THERMAL STRESSES DUE TO DISTURBANCE OF UNIFORM HEAT FLOW BY AN INSULATED HOLE OF AEROFOIL CROSS-SECTION

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The problems of thermal stresses for ovaloid and elliptical cross-sections have been solved by A. L. Florence, J. N. Goodier and G. C. Sih. In this paper the authors have generalised the results to determine the thermal stresses due to disturbance of uniform heat flow by an insulated hole of aerofoil cross-section by making use of complex variable techniques.

Florence & Goodier[1] have solved the problem of thermal stresses due to disturbanc of unform heat flow by an insulated ovaloid hole. Sih[2] has solved this problem for an insulated hole of elliptic cross-section. In this paper the authors generalise the result and determine the thermal stresses in a plate with an aerofoil insulation. Numerical examples for particular aerofoils have been illustrated.

When an infinite plate is under uniform temperature distribution except for the disturbance caused by the presence of an insulated aerofoil hole, thermal stresses are set up in the plate. Let us suppose that the temperature gradient τ, a constant sufficiently far from the hole is directed at an angle ϕ to the x-axis of the hole.

CONFORMAL TRANSFORMATION

The mapping function

$$Z = W(\xi) = W(\rho\, e^{i\theta}) = R\left(\xi + \sum_{1}^{n} b_K\, \xi^{-K}\right) \tag{1}$$

maps the region outside the aerofoil cross-section on to the region exterior to the unit circle γ in the ξ plane.

TEMPERATURE DISTRIBUTION

The temperature distribution around an insulated circular hole of unit radius in the ξ plane is given by

$$T = \tau R \left(\rho + \frac{1}{\rho}\right) \cos(\theta - \phi) \tag{2}$$

The undisturbed temperature flow being at an angle ϕ to the positive ξ axis in the ξ plane, the complex temperature function W is given by the formula

$$W = T + iQ = \tau R \left(\xi e^{-i\phi} + \frac{1}{\xi} e^{i\phi} \right) \tag{3}$$

THERMAL DISLOCATION

If the plate with an aerofoil cross-section insulated in the Z plane is cut along the positive X-axis and allowed to deform under the temperature distribution, T is given by (2) and will be in the state of zero stress. The components of the relative displacement of points $1(r, 0)$ and $2\ (r, 2\pi)$ are given by

$$(u + iv)_2 - (u + iv)_1 = \alpha \int_1^z W dz \tag{4}$$

Substituting (1) and (2) in (4) we easily obtain

$$(u + iv)_2 - (u + iv)_1 = 2 \pi i \alpha \tau R^2 \ (e^{i\phi} - b_1 e^{-i\phi}) \tag{5}$$

CONTINUITY OF DISPLACEMENT

The displacements can be determined from the formula[3]

$$2 \mu (u + iv) = k' \phi (\xi) - \frac{\omega (\xi) \overline{\phi' (\xi)}}{\overline{\omega' (\xi)}} - \overline{\psi} (\xi) \tag{6}$$

where, for the plane stress $k' = \dfrac{3 - \nu}{1 + \nu}$, ν is the Poisson's ratio for the material of the plate, μ is the shear modulus of the material and dashes denote differentiation with respect to ξ. An isothermal state of stress and displacements having dislocation along the positive axis similar to (5) is obtained by choosing the functions ϕ_1 and ψ_1 in the form

$$\phi_1 (\xi) = B \log \xi \tag{7a}$$

$$\psi_1 (\xi) = \overline{B} \log \xi \tag{7b}$$

$$\phi = \phi_1 + \phi_0, \quad \psi = \psi_1 + \psi_0 \tag{8}$$

where B is the complex constant to be determined.

Substituting (7a) and (7b) in (6), the difference between the displacements for points $1\ (r, 0)$ and $2\ (r, 2\pi)$ is given by

$$(u + iv)_2 - (u + iv)_1 = \frac{8 B \pi i}{E} \tag{9}$$

where

$$E = \frac{2 \mu}{1 + \nu}$$

Adding the dislocation (5) and (9) and equating their sum to zero we obtain

$$4 B = - E \alpha \tau R^2 \ (e^{i\phi} - b_1 e^{-i\phi}) \tag{10}$$

The functions $\phi_0 (\xi)$ and $\psi_0 (\xi)$ must satisfy the boundary condition[1]

$$\phi (\sigma) + \frac{\omega (\sigma) \overline{\phi' (\sigma)}}{\overline{\omega' (\sigma)}} + \overline{\psi} (\sigma) + \frac{\overline{B} \sigma \omega (\sigma)}{\overline{\omega' (\sigma)}} = 0 \tag{11}$$

The functions $\phi_0 (\xi)$ and $\psi_0 (\xi)$ being analytic outside the unit circle we take

$$\phi_0 (\xi) = \sum_1^\infty \frac{a_n}{\xi^n}, \quad \psi_0 (\xi) = \sum_1^\infty \frac{a'_n}{\xi^n} \tag{12}$$

where

$$a_j = \alpha_j + i\,\beta_j, \quad \bar{a}_j = \alpha_j - i\beta_j$$
$$a'_j = \alpha'_j + i\,\beta'_j$$

Constants a_0 and a'_0 have been omitted since they do not affect the stress distribution.

STRESS DISTRIBUTION

Let $\rho\rho$ and $\theta\theta$ be the stresses acting normal to the curves ρ = constant and θ = constant respectively in the direction of ρ and θ increasing, Let $\rho\,\theta$ be the shear stress.

The stress distribution in a plate is given by

$$\rho\rho + \theta\theta = 4 \text{ Real part of } \frac{\phi'(\xi)}{\omega'(\xi)} \tag{13}$$

$$\theta\theta - \rho\,\rho + 2\,i\,\rho\,\theta = \frac{2\,\xi^2}{\rho^2\overline{\omega('\xi)}}\left[\overline{\omega\ (\xi)}\left\{\frac{\phi'(\xi)}{\omega'(\xi)}\right\} + \Psi'(\xi)\right] \tag{14}$$

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let $$\frac{\omega(\sigma)}{\omega'(\sigma)} = \frac{\left(\sigma + \frac{b_1}{\sigma} + \frac{b_2}{\sigma^2} + \ldots\ldots + \frac{b_n}{\sigma^n}\right)}{(1 - \bar{b}_1\sigma^2 - 2\bar{b}_2\sigma^3 - \ldots\ldots - n\,\bar{b}_n\,\sigma^{n+1})} = \sum_{-\infty}^{+\infty} A_n\sigma^n \tag{15}$$

Comparing the coefficients of like powers of σ, we find the following set of equations for the distribution of the constants A_n:

$$A_1 - \bar{b}_1 A_{-1} - 2\,\bar{b}_2\,A_{-2}\ldots\ldots\ldots - n\,\bar{b}_n\,A_{-n} = 1$$
$$A_{-1} - \bar{b}_1 A_{-3} - 2\,\bar{b}_2\,A_{-4}\ldots\ldots\ldots\ldots - n\,\bar{b}_n\,A_{-(n+2)} = b_1$$
$$A_{-2} - \bar{b}_1 A_{-4} - 2\,\bar{b}_2\,A_{-5}\ldots\ldots\ldots\ldots - n\,\bar{b}_n\,A_{-(n+3)} = b_2$$
$$A_{-n} - \bar{b}_1 A_{-(n+2)} - 2\,\bar{b}_2\,A_{-(n+3)}\ldots\ldots - n\,\bar{b}_n\,A_{-(2n+1)} = b_n \tag{16}$$

Using (15) and (11) and multiplying by $\frac{1}{2\,\pi i}\ \frac{d\sigma}{(\sigma - \xi)}$ where ξ is a point outside the unit circle, we easily find, after integration, that

$$\phi_o(\xi) = \frac{\sum\limits_{n=1}^{\infty} (C_{-n} - \bar{B}A_{-(n+1)})}{\xi^n} \tag{17}$$

where $$C_{-n} = \sum_{r=1}^{\infty} (r-1)\ \bar{a}_{r-1}\ A_{-(n+r)} \tag{18}$$

To determine the function $\Psi_o(\xi)$ we multiply the conjugate of (11) by $\frac{1}{2\pi i}\ \frac{d\sigma}{(\sigma - \xi)}$

and using (15) we btain after integration

$$\Psi_{\circ}(\xi) = -\overline{\omega(\xi)}\,\frac{\left[\phi'_{\circ}(\xi) + \dfrac{B}{\xi}\right]}{\omega'(\xi)} + B\sum_{1}^{\infty}\overline{A}_{-n}\,\xi^{n-1} + \sum_{1}^{\infty}\overline{C}_{-n}\xi_{n} \tag{19}$$

The constants can be determined from the formula

$$a_k = \frac{1}{2\pi i}\int_{r}\sigma^{k-1}\,\phi(\sigma)\,d\sigma$$

The final expression for the determination of the function and $\phi(\xi)$ and $\Psi(\xi)$ are obtained from (17), (18), (7) and (8) in the form

$$\left.\begin{aligned} \phi(\xi) &= B\log\xi + \frac{\sum_{1}^{\infty}(C_{-n} - \overline{B}A_{n+1})}{\xi^{n}} && (i)\\ \Psi(\xi) &= \overline{B}\log\xi - \overline{\omega(\xi)}\,\frac{(\phi_{o}'(\xi) + B/\xi)}{\omega'(\xi)} + B\sum_{1}^{\infty}\overline{A}_{-n}\,\xi^{n-1}\\ &\quad + \sum_{1}^{\infty}\overline{C}_{n}\,\xi^{n} && (ii) \end{aligned}\right\} \tag{20}$$

PARTICULAR CASES

Case 1

Writing $b_1 = m$ and $b_2 = n$ and all $b_n (= 0$ *in* (1) we find

$$Z = W(\xi) = R_n(\xi + m/\xi + n/\xi^3) \tag{21}$$

From (16), we have

$$\left.\begin{aligned} A_1 - mA_{-1} - 3n\,A_{-3} &= 1\\ A_{-1} - mA_{-3} - 3n\,A_{-5} &= m\\ A_{-2} - mA_{-4} - 3n\,A_{-6} &= 0\\ A_{-3} - mA_{-5} - 3n\,A_{-7} &= n \end{aligned}\right\} \tag{22}$$

Solving (22) we get

$$\left.\begin{aligned} &A_{-2} = A_{-4} = A_{-5} = A_{-6} = A_{-7} = 0\\ &A_{-3} = n,\ A_{-1} = m\,(1 + n)\\ &A_1 = (1 + m^2)\ \ (n + 1) + 3\,n^2 \end{aligned}\right\} \tag{23}$$

From (18) we have

$$C_{-1} = na_1 \tag{24}$$

Using (23) and (24) we get

$$\phi_{\circ}(\xi) = \frac{na_1}{\xi} - \frac{n\,\overline{B}}{\xi^3} \tag{25}$$

Using (25) and (27) we get

$$\Psi_o(\xi) = \frac{-\overline{\omega}(\xi)\ \{\phi_o'(\xi)\ B/\xi\}}{\omega'(\xi)} + nB\,\xi^2 + \text{Const.} \tag{26}$$

The constant a_1 can be determined from 20(*ii*). Finally we obtain

$$\phi(\xi) = B\ \log\ \xi - \frac{n\,\overline{B}}{\xi^2} \tag{27a}$$

$$\Psi(\xi) = \overline{B}\log\xi + nB\xi^2 - \frac{(B^2 + 2n\,\overline{B})\quad(1+m\xi^2+n\xi^4)}{(\xi^4-m\xi^2-3n)} \tag{27b}$$

These results agree with the results obtained by Florence & Goodier[1]

Case II

Writing $b_1 = m$ and all $b_n = 0$ in (1), we find as in the previous case

$$Z = W(\xi) = R\left(\xi + \frac{m}{\xi}\right)$$

$$A_1 = (1 + m^2),\ A_{-1} = m,\ C_{-1} = 0$$

$$\phi(\xi) = B\log\xi \tag{28a}$$

$$\Psi(\xi) = \overline{B}\log + \frac{B(1+m\xi^2)}{(\xi^2-m)} + \text{Const.} \tag{28b}$$

which agree with the result obtained by Florence & Goodier[1] for $m=1$

Case III

Writing all $b_n = 0$ in (1)

we find $Z = R\xi$ and

$$\phi(\xi) = B\log\xi \tag{29a}$$

$$\Psi(\xi) = \overline{B}\log\xi \tag{29b}$$

Now the stress distribution in an insulated circular hole can be determined by using (29) in (13) and (14) which reduce to

$$\rho\rho + \theta\theta = 2\left[\frac{B}{R\xi} + \frac{\overline{B}}{R\,\overline{\xi}}\right] \tag{30}$$

$$\theta\theta - \rho\rho + 2i\,\rho\theta = \frac{2}{\rho^2\,R\xi}\left[-A\,\xi\,\overline{\xi} + \overline{A}\,\xi^2 + 2A\right] \tag{31}$$

From (30) and (31) we get

$$\frac{R}{2}(-\rho\rho + i\rho\theta) = \frac{B\,\overline{\xi}}{\rho^2}\left(\frac{1}{\rho^2} - 1\right) \tag{32}$$

Using equation (10) which reduces to $B = -\frac{1}{4}E\alpha\,\tau R^2\,\epsilon^{i\phi_o}$ since $m = 0$ for a circle and putting $\overline{\xi} = \rho\,e^{-i\theta}$ in (32) it reduces to

$$\frac{R}{2}[-\rho\rho + i\,\rho\,\theta] = -\frac{\rho}{4}\left(\frac{1}{\rho^4} - \frac{1}{\rho^2}\right)E\,\alpha\tau\,R^2\,[\cos(\theta-\phi) - i\sin(\theta-\phi)] \tag{33}$$

Now separating real and imaginary parts, we easily get

$$\rho\rho = \frac{\rho}{2}\left(\frac{1}{\rho^4} - \frac{1}{\rho^2}\right) E\,\alpha\,\tau\, R \cos\,(\theta - \phi) \tag{34}$$

$$\rho\theta = \frac{\rho}{2}\left(\frac{1}{\rho^4} - \frac{1}{\rho^2}\right) E\,\alpha\,\tau\, R \sin\,(\theta - \phi) \tag{35}$$

Also (30) gives

$$\theta\theta + \rho\rho = -\frac{E\,\alpha\tau\,R}{\rho}\cos\,(\theta - \phi) \tag{36}$$

$$\theta\theta = -\frac{E\,\alpha\tau\,R}{2\rho^3}\left[(\rho^2 + 1)\cos\,(\theta - \phi)\right] \tag{37}$$

Therefore for a circle we obtain

$$(\rho\rho,\ \theta\theta,\ \rho\theta) = -\frac{E\,\alpha\tau\,R}{2\rho^3}\left[(\rho^2 - 1)\cos\,(\theta - \phi),\ (\rho^2 + 1)\cos\,(\theta - \phi),\ (\rho^2 - 1)\sin\,(\theta - \phi)\right]$$

When the heat flow is perpendicular to the X-axis then

$$\phi = \frac{\pi}{2}$$

$$(\rho\rho,\ \theta\theta,\ \rho\theta) = -\frac{E\,\alpha\tau\,R}{2\rho^3}\left[(\rho^2 - 1)\sin\theta,\ (\rho^2 + 1)\sin\theta,\ -(\rho^2 - 1)\cos\theta\right] \tag{39}$$

These results agree with those obtained by Sih[2].

REFERENCES

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