_{21} C_{33} + B_{22} C_{32}) - A_{21} B_{11} C_{32}$$

$$P_4 = A_{12} (B_{22} C_{33} - C_{21} B_{31}) - B_{11} (A_{21} C_{33} - C_{21} A_{31}) \tag{13}$$

The necessary conditions for stability of the rocket are

$$P_r > 0 \qquad r = 0, 1, \ldots, 4 \tag{14}$$

An additional condition for the stability of the rocket is obtained by 'Test Function' method[3]:

$$C^2 - a b c + a^2 d < 0 \tag{15}$$

where

$$a = \frac{P_1}{P_0} , \ b = \frac{P_2}{P_0} , \ c = \frac{P_3}{P_0} \text{ and } d = \frac{P_4}{P_0} \tag{16}$$

conditions (14) and (15) together form the sufficient conditions for stability of the rocket.

Condition (15) with the help of (16), can be written as

$$P_3^2 P_o - P_1 P_2 P_3 + P_1^2 P_4 < 0 \tag{17}$$

INITIAL PHASE OF BURNING

Initial phase of burning is characterized by high value of rocket thrust compared to rocket velocity and other parameters. Substituting values from (11) in (13) and retaining highest powers of thrust we have

$$P_0 \simeq 0$$

$$P_1 \simeq m^3 K^2 G$$

$$P_2 \supseteqq m^2 G_j \left(C_D \rho A V K^2 + K_{jD} + \tfrac{1}{2} C_q \rho A l^2 V \right)$$

$$P_3 \supseteqq C_D \rho A V m G_j \left(K_{jD} + \tfrac{1}{2} C_q \rho A l^2 V \right) \tag{18}$$

$$P_4 \supseteqq m G_j \left[\tfrac{1}{2} C_D C_m \rho^2 A^2 l V^3 + mg \cos\theta_o \left\{ C_m \rho A l V (\phi_y - \nu_y) + \tfrac{1}{2} C_l \rho A l^2 \phi_y \right\} \right]$$

substituting (18) in inequality (17) and retaining the highest power of $m\ G_j$ and dropping out the common factor $m^3 G^2{}_j$ we obtain

$$mK^2 C_m \left[\tfrac{1}{2} C_D \rho^2 A^2 l V^3 + mg \cos\theta_o \rho A l V (\phi_y - \nu_y) \right] <$$
$$C_D \rho A V \left\{ K^2{}_{jD} + K_{jD} \rho A V K^2 \right\} + C_q \left\{ K_{jD} + C_D (\rho A)^2 l^2 V^2 + \tfrac{1}{2} C_D{}^2 \right. \tag{19}$$
$$\left. (\rho A V)^3 l^2 K^2 - \tfrac{1}{2} (mKl)^2 g \rho A \cos\theta_o \dot{\phi}_y \right\} + \tfrac{1}{4} C_D (\rho A V)^3 l^4 C_q{}^2$$

we write the inequality in the form

$$A_o C_m < A_1 + A_2 C_q + A_3 C_q{}^2 \tag{20}$$

$$A_o C_m = A_1 + A_2 C_q + A_3 C_q{}^2 \tag{21}$$

The curve can be plotted giving stable and unstable regions. For this we assume

$$\frac{A_1}{A_0} = 0{\cdot}1 , \quad \frac{A_2}{A_0} = 0.2 \text{ and } \frac{A_3}{A_0} = 0{\cdot}4$$

and then (21) becomes

$$C_m = 0{\cdot}1 + 0{\cdot}2\ C_q + 0{\cdot}4\ C_q{}^2$$

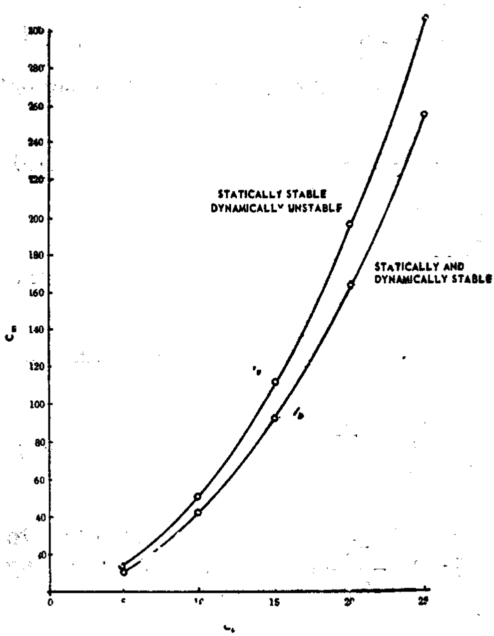

Fig. 2—Stability in YZ plane for initial phase.

Taking C_m as ordinate and C_q as abscissa, curve I_y in Fig. 2 represents the stability dia-gram in Y_Z plane for the initial phase of burning. For region above the curve I_y, the rocket will be statically stable but dynamically unstable whereas for region below the curve I_y, the rocket will be both statically and dynamically stable.

FINAL PHASE OF BURNING

As propellant burns, rocket velocity increases in magnitude and becomes a dominant factor. Substituting values from (11) in (13) and retaining the jet thrust term and highest velocity term for each of the coefficients of (12) we obtain

$$P_o \supseteqq m^3 V K^2$$

$$P_1 \supseteqq m_3\, G_j\, K^2 + V^2 \left(\tfrac{1}{2}\, C_q \;\; \rho A\, m^2 l^2 + C_D \;\; \rho A\, m^2 K^2 + \tfrac{1}{2}\, C_l \;\; \rho A\, m^2 K^2 \right.$$

$$P_2 \supseteqq m^2\, Gj\, (C_D \;\; \rho A V K^2 + K_{jD} + \tfrac{1}{2}\, C_q \;\; \rho A l^2 V) + V^3 (\tfrac{1}{2}\, C_m \;\; m^2 \rho A l + \tfrac{1}{2}\, C_q\, C_D\, (\rho A)^2\, m l^2 + \tfrac{1}{2}\, C_D \;\; C_l \;\; (\rho A)^2\, m K^2 + \tfrac{1}{4}\, m\, C_l \;\; C_q \;\; (\rho A)^2 l^2)$$

$$P_3 \supseteqq C_D \;\; \rho A V\, m Gj\, (K_{jD} + \tfrac{1}{2}\, C_q\, \rho A l^2 V)\, \tfrac{1}{2}\, V^4 (\rho A)^2 \;\; C_D (C_m m l + \tfrac{1}{2}\, C_l \;\; C_q \;\; \rho A l^2)$$

$$P_4 \supseteqq m\, G_j\, [\, 1/2\, C_D\, C_m\, (\rho A)^2\, l V^3 + mg \cos \theta_0 \{ C_m\, \rho\, A l \;\; V\, (\phi_y \;\; - \nu_y \;\;) + 1/2\, C_q \;\; \rho\, A l^2\, \dot{\phi}_y \;\; \}\,] + 1/4\, C_D\, C_m \;\; C_l \;\; (\rho A)^3\, l\, V^5 \qquad (22)$$

In order to obtain the stability inequality we substitute (22) in (17). To get a simplified form of the resulting inequality use is made of the fact that only the thrust and velocity terms are dominant in the final phase of burning. Therefore, retaining the highest order terms with respect to thrust and velocity (here for the sake of definiteness we assume that thrust is of the order of velocity cube) we get the inequality

$$\alpha_o \;\; C_m - \alpha_1\, C_m\, C_q - \alpha_2\, C_q - \alpha_3\, C_q^{\,2} - \alpha_4\, C_q^{\,3}\, \alpha_5 < 0 \qquad (23)$$

where

$$\alpha_0 = 1/4\, C_D\, C_l \;\; (\rho A)^3\, V^5\, m^6\, G_j^2\, l\, K^4 - 1/2\, C_D\, (\rho A)^2\, V^3\, m^7\, G^3{}_j\, l\, K^4 - 1/2\, C_D^{\,2}\, (\rho A)^3\, V^5\, m^6\, G_j^{\,2}\, l\, K^4$$

$$\alpha_1 = 1/2\, C_D\, (\rho A)^3\, V^5\, m^6\, G_j^2\, l^3\, K^2 \qquad (24)$$

$$\alpha_2 = 1/2\, C_D^{\,2}\, (\rho A)^3\, V^3\, m^6\, G_j^3\, l^2\, K^4 + 1/2\, C_D^2\, C_l \;\; (\rho A)^4\, V^5\, m^5\, G^2{}_j\, l^2\, K^4 + 1/2\, K_{jD} C_D\, (\rho A)^2\, V^2\, m^6\, G_j^{\;\;3}\, l^2\, K^2 + 1/2\, C_D^3\, (\rho A)^4\, V^5\, m^5\, G_j^2\, l^2\, K^4 + 1/2\, C_l \;\; C_D\, (\rho A)^4\, V^5\, m^5\, G_j^2\, l^4\, K^2$$

$$\alpha_3 = 1/4\, C_D\, (\rho A)^3\, V^3\, m^6\, G_j^3\, l^4\, K^2 + 1/4\, C_D\, C_l \;\; (\rho A)^4\, V^5\, m^5\, G_j^2\, l^4\, K^2 + \tfrac{1}{2}\, C^2{}_D\, (\rho A)^4\, V^5\, m^5\, G_j^2\, l^4\, K^2$$

$$\alpha_4 = 1/8\, C_D\, (\rho A)^4\, V^5\, m^5\, G_j^2\, l^6$$

$$\alpha_5 = K_{jD}\, (C_D)^2\, (\rho A)^2\, V^2\, m^6\, G_j^3\, K^4 + 1/4\, C_l \;\; C_D^2\, (\rho A)^4\, V^5\, m^5\, G_j^2\, l^2\, K^4$$

Let

$$\alpha_n = \alpha'_n\, \xi \qquad\qquad n = 0, 1, \ldots 5$$

where

$$\xi = C_D\, (\rho A)^2\, V^2\, m^5\, G_j^2$$

Inequality (23) can be written as

$$C_m < \frac{\alpha'_2\, C_q + \alpha'_3\, C_q^{\,2} + \alpha'_4\, C_q^{\,3} + \alpha'_5}{\alpha'_0 - \alpha'_1\, C_q} \qquad (25)$$

Assuming

$$\frac{\alpha'_1}{\alpha'_0} = 0{\cdot}002, \;\; \frac{\alpha'_2}{\alpha'_0} = 1{\cdot}02, \;\; \frac{\alpha'_3}{\alpha'_0} = 0{\cdot}6$$

$$\frac{\alpha_4'}{\alpha_0'} = 0{\cdot}003 \;\;\text{and}\;\; \frac{\alpha_5'}{\alpha_0'} = 0{\cdot}001$$

the equation corresponding to the inequality (25) is given by

$$C_m = \frac{1 \cdot 02\, C_q + 0 \cdot 6\, C^2_q + 0.003\, C^3_q + 0 \cdot 001}{1 - \cdot 002\, C_q} \tag{26}$$

Utilising (26), C_m is plotted against Cq in Fig. 3 to give stability and instability regions for the final phase of burning.

STABILITY OF THE PROJECTED MOTION ON THE XZ—PLANE

So far we have considered the projected motion of the actual three dimensional motion on the YZ—plane. To analyse the stability of the projected motion on the XZ plane we have to proceed with the equations for projected motion on XZ plane exactly in the same manner as has been done in case of projected YZ plane motion. In XZ plane, X—axis being horizontal, the gravity component will be only along Z—axis. The term $mg \cos \theta_o\, v_x$ in (3) is very small compared to $m\, G_j$ and hence it can be neglected and (3) can be written as

$$m\dot{V} = m\ddot{Z}_o = m\, G_j - mg \sin \theta_o - \tfrac{1}{2} C_D\ \rho\, A\, V^2 \tag{27}$$

(27), (6) and (7) determine the projected motion on XZ plane and are independent of the terms with Y suffix. Hence the motion on XZ plane can be treated independent of the motion on YZ plane. (6) and (7) can be obtained from (4) and (5) respectively by dropping out from the latter gravity terms and changing the suffix Y into X and there is no contribution of $mg \sin \theta_o$ term in the perturbation equation. Hence we can obtain the results of stability analysis for the XZ plane from those of stability analysis for the YZ plane by dropping out from the latter gravity terms and changing the suffix Y into X.

Thus for initial phase of burning P_0, P_1, P_2 and P_3 will remain invariant while P_4 will be given by

$$P_4 = \tfrac{1}{2}\, m\, G_j\, C_D\, C_m\, \rho^2\, A^2\, l\, V^3 \tag{28}$$

The stability inequality comes out to be

$$\tfrac{1}{2}\, mK^2\, C_m\, C_D\, \rho^2\, A^2\, lV^3 < C_D\, \rho AV\, \{K^2_{jD} + K_{jD}\, C_D\, \rho A\, V\, K^2\} + Cq\, \{K_{jD}\, C_D\, (\rho AV)^2\, l^2 + \tfrac{1}{2}\, C_D^2\, (\rho AV)^3\, l^2\, K^2\} + \tfrac{1}{4}\, C_D\, (\rho AV)^3\, l^4\, C_q^2 \tag{39}$$

which can again be written as

$$A_o\, C_m < A_1 + A_2\, C_q + A_3\, C^2_q \tag{30}$$

In (30) the value of A_o is less and that of A_2 greater than their values in inequality (20). The values of A_1 and A_3 remain unchanged. Hence, for the curve

$$A_o\, C_m = A_1 + A_2\, C_q + A_3\, C^2_q \tag{31}$$

Let

$$A_1 / A_o = 0.12, \quad A_2 / A_o = 0 \cdot 26 \quad \text{and} \quad A_3 / A_o = 0 \cdot 48$$

(31) is represented by the curve I_x in Fig. 2. The region above the curve I_x is statically stable but dynamically unstable whereas the region below the curve I_x is both statically and dynamically stable.

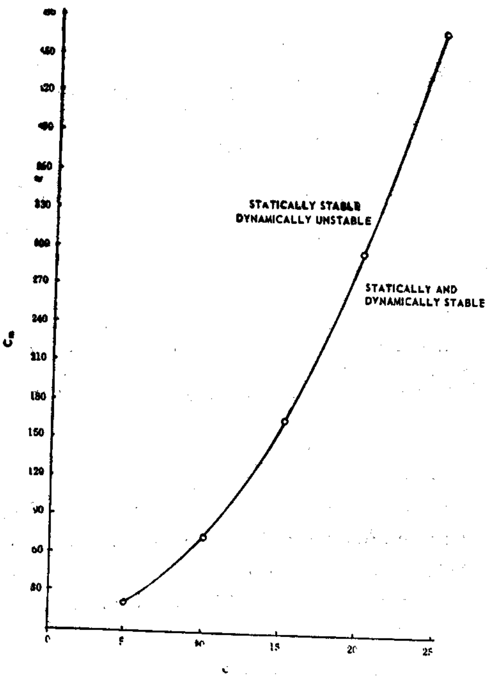

Fig. 3—Stability and unstablity regions in final phase.

In final phase of burning P_o, P_1, P_2 and P_3 come out to be the same as given by (22) but P_4 becomes

$$P_4 = \frac{1}{2}\, m\, G_j\, (\, C_D\, C_m \;\; \rho^2 A^2\; l\; V^3\,) + \frac{1}{4}\, C_D\;\; C_m\;\; C_l\;\; (\rho A)^3\; l\; V^5 \qquad (32)$$

Unlike the case of initial phase of burning, the stability criterion for final phase of burning in XZ plane remains the same as that in YZ plane and is given by (23). Hence the curve in Fig. 3 represents the stability and instability regions for the projected motion on XZ and YZ plane both.

STABILITY OF THREE DIMENSIONAL MOTION

The three dimensional motion of the rocket is stable if its projected motions on the YZ and XZ planes both are stable. Thus for the initial phase of burning the curve I_y (Fig. 2) represents the stability and instability regions of the three dimensional motion also because the two curves don't intersect and the curve I_y lies nearer to the abscissa. For the final phase of burning, the curve of Fig. 3 represents the stability and instability regions for the three dimensional motion also. Hence the regions of stability and instability of the three dimensional motion can be obtained by analysing those of the two dimensional projected motions. For initial phase of burning the analysis of the projected motion on YZ plane is only required but for the final phase of burning any one of the projected motions on YZ or XZ plane can be used.

It is to be observed that stability regions for initial phase of burning are affected by gravity whereas those for final phase of burning are unaffected. It can be physically explained by the fact that in the latter phase of burning for small variation in ρ aerodynamic forces and torques become dominant compared to gravity effects and therefore the latter have negligible effect on the stability regions.

ACKNOWLEDGEMENTS

Authors are thankful to Dr. R.R. Aggarwal for useful discussions and Dr. Kartar Singh for his encouragement in the preparation of this paper.

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