ONE DIMENSIONAL SMALL AMPLITUDE WAVES IN RADIATION GAS DYNAMICS

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Equations for one-dimensional radiation gas dynamics (*RGD*) have been derived. It is shown that the Rosseland approximation, widely used in Astrophysics, is a particular case of the new set of equations. The equation for small amplitude waves is also derived and Lick's equation is found to be a particular case of the present equation when velocity 'c' of light in the medium tends to infinity. In *RGD* there exist radiation induced waves travelling with velocities comparable to 'c' and these waves are followed by modified gas-dynamic-waves. Also when radiation pressure is comparable to gas pressure, we cannot neglect the time derivative in the radiative transfer equation, though it comes with a factor $1/c$. Many other interesting featurs of waves in *RGD* are discussed.

Stokes[1, 18] for the first time, considered the effect of radiation on sound waves in an approximate manner. Recently attention has been drawn towards the radiative effects in acoustic propagation by Prokofyev[2,3], Vincenti & Baldwin[4], Lick[5] and Moore[6]. The radiative effects in the equilibrium and pulsation of stars have been widely discussed by astrophysicists and a full discussion is available in Chandrasekhar[7]; Menzel, Bhatnagar & Sen[8]; Rosseland[9] and S. Flügge[10] and many others. The study of the effect of radiation on the shock wave propagation started with Sachs[11], Zel'dovich[12] etc. and recently a large number of papers have come on shock waves in radiating medium. But the whole discussion has been mainly based on Rosseland approximation to radiative transfer equation, which is valid when the medium is opaque. In this approximation the radiation energy density and radiation pressure are replaced by their corresponding values in thermodynamic equilibrium and thus become functions of temperature only. There is no doubt about the validity (as shown in references 7 and 8) of Rosseland approximation for stellar structure problems but its validity needs investigation for waves in *RGD* (Radiation Gas Dynamics), particularly in shock wave problems, where the variation of temperature with spatial coordinates cannot be regarded to be small and the medium may be transparent.

It is also worth noticing that the time derivative in the radiative transfer equation has always been neglected because it comes, with a factor $1/c$, where c is the velocity of light in the medium and it is very large quantity. But it changes the wave nature of the equations and many important facts are lost. The expressions for radiation energy density and radiation stress tensor in terms of the specific intensity of radiation contain c in denominators and hence if these quantities are regarded comparable to the gas internal energy density and gas pressure, c cannot be taken to be infinite.

In the present paper we have tried to give a simple set of equations for one-dimensional *RGD* based on Schuster-Schwarzchild method of dividing specific intensity into two groups. This corresponds to the Eddington approximation to radiative transfer equation. The Rosseland approximation to radiative transfer equation comes as a particular case of our equations. These equations clearly show the range of influence and domain of dependence in *RGD*. The equation for small amplitude waves is derived and it is shown that the equation discussed by Lick[5] is a particular case of it when $c \to \infty$, We have obtained a solution for a signalling problem of our acoustic equation, valid for very short time. But this is sufficient to show that there exist radiation induced waves travelling with velocities comparable to c and these waves are followed by modified gas-dynamic waves.

EQUATIONS OF *RGD*

We take a set of rectangular axes (x_1, x_2, x_3) fixed in space and neglect viscosity and molecular heat conduction everywhere. In *RGD* we come across radiation stress tensor $p_R{}^{ij}$ radiation energy density E_R and radiation flux vector F_i in addition to gas internal energy E_G per unit mass, gas pressure P_G, mass density ρ and velocity vector u_i of the fluid. $p_R{}^{ij}$ E_R and F_i are connected with specific intensity I, of radiation[7,8] by

$$p_R^{ij} = \frac{1}{c} \int I\, l_i\, l_j\, d\omega \tag{1}$$

$$E_R = \frac{1}{c} \int I\, d\omega \tag{2}$$

and

$$F_i = \int I\, l_i\, d\omega \tag{3}$$

where l_i are direction cosines of I, $d\omega$ is an element of solid angle and the integration is taken over the whole solid angle round the point under consideration. Also we have

$$E_G = \frac{p_G}{(\gamma-1)\,\rho}\ , \ p_G = R\,\rho\,T \tag{4}$$

where R is the gas constant. The equations of continuity, momentum and energy are Flügge[10].

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \tag{5}$$

$$\rho \frac{du_i}{dt} = P_i - \frac{\partial p_G}{\partial x_i} - \frac{\partial p_R{}^{ij}}{\partial x_j} \tag{6}$$

and

$$\rho \frac{d}{dt} \left(E_G + \frac{E_R}{\rho} \right) + (p_G\, \delta_{ij} + p_R{}^{ij}) \frac{\partial u_i}{\partial x_j} + \frac{\partial F_i}{\partial x_i} = 0 \tag{7}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\ ,$$

δ_{ij} are Kronecker deltas and P_i is the body force per unit volume. We shall make the assumption that the source function for radiation is

$$B \equiv \frac{\sigma}{\pi} T^4 \tag{8}$$

so that the equation of radiative transfer is

$$\frac{1}{c} \frac{\partial I}{\partial t} + l_i \frac{\partial I}{\partial x_i} = \alpha\,(B - I) \tag{9}$$

where α is the volume absorption coefficient and c is the velocity of light in the medium. We shall take α to be constant. The equations (1) to (9) are sufficient for *RGD* (Radiation Gas Dynamics) problems.

Now we write the equations for one-dimensional *RGD*, the motion being parallel to x_1 – axis and all flow and physical quantities being independent of x_2 and x_3. Then the expression

$$(p_G\,\delta_{ij} + p_R{}^{ij})\ \frac{\partial u_i}{\partial x_j}\ \text{in (7) becomes}\ (p_G + p^{11}{}_R)\ \frac{\partial u_1}{\partial x_1}.$$

We shall replace, hereafter, l_1, $p_R{}^{11}$, u_1, P_1, x_1 and F_1 by $\mu \equiv \text{Cos}\theta$, p_R, u, P, x and F respectively. Then we have from (1) to (9)

$$p_R = \frac{1}{c}\int I\,\mu^2\,d\omega, \tag{10}$$

$$E_R = \frac{1}{c}\int I\,d\omega, \tag{11}$$

$$F = \int I\,\mu\,d\omega, \tag{12}$$

$$E_G = \frac{p_G}{(\gamma-1)\,\rho},\quad p_G = R\rho T, \tag{13}$$

$$\frac{\partial\rho}{\partial t} + u\,\frac{\rho}{\partial x} + \rho\,\frac{\partial u}{\partial x} = 0, \tag{14}$$

$$\rho\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)u + \frac{\partial}{\partial x}(p_G + p_R) = P, \tag{15}$$

$$\rho\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)\left(E_G + \frac{E_R}{\rho}\right) + (p_G + p_R)\,\frac{\partial u}{\partial x} + \frac{\partial F}{\partial x} = 0, \tag{16}$$

$$B = \frac{\sigma}{\pi}\,T^4 \tag{17}$$

and

$$\frac{1}{c}\,\frac{\partial I}{\partial t} + \mu\,\frac{\partial I}{\partial x} = \alpha\,(B - I). \tag{18}$$

THE NEW SET OF EQUATION

The approximation, first introduced by Schuster and Schwarzchild, consists in assuming that the specific intensity at any point can be written

as

$$\left.\begin{aligned} I(x, t, \mu) &= I_+(x, t) \quad \text{for } \mu > 0 \\ &= I_-(x, t) \quad \text{for } \mu < 0 \end{aligned}\right\} \tag{19}$$

where I_+ and I_- are functions of x and t only. Then from (10) to (12) be can write

$$p_R = \frac{2\pi}{3c}\,(I_+ + I_-) \tag{20}$$

$$E_R = \frac{2\pi}{c}\,(I_+ + I_-) \tag{21}$$

and

$$F = \pi\,(I_+ - I_-). \tag{22}$$

Multiplying (18) by $2\pi \sin\theta\, d\theta$ and $2\pi \cos\theta \sin\theta\, d\theta$ and integrating from $\theta = 0$ to $\theta = \pi$ we have

$$\frac{2\pi}{c} \frac{\partial}{\partial t} (I_+ + I_-) + \frac{\partial F}{\partial x} = 4\pi\alpha B - 2\pi\alpha (I_+ + I_-) \tag{23}$$

and

$$\frac{1}{c} \frac{\partial F}{\partial t} + \frac{2\pi}{3} \frac{\partial}{\partial x} (I_+ + I_-) = -\alpha F \tag{24}$$

where (22) is used. Elimination of $(I_+ + I_-)$ between (20), (21), (23) and (24) gives

$$\left(\frac{\partial^2 F}{\partial x^2} - \frac{3}{c^2} \frac{\partial^2 F}{\partial t^2} \right) = 4\pi\alpha \frac{B}{\partial x} + \frac{6\alpha}{c} \frac{\partial F}{\partial t} + 3\alpha^2 F, \tag{25}$$

$$\frac{1}{c} \frac{\partial F}{\partial t} + c \frac{\partial p_R}{\partial x} = -\alpha F \tag{26}$$

and

$$E_R = 3 p_R \tag{27}$$

The set of equations (13) to (17) and (25) to (27) contain nine equations involving nine unknowns (P is taken to be known) and thus are sufficient to solve any problem of one-dimensional *RGD* provided correct initial and boundary conditions are given. Radiation travels with velocity of light, whereas equation (25) shows that $dx/dt = \pm c/\sqrt{3}$ are two characteristics of these equations. Thus it is true that though the front of any wave in *RGD* travels with velocity c, our present approximation shows that the front of the main disturbance travels with velocity $c/\sqrt{3}$ This is due to the fact that emission from a particle takes place in all directions. In the present paper, whenever we shall talk of waves moving with velocity $c/\sqrt{3}$ we shall mean this main disturbance and it should not be forgotten that the actual front of the wave moves with velocity c.

Rosseland approximation is a particular case of the present approximation and this approximation is obtained by taking steady motion and neglecting d^2F/dx^2 in comparison to $3\alpha^2 F$ in (25) and (26). Then

$$3\alpha^2 F = -4\pi\alpha \frac{dB}{dx} \tag{28}$$

and

$$-\alpha F = c \frac{d\, p_R}{dx} . \tag{29}$$

Elimination of F and integration give

$$p_R = \frac{4\pi}{3c} B + \text{constant}.$$

But $p_R = 0$ at $T = 0$ and hence

$$p_R = \frac{4\pi}{3c} B = \frac{4\sigma}{3c} T^4 \tag{30}$$

and also

$$E_R = \frac{4\sigma}{c} T^4. \tag{31}$$

Another approximation, namely transparent approximation, has also been derived to the radiative transfer equation, Wang[13]. This is obtained by assuming $c = \infty$ and α to be very small, so that from (25)

$$\frac{\partial^2 F}{\partial x^2} = 4\,\pi\,\alpha\,\frac{\partial B}{\partial x} \tag{32}$$

and integrating this we get (neglecting the constant of integration)

$$\frac{\partial F}{x} = 4\,\pi\,\alpha\,B \tag{33}$$

It is doubtful whether the constant of integration can be neglected or not. A correct formulation of transparent approximation is rederived by Murgai[14] for a plume surrounded by an atmosphere at constant temperature.

SMALL AMPLITUDE LINEAR WAVES IN RGD

The importance of linear waves in basic understanding of corresponding non-linear waves cannot be under-estimated. It is quite clear from Whitham's[15] investigations that the treatment of non-linear waves should start with a study of the corresponding linearised theory of small perturbations to fix the broad qualitative features, and then non-linear effects should be built in this basic outline. We shall neglect the body force P and deduce in this section the equation governing the linear waves in RGD. We assume that there is a uniform equilibrium state characterised by

$$u=0,\ p_G = p_{Go},\ \rho = \rho_o,\ T = T_o,\ F = o,\ p_R = p_{Ro} = \frac{4\sigma}{3c}\ T^4_o,\ E_R = 3p_{Ro} \tag{34}$$

The perturbations about this constant state are defined by

$$\left.\begin{aligned} u = u',\quad \rho = \rho_o + \rho',\quad p_G = p_{Go} + p'_G \\ F = F' \text{ and } p_R = p_{Ro} + p'_R \end{aligned}\right\} \tag{35}$$

so that

$$\left.\begin{aligned} T' \equiv T - T_o = T_o\ \left(\frac{p'_G}{p_{Go}} - \frac{\rho'}{\rho_o}\right) \\ E'_R \equiv E_R - E_{Ro} = 3p'_R \end{aligned}\right\} \tag{36}$$

From (14) and (16) we derive

$$\rho\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(E_G + \frac{E_R}{\rho}\right) - \frac{p_G + p_R}{\rho}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho + \frac{\partial F}{\partial x} = 0 \tag{37}$$

Substituting (35), (36) in equations (14), (15), (17), (25) to (27) and (37) we obtain up to first order

$$\frac{\partial \rho'}{\partial t} + \rho_o\frac{\partial u'}{\partial x} = 0, \tag{38}$$

$$\rho_o\frac{\partial u'}{\partial t} + \frac{\partial}{\partial x}\left(p'_G + p_R\right) = 0, \tag{39}$$

$$\frac{\partial}{\partial t}\left\{(p'_G + p'_R) + (3\gamma - 4)\, p'_R\right\} - \frac{\gamma\, p_{Go} + 4(\gamma - 1)\, p'_{Ro}}{\rho_o}\frac{\partial \rho'}{\partial t} + (\gamma - 1)\frac{\partial F'}{\partial x} = 0, \qquad (40)$$

$$\frac{\partial^2 F'}{\partial x^2} - \frac{3}{c^2}\frac{\partial^2 F'}{\partial t^2} = \frac{1\,6\sigma\alpha\, T_o^3}{R\rho_o}\left(\frac{\partial p'_G}{\partial x} - \frac{p_{Go}}{\rho_o}\frac{\partial \rho'}{\partial x}\right) + \frac{6\alpha}{c}\frac{\partial F'}{\partial t} + 3\,\alpha^2 F' \qquad (41)$$

and

$$\frac{1}{c}\frac{\partial F'}{\partial t} + c\,\frac{\partial p'_R}{\partial x} = -\,\alpha F'. \qquad (42)$$

From (39) it follows that there exists a potential function ϕ such that

$$u' = \frac{\partial \phi}{\partial x}\,, \quad p'_G + p'_R = -\,\rho_o\,\frac{\partial \phi}{\partial t}. \qquad (43)$$

From (38)

$$\frac{\partial \rho'}{\partial t} = -\,\rho_o\,\frac{\partial^2 \phi}{\partial x^2}. \qquad (44)$$

Substitution of (43), (44) in (40) to (42) and elimination of F' and p'_R from the three equations, thus obtained, give us

$$\left[\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{3}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - a^2_5\,\frac{\partial^2}{\partial x^2}\right)\frac{\partial \phi}{\partial t}\right] + \left[\left\{\frac{a_1^2}{(\gamma-1)\,c^2} + \frac{6\alpha}{c}\right\}\left(\frac{\partial^2}{\partial t^2} - a^2_5\,\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 \phi}{\partial t^2} + a^2_1\left\{\frac{3\,\gamma-4}{(\gamma-1)\,c^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right\}\left(\frac{\partial^2}{\partial t^2} - a^2_T\,\frac{\partial^2}{\partial x^2}\right)\phi\right] + \left[3\left(\alpha^2 + \frac{a_1^2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}\; a^2_3\,\frac{\partial^2}{\partial x^2}\right)\frac{\partial \phi}{\partial t}\right] = 0, \qquad (45)$$

where

$$a^2_1 = \frac{16\,\sigma\alpha\,(\gamma - 1)\,T_o^3}{R\,\rho_o} = 12\,(\gamma - 1)\,\alpha c\,\frac{p_{Ro}}{p_{Go}}, \qquad (46)$$

$$a^2_T = \frac{p_{Go}}{\rho_o}, \qquad (47)$$

$$a^2_5 = a^2_T\left(\gamma + \frac{a^2_1}{3c\,\alpha}\right) \qquad (48)$$

and

$$a^2_3 = a^2_T\,\frac{3\,\gamma\,\alpha^2 + \dfrac{5\,a^2_1\,\alpha}{c} + \dfrac{a_1^4}{3\,(\gamma-1)\,c^2}}{3\left(\alpha^2 + \dfrac{a^2_1\,\alpha}{c}\right)} \qquad (49)$$

We also define the isentropic sound speed a_s in a system containing matter and radiation by

$$a^2_s = \Gamma\,\frac{p_{Go} + p_{Ro}}{\rho_o}, \qquad (50)$$

where

$$\Gamma = \beta_o + \frac{(4-3\,\beta_o)^2\,(\gamma-1)}{\beta_o + 12(\gamma-1)(1-\beta_o)} \tag{51}$$

and

$$\beta_o = \frac{p_{Go}}{p_{Go} + p_{Ro}}, \tag{52}$$

a_s is the velocity of propagation of waves of small amplitude in *RGD* when flux F' is everywhere zero. Using (46) and (52) we can easily show that

$$\frac{a^2_5}{a^2_T} = \gamma + 4\,(\gamma-1)\;\frac{1-\beta}{\beta}, \tag{53}$$

$$\frac{a^2_3}{a^2_T} = \frac{(4-3\gamma)\,\beta^2 - 12\,(\gamma-1)\,\beta + 16\,(\gamma-1)}{\{1-12\,(\gamma-1)\}\,\beta^2 + 12\,(\gamma-1)\,\beta} = \frac{a^2_s}{a^2_T} \tag{54}$$

$$\underset{\beta\to 1}{Lt}\;\frac{a^2_5}{a^2_s} = 1 \tag{55}$$

and

$$\underset{\beta\to 0}{Lt}\;\frac{a^2_5}{a^2_s} = 3\,(\gamma-1). \tag{56}$$

The left hand side of equation (45) is grouped by three square brackets. Each bracket contains a homogeneous differential operator in x and t and the orders of these operators are five, four and three. If we denote these operators by P_5, P_4 and P_3 we can write (45) as

$$P_5\,\phi + P_4\,\phi + P_3\,\phi = 0. \tag{57}$$

We shall define the solutions $\{\phi\}$ satisfying $P_5\phi=0$, $P_4\,\phi=0$ and $P_3\phi=0$ by fifth order waves, fourth order waves and third order waves respectively. It is evident that the quantities u', p'_G, p'_R, F' satisfy the equation

$$P_5 f + P_4 f + P_3 f = 0 \tag{58}$$

but ρ' satisfies

$$\frac{\partial}{\partial t}\left\{ P_5 f + P_4 f + P_3 f \right\} = 0 \tag{59}$$

and hence we may have a discontinuity in ρ' and T' along the curve $\frac{dx}{dt} = 0$ in $x-t$ plane, all other physical and flow parameters being continuous across it. This corresponds to the contact discontinuity.

It can be shown that the operator P_4 can be written as

$$P_4 \equiv \left(\frac{3a_1^2}{c^2} + \frac{6\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2} - \alpha^2_1\,\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - \alpha^2_2\,\frac{\partial^2}{\partial x^2}\right) \tag{59a}$$

where

$$\alpha_1^2,\ \alpha_2^2 = \frac{A_1 \pm \sqrt{A_1^2 - 4a_1^2 a^2_T\,B_1}}{2B_1}$$

and

$$A_1 = \frac{a_1^4 a_T^{\,2}}{3(\gamma-1)\,\alpha c^3} + \frac{6a_T^{\,2}\,a_1^2}{c^2} + \frac{6\gamma\alpha\,a_T^{\,2}}{c} + a_1^2,$$

$$B_1 = \frac{3a_1^2}{c^2} + \frac{6\alpha}{c}.$$

α_1 and α_2 can be taken to be positive quantitiess satisfying $\alpha_1 > \alpha_2$.

If the terms of order $\frac{a^2_T}{c^2}$ be neglected in comprison with terms of order unity we can write

$$\left.\begin{aligned} \alpha_1^2 &= \frac{c^2}{3} \quad \frac{a^2{}_1/c}{a^2{}_1/c + 2\alpha} \\ \text{and} \qquad \alpha_2^2 &= a^2{}_T \quad , \end{aligned}\right\} \tag{59b}$$

As $c \to \infty$ in (10), (11), (45), (52) and (55) we have

$$p'_R = E'_R = 0, \ \beta = 1, \ a^2{}_5 = a^2{}_s \ ,$$

and

$$-\frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial^2}{\partial t^2} - a^2{}_s \frac{\partial^2}{\partial x^2}\right)\phi - a^2{}_1 \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial t^2} - a^2{}_T \frac{\partial^2}{\partial x^2}\right)\phi$$
$$+ 3\,\alpha^2\left(\frac{\partial^2}{\partial t^2} - a^2{}_s \frac{\partial^2}{\partial x^2}\right)\frac{\partial \phi}{\partial t} = 0, \tag{60}$$

The equation (60) is exactly the same as the acoustic equation discussed in detail by Lick[5], except that the constant $3\,\alpha^2$ in third order operator is replaced by $\frac{9}{4}\alpha^2$. Lick's equation is derived by a completely different method of approximate kernal substitution and the difference in the constant is due to different methods of approach. This agreement shows that Eddington's approximation or Schuster and Schwarzschild method of dividing specific intensity is extremely good. But the most important point to note is that when we make c tend to ∞, the nature of the equation completely changes and it changes from a fully hyperbolic equation to a mixed hyperbolic and parabolic equation and the two waves given by characsteritics $dx/dt = \pm c/\sqrt{3}$ are lost. In our new set of equations these two determine the range of influence and domain of dependence. It is also evident from (45) and (59a) that the fifth order, fourth order and third order waves are governed by equations which are hyperbolic with distinct characteristics and finite values of dx/dt along these characteristics. For the equation (45) we can prescribe five independent initial values but for (60) we can prescribe only three. Though for signalling problem such a difference in initial conditions does not give any trouble, the flow near the x-axis will be completely changed with such a difference in initial conditions for other types of problems.

(45) can be written as

$$\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{3}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - a_5^2 \frac{\partial^2}{\partial x^2}\right)\frac{\partial \phi}{\partial t} + \frac{1}{c}\left(\frac{3a_1^2}{c} + 6\alpha\right)$$
$$\times \left(\frac{\partial}{\partial t^2} - \alpha_1^2 \frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - a^2{}_T \frac{\partial^2}{\partial x^2}\right)\phi$$
$$+ 3\left(\alpha^2 + \frac{a_1^2\,\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2} - a_s^2 \frac{\partial^2}{\partial x^2}\right)\frac{\partial \phi}{\partial t} = 0 \tag{61}$$

To study dispersion relation we use non-dimensional quantities

$$\bar{x} = \alpha\, x\,, \ \bar{t} = \alpha\, a_T\, t\,, \ \bar{c} = \frac{c}{a_T}\,, \ \bar{a}_1^2 = \frac{a_1^2}{\alpha a_T}\,, \ \bar{a}_s = \frac{a_s}{a_T}, \ \bar{a}_5 = \frac{a_5}{a_T}\,,$$
$$\bar{\phi} = \frac{\phi\alpha}{a_T}\,, \ \bar{\alpha}_1 = \frac{\alpha_1}{a_T} \tag{62}$$

so that equation (61) transforms to

$$\left(\frac{3}{\bar{c}^2}\frac{\partial^2}{\partial\bar{t}^2}-\frac{\partial^2}{\partial\bar{x}^2}\right)\left(\frac{\partial^2}{\partial\bar{t}^2}-\bar{a}_5^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial\bar{\phi}}{\partial\bar{t}}$$

$$+\frac{1}{\bar{c}}\left(\frac{3\bar{a}_1^2}{\bar{c}}+6\right)\left(\frac{\partial^2}{\partial\bar{t}^2}-\bar{\alpha}_1^2\frac{\partial^2}{\partial\bar{x}^2}\right)\left(\frac{\partial^2}{\partial\bar{t}^2}-\frac{\partial^2}{\partial\bar{x}^2}\right)\bar{\phi}$$

$$+3\left(1+\frac{\bar{a}_1^2}{\bar{c}}\right)\left(\frac{\partial^2}{\partial\bar{t}^2}-\bar{a}_s^2\frac{\partial^2}{\partial\bar{x}^2}\right)\frac{\partial\bar{\phi}}{\partial\bar{t}}=0 \tag{63}$$

Substituting $\bar{\phi}=e^{i(\bar{\omega}\bar{t}-\bar{k}\bar{x})}$ in (63) we have

$$(C_1+iD_1)\bar{k}^4-(C_2 t+iD_2)\bar{k}^2+(C_3+iD_3)=0 \tag{64}$$

where

$$C_1=\left(\frac{3\bar{a}_1{}^2}{\bar{c}}+6\right)\frac{\bar{\alpha}_1{}^2}{\bar{c}},$$

$$D_1=\bar{a}_5{}^2\,\bar{\omega}$$

$$C_2=\left(\frac{3\bar{a}_1^2}{\bar{c}}+6\right)\left(1+\bar{\alpha}_1^2\right)\frac{\bar{\omega}^2}{\bar{c}},$$

$$D_2=\left(1+3\frac{\bar{a}_5{}^2}{\bar{c}^2}\right)\bar{\omega}^3-3\bar{a}_s^2\left(1+\frac{\bar{a}_1{}^2}{\bar{c}}\right)\bar{\omega},$$

$$C_3=\frac{1}{\bar{c}}\left(\frac{3a_1{}^2}{\bar{c}}+6\right)\bar{\omega}^4$$

and

$$D_3=\frac{3}{\bar{c}^2}\bar{\omega}^5-3\left(1+\frac{\bar{a}_1^2}{\bar{c}}\right)\bar{\omega}^3.$$

The roots of this polynomial are of the form $\pm\bar{k}_1, \pm\bar{k}_2$, where $\bar{k}_1$ and $\bar{k}_2$ are complex quantities. The positive and negative signs before each value suggest the possibility of propagation in positive or negative direction of x-axis with decreasing amplitude. In the absence of radiation we get only one distinct mode of propagation and hence in *RGD*, where two distinct modes of propagation are possible, one mode represents radiation induced waves and the other represents modified gas-dynamic waves.

If $\bar{k}=\bar{k}_R+i\bar{k}_i$ then the velocity of propagation is $\dfrac{\bar{\omega}}{\bar{k}_R}$ and the damping distance is $-\dfrac{1}{\bar{k}_i}$. We choose $\beta_0=0{\cdot}5$, $\gamma=5/3$ and $T_0=10^7$, as these values correspond to the conditions in hot and massive stars. Then we obtain

$$a_T=4{\cdot}08\times10^7 \text{ cm/sec.}, \quad \bar{a}_s=1{\cdot}70, \quad \bar{c}=7{\cdot}35\times10^2$$

$$\bar{a}_5=2{\cdot}08, \quad \frac{\bar{c}}{\sqrt{3}}=4{\cdot}24\times10^2. \tag{65}$$

Table 1 gives distribution of $k_1=\bar{k}_{1R}+i\,\bar{k}_{1i}$, $\bar{k}_2=\bar{k}_{2R}+i\,\bar{k}_{2i}$, $\dfrac{\bar{\omega}}{\bar{k}_{1R}}$ and

$\frac{\bar{\omega}}{\bar{k}_{2R}}$ with $\bar{\omega}$. Figures 1 and 2 give the graphs of various quantities with $\log_{10} \bar{\omega}$.

TABLE 1

$\bar{\omega}$	$\bar{k}_1$			$\bar{k}_2$		
	$\bar{k}_{1R}$	$\bar{k}_{1i}$	$\bar{\omega}/\bar{k}_{1R}$	$\bar{k}_{2R}$	$\bar{k}_{2R}$	$\bar{\omega}/\bar{k}_{2R}$
$Lt\ \bar{\omega} \to 0$	0	0	1·70	0	0	0
10^{-7}	5.92×10^{-8}	-1.47×10^{-18}	1·70	2.56×10^{-5}	-2.56×10^{-5}	3.91×10^{-3}
11^{-4}	5.92×10^{-5}	-1.47×10^{-7}	1·70	8.11×10^{-4}	-8.07×10^{-4}	1.23×10^{-1}
10^{-2}	5.65×10^{-3}	-1.63×10^{-3}	1·77	1.01×10^{-2}	-5.58×10^{-3}	9.90×10^{-1}
10^{-1}	1.58×10^{-2}	-1.45×10^{-2}	6·33	9.99×10^{-2}	-4.30×10^{-3}	1·00
1	4.82×10^{-2}	-4.76×10^{-2}	2.07×10	1·00	-4.53×10^{-3}	1·00
10	1.00×10^{1}	-3.26×10^{-2}	1·00	1.55×10^{-1}	-1.49×10^{-1}	6.45×10^{1}
10^{2}	9.98×10^{1}	-2.83	1·00	5.80×10^{-1}	-3.98×10^{-1}	1.72×10^{2}
10^{4}	5.05×10^{3}	-1.01×10^{3}	1·98	3.10×10^{1}	-9.42	3.23×10^{2}
10^{7}	4.80×10^{6}	-1.09×10^{3}	2·08	2.36×10^{4}	-1.64×10^{1}	4.24×10^{2}
$Lt\ \bar{\omega} \to \infty$	∞	-1.09×10^{3}	2·08	∞	∞	4.24×10^{2}

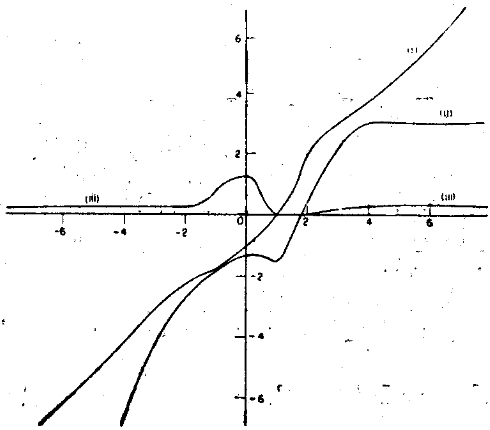

Fig. 1— (i) $\log_{10} \bar{k}_{1R}$ Versus $\log_{10} \bar{\omega}$
(ii) $\log_{10} |\bar{k}_{1i}|$ Versus $\log_{10} \bar{\omega}$
(iii) $\log_{10} \frac{\bar{\omega}}{\bar{k}_{1R}}$ Versus $\log_{10} \bar{\omega}$

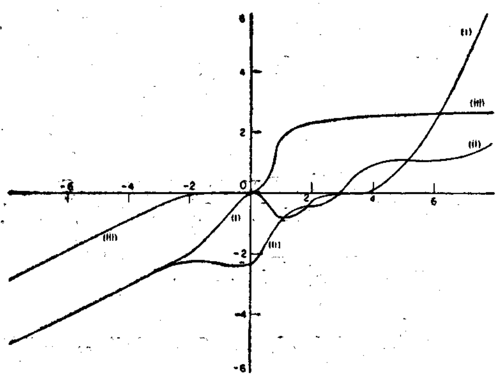

Fig. 2— (i) $\log_{10} \bar{k}_{2R}$ Versus $\log_{10} \bar{\omega}$
(ii) $\log_{10} |\bar{k}_{2i}|$ Versus $\log_{10} \bar{\omega}$
(iii) $\log_{10} \frac{\bar{\omega}}{\bar{k}_{2R}}$ Versus $\log_{10} \bar{\omega}$

We notice from (65) and Table 1 that low frequency waves are third order waves and high frequency waves are fifth order waves.

Signalling Problem:—The signalling problem which we are considering is slightly different from that considered by Lick[5] and Moore[6]. An extremely good qualitative picture is given in Lick's paper while Moore*, under the assumption of γ nearly equal to one, has given a complete history of velocity and temperture profiles.

For time $t < 0$, it is assumed here that the gas is at rest with temperature T_o pressure $p_{G0} + p_{R0}$ and density ρ_o . For $t > 0$ a constant velocity $u' = \beta$ and a constant pressure $p'_G + p'_R = \rho_o\, \epsilon$ are imposed on the boundary $x = 0$. The solution is sought for $x > 0$ and $t > 0$. Thus our differential equation is (45) with the following initial and boundary conditions :

$$\text{At } t = 0 \;,\; \phi = \phi_t = \phi_{tt} = \phi_{ttt} = \phi_{tttt} = 0 \qquad for\ x > 0 \tag{66}$$

$$\text{At } x = 0 \quad \left.\begin{aligned} \frac{\partial \phi}{\partial x} &= \beta \\ \frac{\partial \phi}{\partial t} &= -\epsilon \end{aligned}\right\} \; for\ t > 0 \tag{67}$$

If a Laplace transform with respect to time is applied to equation (45) we obtain

$$\delta_1 \frac{d^4 \bar\phi}{d x^4} - \delta_2 \frac{d^2 \bar\phi}{d x^2} + b_2^2\, p^3\, \bar\phi = 0 \tag{68}$$

where

$$\bar\phi = \int_0^\infty e^{-pt}\, \phi\, dt \;, \tag{69}$$

$$\delta_1 = a_5^2\, p + a_1^2\, a_s^2 \;, \tag{70}$$

$$\delta_2 = \left(1 + 3\frac{a_5^2}{c^2}\right) p^3 + \left[\left\{ \frac{a_1^2}{(\gamma-1)\, c^2} + \frac{6\alpha}{c} \right\} a_5^2 + \left\{ 1 + \frac{3\gamma - 4}{\gamma - 1}\, \frac{a^2_T}{c^2} \right\} a_1^2 \right] p^2 + a_T^2 \left(3\gamma\alpha^2 + \frac{5\, a_1^2\, \alpha}{c} + \frac{a_1^4}{3(\gamma-1)\, c^2} \right) p \tag{71}$$

and

$$b_2^2 = \frac{3}{c^2}\, p^2 + \left[\left\{ \frac{a_1^2}{(\gamma-1)\, c^2} + \frac{6\alpha}{c} \right\} + \frac{3\gamma - 4}{\gamma - 1}\, \frac{a_1^2}{c^2} \right] p + 3\left(\alpha^2 + \frac{a_1^2\, \alpha}{c} \right) \tag{72}$$

The solution of equation (68) finite for all positive values of x, can be found in the form

$$\bar\phi = A_1\, e^{\gamma_1 x} + A_2\, e^{\gamma_2 x} \tag{73}$$

where

$$\gamma_{1,2} = -\left[\frac{\delta_2 \pm \sqrt{(\delta_2^2 - 4\, b_2^2\, p^3\, \delta_1)}}{2\, \delta_1} \right]^{\frac{1}{2}} \tag{74}$$

*Moore's paper appeared after the completion of the present investigation.

A_1 and A_2 are found by using the values of $\bar{\phi}$ and $\frac{d\bar{\phi}}{dx}$ at $x = 0$, obtained from the Laplace transforms of the boundary conditions (67). The result is

$$A_1 = \frac{p\beta + \gamma_2 \epsilon}{p^2(\gamma_1 - \gamma_2)}, \quad A_2 = \frac{p\beta + \gamma_1 \epsilon}{p^2(\gamma_2 - \gamma_1)} \tag{75}$$

The solution for small time is obtained by the usual method of expansions for large p. We have for large p

$$\gamma_1 = -\frac{p}{a_5} - \frac{a_1^2}{a_5}\frac{a_5^2 - a_T^2}{2a_5^2}\left\{\frac{1 - \frac{3\gamma - 4}{\gamma - 1}\frac{a_5^2}{c^2}}{1 - 3\frac{a_5^2}{c^2}}\right\} + O\left(\frac{1}{p}\right) \tag{76}$$

$$\gamma_2 = -\frac{\sqrt{3}}{c}p - \sqrt{3}\left\{\alpha - \frac{a_1^2}{2(\gamma - 1)c}\frac{\frac{a_5^2 - a_T^2}{c^2}}{1 - 3\frac{a_5^2}{c^2}}\right\} + O\left(\frac{1}{p}\right) \tag{77}$$

$$A_1 = -\frac{\beta - \frac{\sqrt{3}\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\frac{1}{p^2} + O\left(\frac{1}{p^3}\right) \tag{78}$$

and

$$A_2 = \frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\frac{1}{p^2} + O\left(\frac{1}{p^3}\right) \tag{79}$$

For RGD, it follows from (46) that $\frac{a_1^2}{c}$ is finite and of order unity, but $\frac{a_5^2}{c^2}$, $\frac{a_T^2}{c^2}$ can be neglected in comparison with terms of order unity. If so we have for small t, the first approximation

$$\phi = -\frac{\beta - \frac{\sqrt{3}\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\frac{1}{p^2}\, e^{-\frac{x}{a_5}p - \frac{a_1^2}{a_5}\frac{a_5^2 - a_T^2}{2a_5^2}x}$$

$$+ \frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\frac{1}{p^2}\, e^{-\frac{\sqrt{3}x}{c}p - \sqrt{3}\alpha x} \tag{80}$$

whence

$$\phi = -\frac{\beta - \frac{\sqrt{3}\,\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\left(t - \frac{x}{a_5}\right) e^{-\frac{a_1^2}{a_5}\frac{a_5^2 - a_T^2}{2\,a_5^2}x}$$

$$+ \frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\left(t - \frac{\sqrt{3}}{c}x\right) e^{-\sqrt{3}\,\alpha x} \qquad \text{for } x < a_5 t \qquad (81)$$

$$= \frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}\left(t - \frac{\sqrt{3}}{c}x\right) e^{-\sqrt{3}\,\alpha x} \qquad \text{for } a_5 t < x < \frac{ct}{\sqrt{}}$$

$$= 0 \qquad \text{for } \frac{ct}{\sqrt{3}} < x$$

Since β and ϵ are small quantities, ϕ is of second order in β, ϵ and t. But its derivatives contain first order terms and we have from (43), up to first order in small quantities,

$$\mu' = \frac{1}{a_5}\frac{\beta - \frac{\sqrt{3}\,\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} e^{-\frac{a_1^2}{a_5}\frac{a_5^2 - a_T^2}{2\,a_5^2}x}$$

$$- \frac{\sqrt{3}}{c}\frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} e^{-\sqrt{3}\,\alpha x} \qquad \text{for } x < a_5 t \qquad (82)$$

$$= -\frac{\sqrt{3}}{c}\frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} e^{-\sqrt{3}\,\alpha x} \qquad \text{for } a_5 t < x < \frac{ct}{\sqrt{3}}$$

$$= 0 \qquad \text{for } \frac{ct}{\sqrt{3}} < x$$

and

$$\frac{p_R' + p_G'}{\rho_0} = \frac{\beta - \frac{\sqrt{3}\,\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} e^{-\frac{a_1^2}{a_5}\frac{a_5^2 - a_T^2}{2\,a_5^2}x} - \frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}}$$

$$\times e^{-\sqrt{3}\,\alpha x} \qquad \text{for } x < a_5 t\,, \qquad (83)$$

$$= -\frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} e^{-\sqrt{3}\,\alpha\, x} \qquad \textit{for } a_5 t < x < \frac{ct}{\sqrt{3}},$$

$$= 0 \qquad \textit{for } \frac{ct}{\sqrt{3}} < x$$

The solution of this problem for the equation (60) which is similar to the equation used by Lick, can be obtained fom (81) to (83) by taking $c \to \infty$, in which case $a_5 \to a_s$. Hence we have from (81)

$$\phi = -\frac{\beta}{\frac{1}{a_s}} \left(t - \frac{x}{a_s} \right) e^{-\frac{a_1^2}{a_s} \frac{\gamma - 1}{2\gamma} x} + \frac{\beta - \epsilon/a_s}{1/a_s}\, t\, e^{-\sqrt{3}\,\alpha\, x} \qquad \textit{for } x < a_s t \tag{84}$$

$$= \frac{\beta - \epsilon/a_s}{1/a_s}\, t\, e^{-\sqrt{3}\,\alpha\, x} \qquad \textit{for } x > a_s t .$$

The solution (84) can be obtained also by proceeding with equation (60) without any reference to solution (81). We make the following conclusions from (81) to (84).

(*i*) Just after the start of disturbance at $x=0$, the fifth order waves in (45) dominate and the disturbance propagates with characteristic speeds $\frac{c}{\sqrt{3}}$ and a_5. But due to the presence of lower order terms they are exponentially damped.

(*ii*) When c is finite the waves can be divided into two groups. The first group consists of radiation induced waves moving with velocities comparable to that of light in the medium. The effect of radiation is primary for these waves and the changes in gas-dynamic variables is secondary. The precursor radiation, as called by Lick[5] and Moore[6] is essentially radiation induced wave. The second group consists of modified gas-dynamic wave and the effect of radiation on these waves is secondary. In (81) the radiation induced waves and modified gas dynamic waves are given respectively by

$$\frac{\beta - \frac{\epsilon}{a_5}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} \left(t - \frac{\sqrt{3}}{c} x \right) e^{-\sqrt{3}\,\alpha\, x}$$

and

$$-\frac{\beta - \frac{\sqrt{3}\,\epsilon}{c}}{\frac{1}{a_5} - \frac{\sqrt{3}}{c}} \left(t - \frac{x}{a_5} \right) e^{-\frac{a_1^2}{a_5} \frac{a_5^2 - a_T^2}{2 a_T^2} x}$$

(*iii*) The velocity of propagation of the front of modified gas dynamic waves is a_5 for small t and this is quite different from isentropic sound speed a_s (see (53), (55), (56)) One will be tempted to drop the terms $\frac{3}{c^2} \frac{\partial^2 F}{\partial t^2}$ and $\frac{6\alpha}{c} \frac{\partial F}{\partial t}$ in (25) and $\frac{1}{c} \frac{\partial F}{\partial t}$ in (26) because c is large, but if we do so, from the equation corresponding to (45) and find the solution corresponding to (81) we shall obtain

$$\phi = -\frac{\beta}{\frac{1}{a_5}} \left(t - \frac{x}{a_5} \right) e^{-\frac{a_1^2}{a_5} \frac{a_5^2 - a_T^2}{2 a_5^2} x}$$

$$+ \frac{\beta - \frac{\epsilon}{a_5} t}{\frac{1}{a_5}} \cdot e^{-\sqrt{3\left(\alpha^2 + \frac{a_1^2 \alpha}{c}\right)}\, x} \qquad \text{for } x < a_5 t \qquad (85)$$

$$= \frac{\beta - \frac{\epsilon}{a_5} t}{\frac{1}{a_5}} \cdot e^{-\sqrt{3\left(\alpha^2 + \frac{a_1^2 \alpha}{c}\right)}\, x} \qquad \text{for } a_5 t < x$$

Comparing (81) and (85) we see that the damping distance $\frac{1}{\sqrt{3\left(\alpha^2 + \frac{a_1^2 \alpha}{c}\right)}}$ for radiation induced waves in (85) is very much different from that in (81) when $\frac{a_1^2}{c}$ as given by (46) is finite for *RGD* problems and thus, the neglect of these terms, affects the solution considerably, mainly in the value of $\frac{p_G' + p_R'}{\rho_0}$ given by (83). Thus when radiation pressure and energy density are taken into account, the neglect of $\frac{3}{c^2} \frac{\partial^2 F}{\partial t^2}$ and $\frac{6\alpha}{c} \frac{\partial F}{\partial t}$ in (25) and $\frac{1}{c} \frac{\partial F}{\partial t}$ in (26) will affect the solution considerably.

GENERAL DISCUSSION OF SMALL AMPLITUDE WAVES IN RGD

From the investigations of Whitham[15], Lick[5] and Moore[6] we make the following conclusions about the qualitative picture of waves governed by the equation (45).

For small time the fifth order waves dominate. The initial waves travel at two speeds a_5 and $\frac{c}{\sqrt{3}}$, where $a_5 \neq a_s$ and they are exponentially damped due to lower order terms. After very long time the third order waves dominate. These waves travel now at isentropic speed a_s but they are diffused due to second and higher order terms. For some intermediate time, the fourth order waves dominate. The waves travel at speeds α_1 and α_2 but these speech are different from isothermal speed a_T. Also the fourth order waves are diffused due to the fifth order terms and are exponentially damped due to the third order terms.

Both the fifth and the fourth order waves contain radiation induced waves and modified gas-dynamic waves but only modified gas-dynamic waves are present in third order waves.

It is important to note that not all waves of (45) are diffused or exponentially damped. For substituting $\phi = f(\xi)$, where $\xi = \omega t - kx$ with arbitrary values of ω and k, we find that f satisfies a fifth order ordinary differential equation in ξ with constant coefficients. Thus, given arbitrary frequency and wave number there exists wave forms, limited by exponential functions, which can propagate with speed $\frac{\omega}{k}$ without change in shape Courant & Hilbert[17].

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